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THE ROLE OF CONTINUOUS PROCESSES IN COGNITIVE DEVELOPMENT

JOHN L. PFALTZ

Abstract. Many scientists observing the cognitive development in children have noted distinct phases in the way they learn. One phase appears to be a gradual accumulation of experience. Another phase appears to be a reorganization of those experiences to make them more useful. In this paper we show how mathematical closure concepts can be used to abstractly model these cognitive processes.

Closed sets, which we will call knowledge units, represent tight collections of experience, facts, or skills, *etc.* Associated with each knowledge unit is the notion of its generators consisting of those attributes which characterize it.

Finally, we provide a rigorous mathematical model of these different kinds of learning in terms of continuous and discontinuous transformations. There are illustrations of both kinds of transformation, together with necessary and sufficient criteria for certain kinds of transformation to be continuous. By using a rigorous definition, one can derive necessary alternative properties which may be more easily observed in experimental situations.

The formal mathematics is illustrated with reference to Lev Vygotsky's view of cognitive psychology, but it is not a verification of his model. We believe that this concept of "continuity" can be refined to test, and possibly verify, his and other models of cognitive behavior.

1. Introduction

It is generally accepted that mental cognition occurs in the brain, which is itself comprised of a network of neurons, axons, and synapses. Neuroscientists have a rather clear understanding of the physical layout of the brain, including which portions are responsible for individual mental functions [12]. But, how mental processes actually occur is still elusive. Nevertheless, it is clear that the response to external stimuli occurs in a reactive network. Thus, if we want to understand cognitive behavior we must, at some level, come to grips with network behavior.

A recurring theme in the studies of Lev S. Vygotsky [5], one of the founding fathers in childhood cognitive psychology, is that of a distinction between two kinds of learning. Vygotsky speaks of periods of *development* in which the mind seems to be, more or less, passively assimilating its inputs from the external world, and *qualitative transformations* leading to higher psychological functions. He later equates this dichotomy with, the politically correct terms in prewar Russia, *evolution* and *revolution*: "scientific thought ... sees revolution and evolution as

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two forms of development that are mutually related and mutually presuppose each other" ([5], p. 73). "The history of child behavior is born from the interweaving of these two lines" ([5], p. 46). Roth [31] asserts that Vygotsky used the term *neoformation* to denote the qualitative change in one's knowledge, in contrast to the quantitative accumulation that occurs during development.

In this paper, we will use the mathematical terms *continuous* and *discontinuous* to differentiate between these two kinds of learning. It is the difference between smooth incremental assimilation, or development, and transitional reorganization. We are not the first to employ these two mathematical terms. Han van der Maas and Peter Molenaar [39] made use of catastrophe theory to model the difference between smooth continuous aggregation of knowledge and abrupt reorganization of it. Paul van Geert [40] used classical systems modeling to do the same thing. Both employed the classic mathematical model of a real functional output over a multivariate system of numeric inputs. What will be novel in our approach is that we do not define continuity in a numerical fashion; it is defined solely in terms of discrete network behavior.

A desirable consequence of this approach is that the modeling is scale invariant. Networks have different *granularities*. The granularity of the neural networks of the mind is very different from the granularity of large scale, social networks. An example of this is the rather small social network of 379 collaborating scientists, constructed by M.E.J. Newman [17], that is illustrated in Figure 1.[1] In contrast,

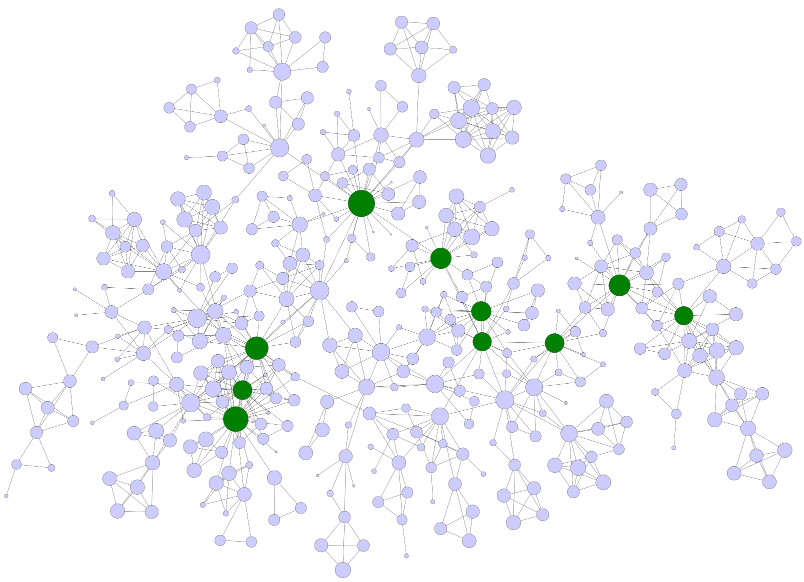

Figure 1. Newman's 379 node collaboration network.

the smallest network granularity, of which we are aware, is one composed of polymer strings forming chordless cycles (described in Section 6.2) which are thought

[1]Figure reprinted with permission from M.E.J.Newman, Phys. Rev. E 74, 036104 (2006). Copyright (2006) by the American Physical Society.

to regulate the transport of proteins through the boundary of the nucleus within a cell [10, 20, 42].

