Finding the Mule in the Network

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***Abstract*—There exist a variety of procedures for identifying clusters in large networks. This paper focuses on finding the connections between such clusters.**

We employ the concept of closed sets to reduce a network down to its fundamental cycles. These cycles begin to capture the global structure of the network by eliminating a great deal of the fine detail. Nevertheless, the reduced version is completely faithful to the original. No connection in the reduced version exists unless it was in the original network; connectivity is preserved.

Reductions of as much as 80% can be observed in real networks. Just reducing the size makes comprehension of the network much easier.

I. Introduction

A network, or graph, is a set P of n points, or nodes, or vertices, or individuals together with an $n \times n$ relation $\mathcal{A}$ defining the associations, or edges, between the points or individuals. Figure 1 is a representative example. Each non-zero entry,

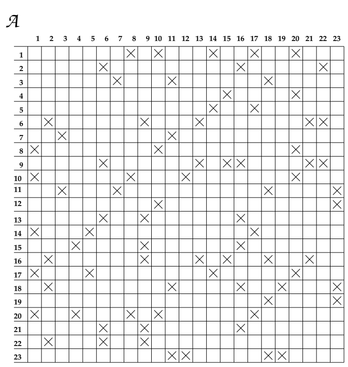

Fig. 1. An adjacency representation of a small 23 node network.

such as $(1, 8) \in \mathcal{A}$, denotes an edge, or connection, between the node, or point, 1 and the node 8. For the purposes of this paper we will assume that $\mathcal{A}$ is symmetric, or equivalently, that the network is undirected.

Networks play an important role in the study of many phenomena. Social networks describe the interactions between individuals [2], [3], [21]; the internet is a familiar feature of our every day computer life [11], [13]; genetic expression can be modeled as proteins in a regulatory network [4], [9]. In these examples, and others, the structure of these networks plays a vital role. But what is a network's "structure"?

Frequently, some of the nodes form "clusters", or "communities", with many intra-cluster connections. The identification of such clusters has been the focus of considerable research; there is an immense amount of literature in disparate fields, of which [1], [5], [15], [17], [18] is but a small fraction. The focus of our research, however, is the identification of those nodes, or connections, that "go between" different clusters. They are often harder to discover. In the drug trade, a "mule" is a go-between that brings drugs from a cluster of producers to a community of users. Hence the term "mule" in our title. In real life, mules are equally hard to discover. To find mules we must understand the network structure.

Possibly the most effective approach to understanding the overall structure of a network, when it works, is to graphically represent it [6]. Many times this allows a visual comprehension of its essential features. Figure 2 is one such representation of Figure 1. Readily, this particular rendition

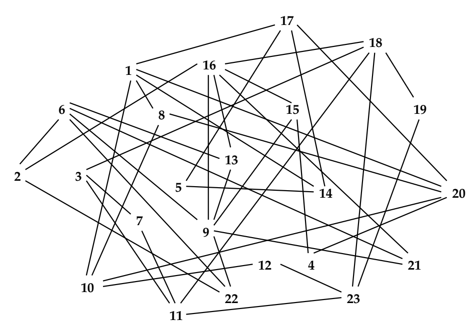

Fig. 2. A graphic representation of Figure 1.

yields little insight! One can improve the picture considerably by iteratively moving nodes and their connections, so as to shorten the connecting edges and thus cluster related nodes. We'll see such a reorganized drawing later as Figure 8 in Section III. Unfortunately however, most interesting networks are "large", far too large to effectively organize in such an iterative fashion without some prior understanding of the underlying structure.

A common approach is to try to spatially cluster those highly interconnected nodes and separate them in the image. Then one should be able to spot the connecting edges and outlier points. But, as suggested by much of the literature cited above, clustering is not an automatic process. For example, many techniques must first input the estimated number of clusters to function effectively.

The approach described in this paper involves reducing the network to its "fundamental cycles". These are cycles of length greater than 3 with no cross connections. The reduction process and the nature of fundamental cycles is explored in Section III. The network shown in figures 1 and 2 will be used as a running example in this section. But, it is far to small to be considered realistic. In Section IV we explore this reduction process as it is applied to larger, real life networks. In Section V, we examine some of the underlying mathematics.

II. Neighborhood Closure

A collection of sets $\mathcal{C} = \{Y_i\}$ is a **closure system** on a set P provided (a) $P \in \mathcal{C}$, and (b) $Y_i \cap Y_k \in \mathcal{C}$ for all i, k. We use the neighborhood concept to establish a closure system on a network, or graph, $N = (P, \mathcal{A})$. Non-zero entries (x, z) in $\mathcal{A}$ denote some form of connection between x and z. They constitute edges in a graphical representation of N such as figures 2 or 3.