In Section 2, we will introduce the idea of an experiential operator, ρ, which expresses the relationship between the elements of a network. The elements can be raw visual stimuli, at a lower level, or concepts and ideas, at a higher level.

In Section 3 we introduce the concept of *closure*, which identifies closely related elements. Closure is central to our mathematics. We conjecture that closure is a formal analog of *naming*, which appears in Vygotsky's approach, and that closure, or naming, is a process that is essential for abstract, or higher level, thinking.

For want of a better term we call closed sets, *knowledge units*. Properties of these knowledge units are developed in Section 4.

It is not until Section 5 that we actually encounter network transformations, and define the concept of *continuity*, and through its negation, *discontinuity*. We will examine several continuous and discontinuous network transformations and provide necessary and sufficient conditions for a simple transformation to be continuous. This section is the meat of the paper.

Section 6 sketches some of the techniques we have found valuable for dealing with larger networks such as Figure 1. One of these tools is a *reduction* process which reduces the complexity of the network, and which can also be interpreted as a paradigm of cognitive abstraction.

2. The experiential operator

Let U denote the finite universe of all awarenesses, sensations, *etc.* that an individual might experience, $U = \{\ldots, w, x, y, z\}$.[2] We denote sets by $\{\ldots\}$ and by upper case letters. Thus, $Y = \{x, z\}$ is a set of two possible experiences in U. Y is said to be a subset of (or contained in) U, denoted $Y \subseteq U$.

Experiences are related to one another. If z is related to x, say for example that z can be experienced having once experienced x, we denote the relationship by $x \rho z$. Relationships may, or may not, be symmetric; we need not have $z \rho x$. Based on known neural morphology [12], most neural cells have many inputs and relatively few outputs, so we can assume most relationships to be asymmetric. Relationships come in a great many varieties. Experiential events can be simultaneous or sequenced in time; can be adjacent or distant in space; can be synonyms or antonyms in a lexical space; or can be friendly or threatening in an emotional space. But for this paper, we assume only one generic relationship. By ρ we mean that some relationship exists. Throughout this paper, we are going to let the term "experience" be generic. We might have related visual stimuli comprising a visual object, or related skills comprising a skill set, or related facts comprising an area of knowledge. All will be regarded as experiential.

Relationships are frequently visualized by means of graphs, or networks, such as Figure 2. Here an edge between x and z denotes $x \rho z$. If no arrow head is present, it is assumed that the relation is symmetric.

[2]This finiteness constraint can be relaxed somewhat, but there is relatively little yield for the resulting complexity.

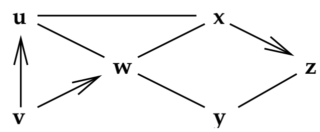

Figure 2. A very small network depicting the relationships, ρ, between 6 experiential elements.

While graphs can provide a valuable intuition, we prefer to regard relationships as operators that map subsets of U onto other subsets in U. Thus, we will denote ρ by the expression $\{x\}.\rho = \{z\}$, that is, ρ operating on x yields z, or because we tacitly assume x is related to itself, $\{x\}.\rho = \{x, z\}$. In Figure 2, $\{x\}.\rho = \{u, w, x, z\}$ and $\{z\}.\rho = \{y, z\}$. Using this kind of suffix notation is a bit unusual, but it has value. One reason for preferring an operator notation is that, in order to experience c, it may be necessary to first experience both a and b, that is, $c \in \{a, b\}.\rho$, but $c \notin \{a\}.\rho$ and $c \notin \{b\}.\rho$. For example, for a neuron c to respond, it may need signals from both a and b. So properly, ρ is a function on sets, not individual elements of U. The relation ρ of Figure 2 is said to be *graphically representable* because, for all z, if $z \in Y.\rho$, there exists $y \in Y$ such that $z \in \{y\}.\rho$. Not all relations have this property. A second reason is that in later sections we will compose the functional operators, and suffix notation lets us read the composition in a natural left to right manner.

To formalize this, we let 2^U denote all possible combinations of "experiences" in the universe U. Mathematically, it is called the *power set* of U. The relationship operator, ρ, maps subsets $Y \subseteq U$ into other subsets, $Z = Y.\rho \subseteq U$. By convention we assume that every experience is related to itself, so that, for all Y, $Y \subseteq Y.\rho$. Consequently, ρ is an *expansive* operator. This is precisely what we want; ρ denotes the possibility of expanding one's realm of experiences. For example, having the experiences x and y, it may be possible to also experience z, or $\{x, y\}.\rho = \{x, y, z\}$.

We will also assume that a greater collection of experience will permit a greater awareness of possible new experience. That is, $X \subseteq Y$ implies $X.\rho \subseteq Y.\rho$. Then, ρ is said to be a *monotone* operator. Monotone, expansive operators have been called *dominance* operators in the literature [14, 28]. We prefer to regard ρ as an *experiential* operator that expands an individual's awareness.