Let $Y \subseteq P$, the **neighborhood** of Y, denoted $Y.\eta$, is $Y.\eta = \{z \in P{-}Y \mid (y_i, z) \in \mathcal{A}$ for some $y_i \in Y\}$. That is, $Y.\eta$ consists of all points not in Y that are connected to at least one element of Y. By the **region** dominated by Y, denoted $Y.\rho$, we mean $Y.\rho = Y \cup Y.\eta$, or Y together with its surrounding neighborhood. Finally, we define the **closure** of Y, denoted $Y.\varphi_\eta$, by $Y.\varphi_\eta = \{z \in Y.\eta \mid z.\eta \subseteq Y.\rho\}$, that is, those neighbors z whose own neighborhood is completely dominated by Y. A set Y is **closed** if $Y.\varphi_\eta = Y$. The collection of sets $\{Y_i.\varphi_\eta\}, Y_i \subseteq P$ constitute a closure system. Readily, $P.\varphi_\eta = P$ and it is not hard to show that $Y_i.\varphi_\eta \cap Y_k.\varphi_\eta$ is closed.

In Figure 3, $\{a\}.\eta = \{b, c, d\}$. In our work, all

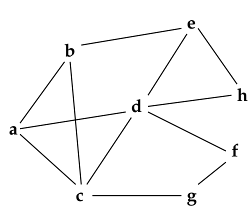

Fig. 3. An undirected graph or network.

the elements are sets; thus we regard the "point" a to be a singleton set $\{a\}$. But, repeated use of the delimiters $\{, \}$ becomes tedious, so we will normally elide them and write $a.\eta = bcd$, unless we want to specifically emphasize the set nature.

The region dominated by d, or $d.\rho$, is $abcefh$. Readily $h.\eta \subseteq d.\rho$, so $h \in d.\varphi_\eta$. Similarly, $a.\eta \subseteq c.\rho$, so $a \in c.\varphi_\eta$. In the network of Figure 3, $c.\varphi_\eta = ac, d.\varphi_\eta = dh$, and $x.\varphi_\eta = x$ for all $x \neq c$ or d. Consequently, we see that the individual points of a network may, or may not, be closed. If every singleton set (*i.e.* point) is closed, the network is said to be **irreducible**.

There is a much more detailed treatment of neighborhood closure in [20] where it is used to define continuous network change. But, we will not need it here. Instead we focus solely on those nodes z for which there exist a singleton node y such that $z \in \{y\}.\varphi_\eta$. We say z is **subsumed** by y.

III. Network Reduction

In classical mathematics, open sets are used to establish the topology of continuous manifolds. We find closed sets are more valuable tools for understanding discrete structures like a network.

If y subsumes z in N, then any closed set structure involving z must equally include y, so z contributes little to our understanding of N. We can delete z and all incident edges with almost no loss of information. If the subsumed points a, h and incident edges are removed from Figure 3 we obtain Figure 4. The deleted nodes and edges are indicated

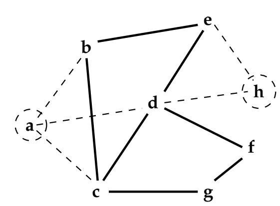

Fig. 4. Figure 3, less its subsumed points.

by dashed lines. Each of the remaining nodes, *e.g.* $\{b\}$, is a closed set.

We have automated this rather simple process of removing subsumed points from a network. The core of the reduction process, which we will denote by ω, is the following loop: it iteratively removes

```
for_each y in P
   {
   for_each z in y.nbhd
      {
      if (z.nbhd contained_in y.region
         {        // z is subsumed by y
         for_each x in z.nbhd
            remove edge (x, z)
         remove z from network
         }
      }
   }
```

Fig. 5. Key loop in reduction process.

subsumed points and edges until all singleton sets ($i.e.$ nodes) are closed and the network is reduced.

Applied to the network of Figure 2, we obtain the network of Figure 6 with only 11 nodes and 13 edges.

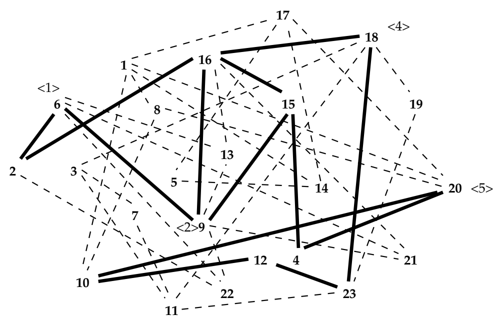

Fig. 6. The reduced network of Figure 2.

A. Fundamental Cycles

The striking characteristic of the irreducible subnetworks of figures 4 and 6 is that the remaining nodes and connecting edges form cycles. The two cycles of Figure 4 are $< b, c, d, e, b >$ and $< c, d, f, g, c >$. A cycle $< y_0, y_1, \ldots, y_n = y_0 >$ of length $n \geq 4$ is said to be a **fundamental cycle** if $y_{i+1} \in y_i.\eta$, for $0 < i < n$, and $y_k \in y_i.\eta$ implies $k = i + 1$. That is, there are no "cross connections". The two cycles of length 4 in Figure 4 are fundamental. A graph/network composed only of fundamental cycles is in many ways the antithesis of "triadic closure" in social theory [12], [16] and "chordal graphs" in graph theory [10], [14]. The latter are graphs with no chordless cycles of length greater than 3.

The cyclic structure of Figure 6 is more evident when redrawn as Figure 7. Here, the fundamental

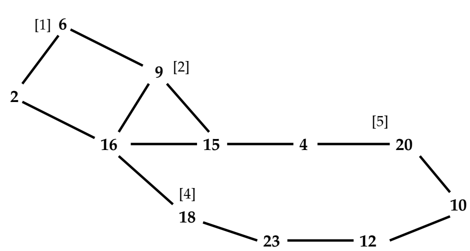

Fig. 7. The fundamental cycles of Figure 6.

cycle $< 16, 15, 4, 20, 10, 12, 23, 18, 16 >$ of length 8 is quite prominent. Longer fundamental cycles convey more information about the global structure of the network.

Our program keeps a count of the numbers of nodes subsumed directly, or indirectly, by each node. The numbers in angle brackets, $< n >$, indicate this. The number, $< 5 >$, of subsumed nodes associated with node 20, and $< 4 >$ associated with node 18 suggest that there may be small clusters attached to the cycle at these points. The count of subsumed nodes, $< 1 >$ and $< 2 >$, associated with nodes 6 and 9 are less convincing.

In Figure 8 we have reconstituted the original network around the two fundamental cycles of Figure 7. It is similar to one found in [17]. This is

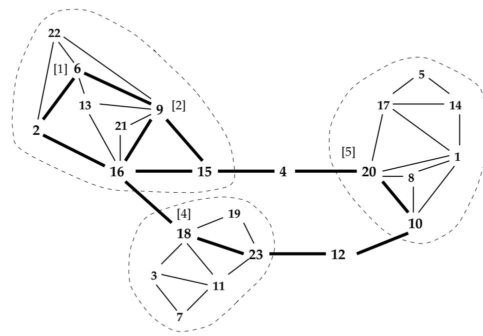

Fig. 8. Figure 2 reconstituted around its fundamental cycles.

accomplished by iteratively expanding each point on a fundamental cycle. For example, nodes 13 and 21 were the two points subsumed by node 9. Node 6 subsumed 22.

In Figure 8 we have circled the small clusters with dashed lines. In the program they are simply denotes as sets of nodes. It is fairly obvious that 4 and 12 are the "mules" in this small network. The cross connection between 16 and 18 may also be of interest.

IV. Network Reduction in Real Life

A more striking example is the reduction of the 379 node network of collaborating scientists constructed by M.E.J. Newman [18]. The reader is encouraged to view an annotated version at `www.umich.edu/m̃ejn/centrality`.

After 3 iterations identifying and removing subsumed points, our program produced the graph of Figure 9. There are 65 points in this reduced network. This network is much more comprehensible, both visually and algorithmically, than the original. Values $< n >$ denote the number of points subsumed by each node.

Our graph reduction algorithm removes the pendant chordal subgraphs that were present in the original. Basically, only the key fundamental cycles remain, together with those triangles all of whose points lie on some chordless cycle of length ≥ 4. But the information loss is minimal. As seen above, the subsumed portions can be regenerated if necessary. Of more significance to us are those retained nodes, and paths in Figure 9, which "connect" the communities of intellectual activity.