3. Closure operators and knowledge units

Certain collections of experiences, of facts, of abilities, appear to be more robust than others. They go by many names in the literature. A cluster of perceived visual stimuli may be called an *external entity*, or *object*. If the granularity of the base experiential elements, U, is coarser, say that of *skills* or *facts*, we might call a cluster of abilities an *area of expertise*, such as *horseshoeing*; or a cluster of facts might be regarded as a *discipline*, such as *medieval history* or *high school algebra*. With so many possible terms and interpretations, we choose to use a more neutral term. We will call such clusters *knowledge units* without trying to specify precisely what such a unit is. In this section we will postulate that this organizing process can be approximately modeled by a mathematical *closure* operator.

The term "closure" has many interpretations within the psychological lexicon, but to our knowledge, Vygotsky never used this word. However, "signs", including words and speech he felt to be crucial in a child's development; they provided a kind of closed wrapper around clusters of the child's experiences. We will use "closure" in a very specific, but unfamiliar, way. Imagine that we have a collection, C, of objects within a relational system, and a separate object, z. If all of the external objects that are related to z are also related to those of the collection C, then, from the relationship point of view, z can be included in C with no change of information. That is an example of "closure".

We can make the idea of closure more precise. An operator φ is said to be a *closure operator* if for all $X, Y \subseteq U$,

$Y \subseteq Y.\varphi$	φ is *expansive*,
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$	φ is *monotone*, and
$Y.\varphi.\varphi = Y.\varphi$	φ is *idempotent.*

There is an extensive literature on closure and closure operators of which [4, 9, 11, 18, 22, 25] are only representative.

Since ρ is both expansive and monotone, it is almost a closure operator itself. But, ρ need not be idempotent. In Figure 2, we have $u.\rho = \{uwx\} \subset \{uwxyz\} = u.\rho.\rho$.[3] However, we can always define a closure operator φ_ρ with respect to ρ. Let,

$$Y.\varphi_\rho = \{z \in Y.\rho \mid \{z\}.\rho \subseteq Y.\rho\} \tag{3.1}$$

Equation (3.1) actually embodies the more intuitive paragraphs above. Y closure consists of the union ($\cup$) of all z in $Y.\rho$ such that $z.\rho$ is contained in $Y.\rho$. Readily, if $z \in Y$ then $z.\rho \subseteq Y.\rho$. We call φ_ρ the *experiential closure* because it is determined by the experiential operator ρ. Note that any relationship, ρ, of any type can give rise to a closure operator, φ_ρ, although it need not be unique. One can have $\varphi_{\rho_1} = \varphi_{\rho_2}$, $\rho_1 \neq \rho_2$. Moreover, one may have closures on the set of experiential elements that are totally unrelated to ρ.

Proposition 3.1. *φ_ρ is a closure operator.*

Proof. Readily, $Y \subseteq Y.\varphi_\rho$ by definition. Let $X \subseteq Y$ and let $z \in X.\varphi_\rho$. By (3.1) $z.\rho \subseteq X.\rho \subseteq Y.\rho$ hence $z \in Y.\varphi_\rho$.
Now, suppose $\exists z \in Y.\varphi_\rho.\varphi_\rho - Y.\varphi_\rho$. By (3.1) $\exists X \subseteq Y.\varphi_\rho$ such that $z.\rho \subseteq X.\rho$. Since $X \subseteq Y.\varphi_\rho, z \in X.\rho \subseteq Y.\rho$, so $z \in Y.\rho$. But, $z \notin Y.\varphi_\rho$ implying $\exists w \in z.\rho, w \notin Y.\rho$. However, $w \in z.\rho \subseteq X.\rho$ contradicting our assumption that $X \subseteq Y.\varphi_\rho$ since $X.\rho \not\subseteq Y.\rho$. □

In the network of Figure 2, observe that $\{u\}$ is closed, but $\{y\}$ is not, because $\{z\}.\rho = \{yz\} \subseteq \{wyz\} = \{y\}.\rho$, so $\{y\}.\varphi_\rho = \{yz\}$. Neither is $\{w\}$ closed, because $\{w\}.\varphi_\rho = \{uw\}$. So, singleton elements need not be closed.

Normally, we omit the subscript ρ from the closure symbol φ because most results are valid for all closure operators. Only if some property of the relational closure is required, will we use the symbol φ_ρ.

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[3] We sometimes elide the curly braces on sets when the meaning is clear; that is $u.\rho = \{u\}.\rho$.

A set Y is said to be *closed* if $Y.\varphi = Y$. Because φ is expansive, U itself must be closed. A well-known property of all closure systems is that, if X and Y are closed, then their intersection $X \cap Y$ must be closed; or equivalently, $X.\varphi \cap Y.\varphi = (X \cap Y).\varphi$.

The concept of *closure* and *closed sets* is absolutely essential for this paper.