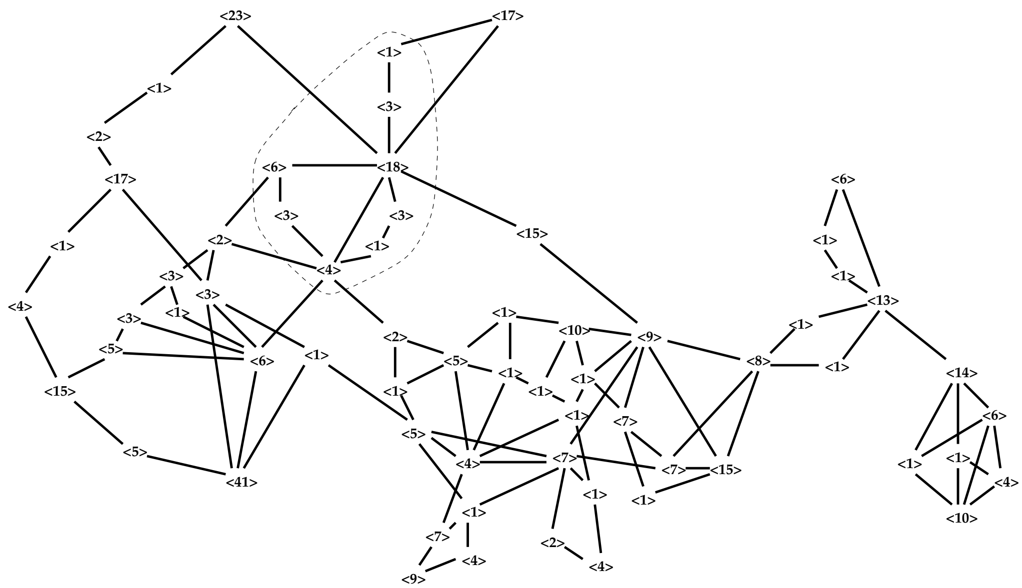

Fig. 9. The reduced version of Newman's 379 node collaboration network.

In his book [8], Malcolm Gladwell talks of connectors which are the central nodes of communities, or clusters, comprised of many connections. What we see in Figure 9 are those singular "connectors" $between$ clusters which are rarer, and harder to find in $\mathcal{A}$ by more usual numerical methods. For example, there is a connection/edge between the node at the top, denoting a cluster of $< 23 >$ individuals with a node representing $< 18 >$ persons. Similarly, there is a path towards the lower left through nodes labeled $< 1 >$ and $< 2 >$ to a community of $< 17 >$ individuals. These two nodes are likely candidates to be mules.

Because of the arbitrary order in which nodes are subsumed, which was exhibited to some extent in the expansion of Figure 8, the values $< n >$ should not be taken too literally. For example, one could argue that the nodes encircled with the dashed line actually comprise a single community. But, the node $< 18 >$ would be a natural candidate for automatic re-expansion, subject to whatever independent community criteria is appropriate for the application.

A. Process Performance

We have not had the opportunity to process enough large networks to accurately measure the performance of our reduction algorithm. But, it appears to be essentially linear in the number, n, of nodes of the network.

First, all operations are performed on sets, or collections of sets. We use a C++ software package that represents every set, or collection, as a simple bit string. Individual points and edges are only involved in conversion to/from singleton sets and neighborhoods during input and output. Consequently, an operation such as

```
if (z.nbhd contained_in y.region)
```

involves only the intersection (logical and) of two bit strings followed by a test for equality. The code does not loop over either set.

Next, we observe that the process of Figure 5 loops over all n nodes, y, of the network. And then the inner loop runs over its neighborhood, $y.\eta$. If N is a complete graph then $|y.\eta| = n - 1$ and the order of ω would appear to be n^2 in this worst case. But, since nodes are removed from P within the loop, all of the nodes of $Y.\eta$ will be subsumed and the outer loop will execute but once. It still appears to be linear.

In practice, the size of neighborhoods is bounded, say $|y.\eta| \neq k$, and average neighborhood size is quite a bit smaller. Consequently execution of the reduction loop in Figure 5 is bounded by $k \cdot n$.

However, the subsumption loop of Figure 5 may have to be iterated. A point z may not be subsumed until after other points in its neighborhood, $z.\eta$, have been subsumed and removed. This is dependent on the order in which the points y of the outer loop are accessed. We are convinced, but have no example, that there exists a worst case ordering of the points such that only one node is subsumed in each iteration. Thus the worst case behavior could be of order n^2.

However, in practice, no network reduction has required more than 3 iterations, and each iteration is over a set P of decreasing size. As indicated above, we believe this process is effectively linear and thus scalable.

V. Mathematical Details

Although network reduction is a process that largely speaks for itself, its mathematical underpinnings can be of interest. We explore these details here.

A graph, or subgraph, is said to be **chordal** if it contains no cycles of length greater than 3 without a chord (edge) joining two of its points [10], [14]. Any complete graph, K_n, is chordal. Every tree is chordal. In fact, chordal graphs can be regarded as tree-like structures of point connected, or edge connected, complete graphs K_n. A cycle $C = < y_0, \dots y_n = y_0 >$ is **chordless** if no subset of its points $\{y_i, \dots y_k\}, 0 \leq i, k \leq n$ forms a cycle. We will call a chordless cycle, such as C, an n-cycle, where n denotes both the number of points and number of edges. Thus, a graph is chordal if it contains no n-cycles, $n \geq 4$.

An n-cycle C is **fundamental** if for all $y_i \in C$, $\{y_i\}$ is closed. The key elements of Figure 8 are its fundamental 4-cycle $< 2, 6, 9, 16, 2 >$ and its 8-cycle $< 16, 15, 4, 20, 10, 12, 23, 18, 16 >$. These fundamental cycles define the topology of the network in much the same manner that 1-cycles can be used to define the topological structure of manifolds [7].

Proposition 5.1: Let G be a finite network and let $G' = G.\omega$ be a reduced version, then G' is irreducible.

Proof: Suppose $\{y\}$ is not closed. Then $\exists z \in y.\varphi_\eta$ implying $z.\rho \subseteq y.\rho$ or that z is subsumed by y contradicting termination of the reduction code.

Proposition 5.2: Let G be a finite network with $G' = G.\omega$ an irreducible version. If $y \in G'$ is not an isolated point then either

(1) there exists a fundamental n-cycle C, $n \geq 4$ such that $y \in C$, or

(2) there exist fundamental n-cycles C_1, C_2 each of length ≥ 4 with $x \in C_1$ $z \in C_2$ and y lies on a path from x to z.

Proof: Let $y_1 \in P_{G'}$. Since y_1 is not isolated, let $y_0 \in y_1.\eta$, so $(y_0, y_1) \in \mathcal{A}$. With out loss of generality, we may assume $y_0 \in C_1$ a cycle of length ≥ 4. Since y_1 is not subsumed by y_0, $\exists y_2 \in y_1.\eta, y_2 \notin y_0.\eta$, and since y_2 is not subsumed by y_1, $\exists y_3 \in y_2.\eta$, $y_3 \notin y_1.\eta$. Since $y_2 \notin y_0.\eta$, $y_3 \neq y_0$.
Suppose $y_3 \in y_0.\eta$, then $< y_0, y_1, y_2, y_3, y_0 >$ constitutes a n-cycle $n \geq 4$, and we are done.
Suppose $y_3 \notin y_0.\eta$. We repeat the same path extension. $y_3.\eta \not\subseteq y_2.\eta$ implies $\exists y_4 \in y_3.\eta$, $y_4 \notin y_2.\eta$. If $y_4 \in y_0.\eta$ or $y_4 \in y_1.\eta$, we have the desired cycle. If not $\exists$ $y_5, \dots$ and so forth. Because G is finite, this path extension must terminate with $y_n \in y_i.\eta$, where $0 \leq i \leq n-3$. Let $x = y_0, z = y_n$. ∎

The points of those chordal subgraphs still remaining in Figure 8 such as the triangle $< 9, 15, 16 >$, are all elements of other fundamental cycles as predicted by Proposition 5.2.

Proposition 5.3: Let $G' = G.\omega$ be an irreducible version of a finite network G. A path (possibly of length 0) exists between $w' = w.\omega$ and $x' = x.\omega$ in G' if and only if there exists a path between w and x in G.

Proof: First, let there be a path from w to x in G. If z is on some path between w and x, and z has been subsumed by y in the reduction process, then $z.\eta \subseteq y.\eta$ implies there exists a path between w and x through y.
Now, assume there exists a path between $w' = w.\omega$ and $x' = x.\omega$ in G'. Suppose $w' = x'$ then w and x have been subsumed by some single point, say y. Since subsumption, of say w, requires $w \in y.\eta$, and similarly for x, y must lie on a path between w and x in G.
Now, suppose $w' \neq x'$. The reduction process never adds an edge, so the edges in the path between w' and x' in G' are edges in G, and there exists a path from w in the tree-like chordal graphs that mapped onto w' and similarly a path from x in the tree that mapped on to x'; thus a path between w and x in G. ∎

Corollary 5.4: If a finite network G is connected then its reduced network $G' = G.\omega$ is connected.