By a *knowledge unit*, K_i, we mean a set closed with respect to φ_ρ in U. That is, the elements of K_i are a tightly bound collection of related experiences that will be regarded as a unit of *knowledge awareness*. In Figure 2, because $\{uw\}$ is closed, it is a knowledge set, K_1. The set $\{uvw\}$ is also closed, and thus also a knowledge unit, K_2. Here, $K_1 = \{uw\} \subset \{uvw\} = K_2$. We can think of increasing knowledge awareness with increasing experience or capability.

There is a strong correlation between closed concepts and the nouns of a language. The role of "naming things" in cognitive function has been remarked by many. Vygotsky was an early proponent of the role of *signs*, or names, in facilitating memory, cognition and in problem solving. For example, he asserts that

> "The role of language in perception is striking because of the opposing tendencies implicit in the nature of visual perception and language. The independent elements in a visual field are simultaneously perceived; in this sense, *visual perception is integral.* Speech, on the other hand, requires sequential processing. Each element is separately labeled and then connected in a sentence structure, *making speech essentially analytical.*" ([5], p. 33).

Typically, the nouns of a language denote closed knowledge units. However, the converse need not be true. Closed knowledge units need not have names.

3.1. Examples of experiential closure

The formal definition of relationship-defined closure, φ_ρ, as well as the more general definition with respect to expansive, monotonicity, and idempotency, conveys little intuitive sense of its being. Here we will examine some examples and tie them into knowledge acquisition.

Two more common relationships, ρ, in the human experience are spatial and temporal co-occurance, not an arbitrary graph such as Figure 2. Consider the retina of the eye, where the close packing of cells (frequently called "pixels", and here shown as hexagonal, even though the retina is never quite so regular) endows each receptive cell with 6 neighbors. Figure 3(a) illustrates a portion of the retinal structure with 13 excited cells (black dots) which we will denote by Y. We seek the spatial closure of Y. Readily, the quiescent cell labeled "a" belongs to $Y.\varphi$, since $\{a\}.\rho \subseteq Y$. The quiescent cells labeled "b", "c" and "d" are also elements of $Y.\varphi$ since $\{b\}.\rho \subseteq Y.\rho$, as are $\{c\}.\rho$, and $\{d\}.\rho$, as can be seen by comparing their neighborhoods (dashed lines) with that of Y's neighborhood in Figure 3(b). The black dots denote $Y.\varphi$.

The spatial operator, ρ, need not be limited to just the immediate neighbors. Figure 3(c) illustrates the situation when the radius of expansion is 2. Now cells "x" and "y" are included in the closure. It was shown in [30] that this spatial closure operator can be implemented in parallel by "expanding" each stimulated

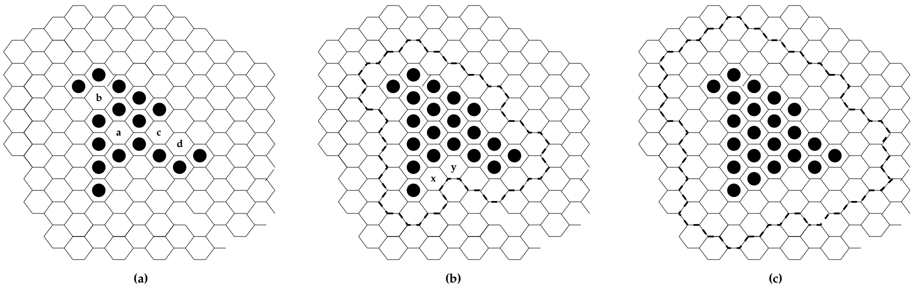

Figure 3. Closures of the pattern Y in (a).

element in Y by n steps, then expanding its complement by n steps, thus contracting Y. As n becomes large, this begins to approximate a convex hull operator.[4]

Since it is assumed that virtually all processing of information passing back from the retina to the visual cortex occurs in parallel; that spatial retinal relationships are preserved in this visual pathway; and that this pathway consists of alternating odd/even cell layers [33], it is plausible to regard this example as an actual, but vastly oversimplified, cognitive process.

Whether or not this closure operator mimics one actually found in nature, it does illustrate how a closure operator can extract "identifiable" objects from a pattern.

In the preceding example the spatial relationship, ρ, was uniform with respect to all axes. It need not be. The following example is a bit more complex.

It is known that there exist specialized orientation-specific cells, often called "Hubel-Wiesel cells", in the retinal layer [15,43]. Let Y denote a set of stimulated uni-directional cells as shown in Figure 4(a), and let the spatial relation, ρ, consist of 4 distinct regions, $\rho_1, \rho_2, \rho_3, \rho_4$ as shown in Figure 4(b).
The closure operator, φ^*, will be defined by[5]

$x \in Y.\varphi^*$ if and only if $x \in Y$, or

(a) $\{x\}.\rho_1 \cap Y \neq \emptyset$ and $\{x\}.\rho_2 \cap Y \neq \emptyset$,

(b) $\{x\}.\rho_3 \cap Y = \emptyset$ and $\{x\}.\rho_4 = \emptyset$.

where Ø denotes the "empty set" consisting of no elements.