Note, however, that the reduced graph G' may be a single point. In particular, this will be the case whenever G is a tree, or a chordal graph.

If x and y can mutually subsume each other in P, then $x.\eta = y.\eta$. But the converse need not be true as illustrated by Figure 10. The graph (a) is irreducible,

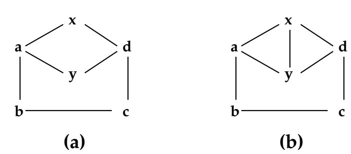

Fig. 10. Mutual subsumption illustrated by two graphs.

all singleton points z are closed. The graph (b) will become irreducible with the deletion of either x or y. In graph (b) $x.\rho = y.\rho$, the necessary and sufficient condition for mutual subsumption.

Two graphs, or networks, $G = (P, \mathcal{A})$ and $G' = (P', \mathcal{A}')$ are said to be **isomorphic**, or $G \cong G'$, if there exists a bijection, $i : P \to P'$ such that for all $x, y \in p$, $(i(x), i(y)) \in \mathcal{A}'$ if and only if $(x, y) \in \mathcal{A}$. That is the mapping i precisely preserves the edge structure, or equivalently its neighborhood structure, that is $i(y) \in i(x).\eta'$ if and only if $y \in x.\eta$.

As indicated by Figure 10, the order in which points, or more accurately the singleton subsets, of G are encountered can alter which points are subsumed and subsequently deleted. Nevertheless, the reduced graph $G.\omega$ will be unique, upto isomorphism.

Proposition 5.5: Let $G.\omega$ and $G.\omega'$ be irreducible images of a finite network G, then $G.\omega \cong G.\omega'$.

Proof: Let $y_0 \in G.\omega$, $y_0 \notin G.\omega'$. Then y_0 is subsumed by some point y_1 in $G.\omega'$ and $y_1 \notin G.\omega$ else because $y_0.\rho \subseteq y_1.\rho$ implies $y_0 \in \{y_1\}.\varphi_\eta$ so $G.\omega$ would not be irreducible.
Similarly, since $y_1 \in G.\omega'$ and $y_1 \notin G.\omega$, there exists $y_2 \in G.\omega$ such that y_1 is subsumed by y_2. Now we have two possible cases; either $y_2 = Y_0$, or not.
Suppose $y_2 = y_0$ (which is most often the case), then $y_0.\rho \subseteq y_1.\rho$ and $y_1.\rho \subseteq y_0.\rho$ or $y_0.\eta = y_1.\eta$. Hence $i(y_0) = y_1$ is part of the desired isometry, i.
Now suppose $y_2 \neq y_0$. There exists $y_3 \neq y_1 \in G.\omega'$ such that $y_2.\rho \subseteq y_3.\rho$, and so forth. Since G is finite this construction must halt with some y_n. The points $\{y_0, y_1, y_2, \ldots y_n\}$ constitute a complete graph Y_n with $\{y_i\}.\rho = Y_n.\rho$, for $i \in [0, n]$. In any reduction all $y_i \in Y_n$ reduce to a single point. All possibilities lead to mutually isomorphic maps. ■

Proposition 5.6: Let $G' = G.\omega$ be an irreducible image of a finite network G. For all $Y \subseteq P_G$, $Y.\varphi_\eta.\omega \subseteq Y.\omega.\varphi_\eta{}'$.

Proof: Let $z \in Y.\varphi_\eta$. We must show that either $z \in Y$ or $z \in Y.\omega$ or $z \in Y.\omega.\varphi_\eta{}'$.
If $z \notin Y$, then $\exists y_0 \in Y, z \in y_0.\eta$. If $z \notin Y.\omega$, then $\exists x \in z.\eta, x \notin y_0.\eta$ so $\exists y_1 \in Y, x \in y_1.\eta$. Whether y_0, z, x, y_1 are distinct in $G.\omega$ or not, by Prop. 5.3, $x \in Y.\omega.\eta'$ so $z \in Y.\omega.\varphi_\eta{}'$. ■

An operator, such as ω, satisfying the containment property of Proposition 5.6 is said to be **continuous** [19], [20]. Continuous operations on discrete spaces have a number of interesting properties; but they are not relevant to this paper. Nevertheless, that this reduction process is a continuous transformation of the network N onto its irreducible version $N.\omega$ is quite satisfying.

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