The application of φ^* to Figure 4(a) is illustrated in Figure 5. There is an evident gap in the middle of the resulting "line", L. Smoothing operators, such as our previous example, can eliminate these small omissions and eliminate isolated signals. Elongating the regions ρ_1 and ρ_2 can support identification of occluded lines.

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[4]Readily, as n grows large, $Y.\varphi$ more closely resembles the hexagonal convex hull of Y. But, it is not hard to create counter examples where the intersection of two closures $X.\varphi$ and $Y.\varphi$ will be closed, but will not be convex hulls. In particular, the "convex hull" of two distinct points, $\{x, y\}.\varphi$ need not be a connected "line".

[5]The concept of "experiential closure", φ_ρ, defined by 3.1 in the preceding section has the virtue of being a closure operator regardless of the relational operator ρ. Other closure operators, such as φ^* can be defined; but proof of the closure properties must be given on an individual, case by case basis. In the case of φ^*, (a) ensures that x must lie "between" elements of Y, and (b) ensures that x is not "beside" other elements of Y. So φ^* is idempotent.

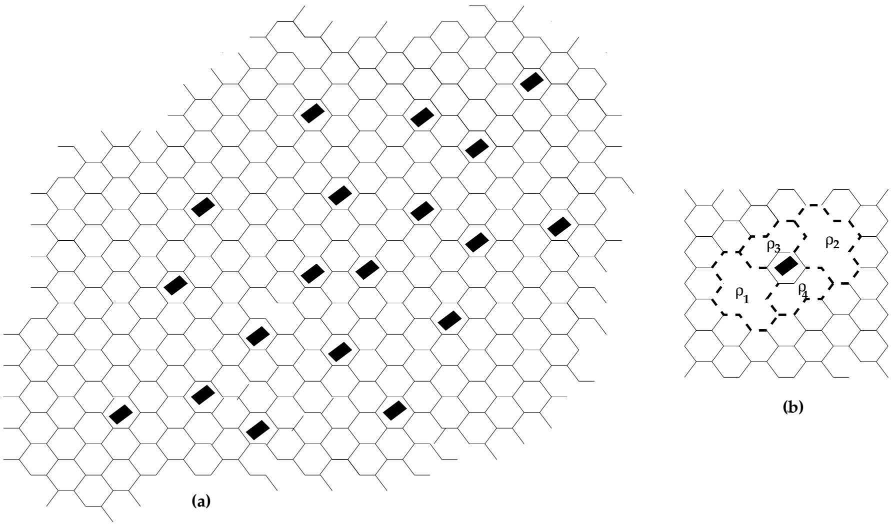

Figure 4. (a) A scattering of excited orientation sensitive Hubel-Wiesel cells, (b) a 4 part spatial neighborhood.

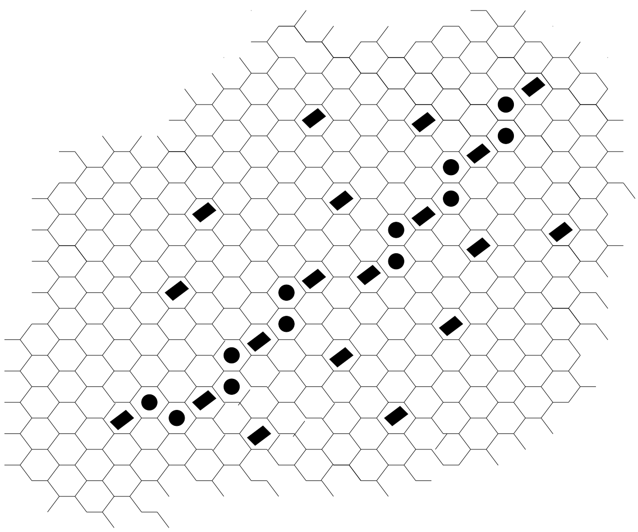

Figure 5. Closure of Figure 4(a) using φ^* based on ρ of Figure 4(b).

Readily, φ^* extracts "lines" from the raw input of individual directional edge detector cells. This illustrates, in a rather primitive example, the potential of closure to designate, or "name", higher level constructs from lower level experiential data. This can be one step in "the main question of vision: how to arrive at a global interpretation of a scene from the local, atomic information contained in an image?" [41].

These two examples of closure operators have been set within the context of visual cognition. They do not necessarily imply that these black and white "cartoon" examples mimic actual visual processes. Real visual cognition can be far more complex, for example, we see in certain multiple frequencies (color). But, it does establish that closure concepts are compatible with known aspects of visual physiology. Similarly, we believe closure concepts are compatible with the cognitive behaviors observed by Vygotsky, and others.

4. Generators and knowledge lattices

If K is a closed knowledge unit there exists at least one set $Y \subseteq K$ such that $Y.\varphi = K$. (It may be K itself.) Y is said to be a *generator* of K. A reasonable interpretation of generating sets is that these are a set of features of K that serve to characterize K.

Readily, any set Y is a generator of $Y.\varphi$, as is any set Z, $Y \subset Z \subseteq Y.\varphi$. If for some $X \subset Y$, $X.\varphi \subset Y.\varphi = K$ then Y is said to be a *minimal generator* of K.[6] In general, a closed set K may have several minimal generating sets, denoted $K.\Gamma = \{Y_1, \ldots Y_m\}$ where $Y_i.\varphi = K, 1 \leq i \leq m$. For example, in Figure 2, $\{uy, uz, wy, wx, wz, xy, xz\}$ are all minimal generators of $\{uwxyz\} \subset \{uvwxyz\}$.

While linguistic nouns normally correspond to closed knowledge units, generators are often described by adjectives, or collections of adjectives.

4.1. Knowledge lattices

It is assumed that our knowledge is structured. One way of doing this is to partially order the knowledge units by containment to form a lattice.[7] Because U itself must be closed (φ is expansive) and because $X \cap Y$ must be closed, any collection of closed sets can be partially ordered by containment to form a lattice. We call them *knowledge lattices*, denoted $\mathcal{L}_\rho$. Figure 6 illustrates the knowledge lattice, $\mathcal{L}_\rho$, associated with the experiential operator, ρ, of Figure 2. Doignon and Falmange called such lattices "knowledge spaces" [7, 8]. This idea of *knowledge spaces* has generated a considerable amount of psychological literature.[8] Ganter and Wille [11] regard a lattice of closed sets as a "concept lattice". In both theories the lattice structure is central; for us, it will be important, but ancillary.

A closed set K_m in $\mathcal{L}_\varphi$ is said to *cover* K_i if $K_i \subset K_m$ and there exists no set K_j such that $K_i \subset K_j \subset K_m$. That is, K_m is the next set above K_i in the lattice.[9] We can think of the *difference*, $K_m - K_i$, as being the skill/experience set differentiating an individual with knowledge unit K_i from one with K_m. In Figure 6, $\{uwxyz\} - \{yz\} = \{uwx\}$ which is not a closed set, and $\{uvw\} - \{uv\} = \{w\}$. Explicitly showing the set differences as we have done in 3 instances in Figure 6 can be an aid to understanding Proposition 4.1 which follows.

[6] If for all closed sets K, there is a *unique* minimal generating set, the closure operator is said to be *antimatroid.* While antimatroid knowledge systems, such as [8] and [11], are mathematically most interesting, they seem, in practice, to be most rare.

[7] Set containment, $\subseteq$ is a "partial order". Because $X \subseteq Y$ and $Y \subseteq Z$ implies $X \subseteq Z$, it is an ordering; but because we may have $X \not\subseteq Y$ and $Y \not\subseteq X$, it is only partial.

[8] Over 400 references can be found at the web site <cord.hockemeyer@uni-graz.at>.

[9] Because U is discrete, there is a "next" set above K_i in $\mathcal{L}$, unless $K_i = U$.

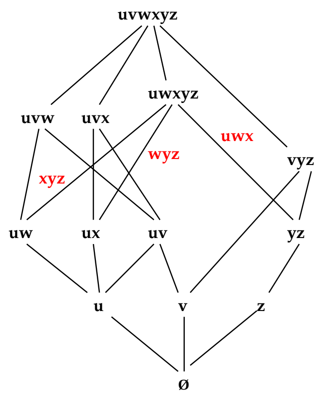

Figure 6. Closed set lattice of Figure 2.

There is a key relationship between the structure of the closed knowledge units in a knowledge lattice and their generators. The generators of a knowledge unit are precisely those features which differentiate it from other knowledge units in the lattice, or ...

Proposition 4.1. *If a closed set K covers the closed sets $K_1, \ldots, K_m$ in $\mathcal{L}_\rho$, then X is a generator of K if and only if $X \cap (K - K_i) \neq \emptyset$ for all $1 \leq i \leq m$.*

Proof. A rigorous proof can be found in [16], here we present a more intuitive argument.
A knowledge unit is the smallest closed set containing some set, X, of experiences. Suppose X is a generator of K. Now, if X does not embrace at least one element from $K - K_i$, then $X.\varphi = K_i$, not K.
Conversely, K contains a number of knowledge units, K_i, and if X includes at least one experience that differentiates each one from K, then X must characterize K; it must be a generator. □

This means that, if one knows the generators of a closed knowledge unit, one knows the closed sets it covers and, conversely, given the lattice of closed sets, one can determine all the minimal generators. It is worthwhile convincing oneself of this unusual result by actual trial. In Figure 6, $\{uwxyz\}$ covers $\{uw\}$, $\{ux\}$ and $\{yz\}$ with respective differences being $\{xyz\}$, $\{wyz\}$ and $\{uwx\}$. Using Figure 2, convince yourself that each of the sets $\{uy\}$, $\{uz\}$, $\{wx\}$, $\{wy\}$, $\{wz\}$, $\{xy\}$, and $\{xz\}$ each of which intersect all three set differences are actually generators of $\{uwxyz\}$.

Suppose U consists of visual stimuli. If X generates K, a closed set of related stimuli, constituting a visual object, then X consists of those visual attributes that characterize the object; and differentiate it from other similar objects, K_i. On the other hand, if K represents an ability level in high-school algebra, as in [8], then X represents those skills necessary to advance from lesser sets of abilities, K_i to K. Finally, if K represents knowledge of the Napoleonic wars, then questions embodying the facts in a generator, X, would comprise an excellent test of the student's knowledge. The concept of generators resonates with many educational themes depending on the network granularity.

Experiential networks are real. The neural networks of the mind are real; our social networks are real; the related collections of facts we call knowledge are real. Our rendition of these real networks by ρ may be an oversimplification; but it is an abstract depiction of real phenomena. In contrast, these *knowledge lattices* are *not* real. They have no existential counterpart that we know of. They are purely a mathematical construct designed to help us understand the organization and structure of real networks; and in the next section, to help us understand how their structure can change under dynamic transformation. This is an important distinction. While in this section, and the next, we may seem to be fixated on these *knowledge lattices*, we are really most concerned about the underlying network of experiential relationships.

Do the concepts of *closure* and *generators* correspond to real phenomena? Even though we have no compelling proof, we believe they do. It seems clear that our minds are capable of identifying and labeling, in some fashion, related collections of experiential input. Replacing a cluster of primitive experiential elements with a single label can optimize neuron use because it facilitates the internal representation at a coarser granularity. It is necessary for "abstract" thought. It permits the creation of a network based on a new relation, say $\bar{\rho}$, between closed, identifiable clusters. Vygotsky asserts that "the very essence of human memory consists in the fact that human beings actively remember with the help of signs" ([5] p. 51). He describes *signs* as a key tool for the "internalization of higher psychological functions" ([5], chapter 4). The concept of "closure" appears to be a somewhat formal abstraction of what appears to be, in practice, an important labeling process.

Similarly, it seems apparent that the mind, on many levels, apprehends objects and abstractions of the real world by abbreviated collections of salient features. This, too, represents an economical use of neurons — which must be important to all organisms. Whether *generators* exactly model this phenomenon is unclear; but surely they represent an abstraction of this capability.

Our imposition of a formal lattice structure as a mathematical device to comprehend the organization of experiential networks may be a major contribution of this paper. In the following sections we will see where this leads us.

5. Transformation as learning

The notion of *transformation* is a familiar one in educational psychology; for example, the process of *internalization* has been described as a "series of transformations" [5]. In this section we will develop the idea of transformation as a mathematical function. Most are familiar with polynomial functions, which describe numerical change — the speed of a falling object is a quadratic function of its time of flight. But now a transformation will be a function that describes a change of *structure*. It requires a different mathematical mindset. It is one reason why we use suffix notation.

By a *transformation*, $U \xrightarrow{f} U'$, we mean a function f which, to every set $Y \subseteq U$ assigns a set $Y.f = Y' \subseteq U'$. (We use Y' to denote the image of Y in U'). Of most interest will be the effect, $K.f$, of transforming closed knowledge units, and how the transformation will affect their relationship with other knowledge units, $K_i.f$. The importance of using a power set as the domain and codomain of a transformation is that elements, y and z, can be functionally removed from the system. For example, consider the transformation f depicted by Figure 7, which simply deletes the elements y and z from the network of Figure 2. Here,

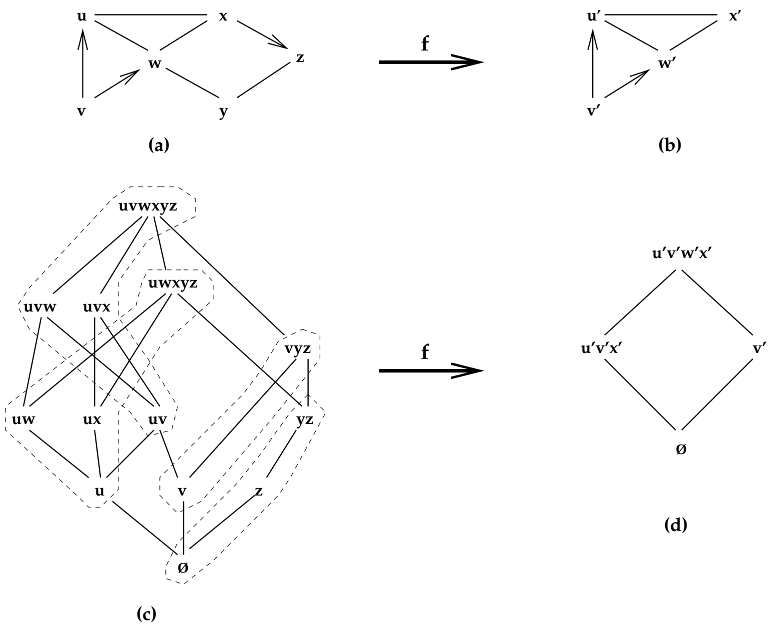

Figure 7. A transformation $U \xrightarrow{f} U'$ that removes y and z from the network of Figure 2.

$U' = U - \{yz\}$ in Figure 7(b). Consequently, $\{yz\}.f = \emptyset$ so, by monotonicity, $\{y\}.f = \{z\}.f = \emptyset$, as well. Thus, f is defined by $Y.f = Y' = Y - \{y, z\}$ for all sets $Y \subset U$.

In the mathematics of the real line, the behavior of functions is typically visualized by the familiar graph plotting the value $y = f(x)$ for all x along the x-axis. When the function is defined on sets of discrete elements, a different approach must be taken. We prefer to illustrate its behavior by what happens to the closed set/knowledge lattice. Although f is defined for all sets, $Y \subseteq U$, we use only these closed sets, or knowledge units, to visualize the process. Figure 7(a) $\xrightarrow{f}$ (b) shows how the experiential network is affected. Figure 7(c) $\xrightarrow{f}$ (d) illustrates its behavior with respect to the knowledge lattice. Here we have circled those closed sets which map on to the same set in $\mathcal{L}_{\rho'}$. For example, the closed sets $\{\{uvwxyz\}, \{uvw\}, \{uvx\}, \{uv\}\}$ all map onto $\{u'v'w'x'\}$ in $\mathcal{L}_{\rho'}$. This transformation, f, is a classic example of a smooth, well-behaved lattice morphism.

Learning is more often associated with the acquisition of additional experiential awareness. Because transformations are defined over 2^U, not U itself, the empty set, Ø can also map to a non-empty set. This provides a functional way of expanding the experiential network, and thus knowledge structures. It is basic to our model of "learning".

Consider the transformation $U \xrightarrow{g} U'$ where $U' = U \cup \{t'\}$. Here we have $\emptyset.g = \{t'\}$, and $\{uvw\}$ in $\mathcal{L}_\rho$ corresponds to $\{t'u'v'w'\}$ in $\mathcal{L}_{\rho'}$. These pairs of new

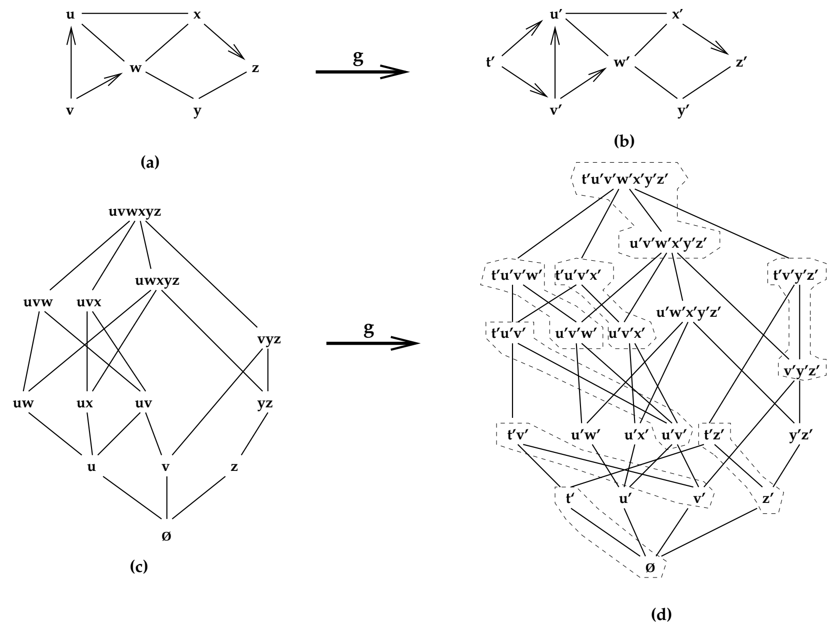

Figure 8. A transformation g that adds t to the system of Figures 2 and 6.

sets are enclosed with dashed lines in Figure 8(d). Notice that the presence of t' in U' with its connections to u' and v' enlarges the closure structure of U'; but it seems to "slide" in gracefully.

A transformation $U \xrightarrow{f} U'$ is said to be *monotone* if, for all sets X, Y in U, $X \subseteq Y$ implies $X.f \subseteq Y.f$. Monotonicity is essential throughout the following mathematical approach.[10] Observe that the transformations f and g are both monotone in that $K_i \subseteq K_m$ in $\mathcal{L}_\varphi$ implies $K_i.f \subseteq K_m.f$ in $\mathcal{L}'_\varphi$. These are well-behaved lattice morphisms. Not all monotone transformations need be well-behaved. In Figure 9, the element t' is once again added to create U', but it is embedded differently with respect to the connections of ρ'. It is still monotone, but something seems to be wrong with the corresponding lattice morphism, h. The closed sets in $\mathcal{L}_{\rho'}$ seem to be augmented with $\{t'u'\}$, not $\{t'\}$ as expected. They have been circled because they appear to be relevant, but they are not consistent

[10]In artificial intelligence (A.I.), learning is said to be "monotonic" if no new piece of information can invalidate any existing "knowledge" as represented by a set of rules. That concept of knowledge involves a notion of logical contradiction, not just the simple inclusion or deletion of experiential input. There is an abundance of literature about A.I. architectures which support both monotonic and non-monotonic reasoning [1, 23, 32]. Our use of the term is rather different.

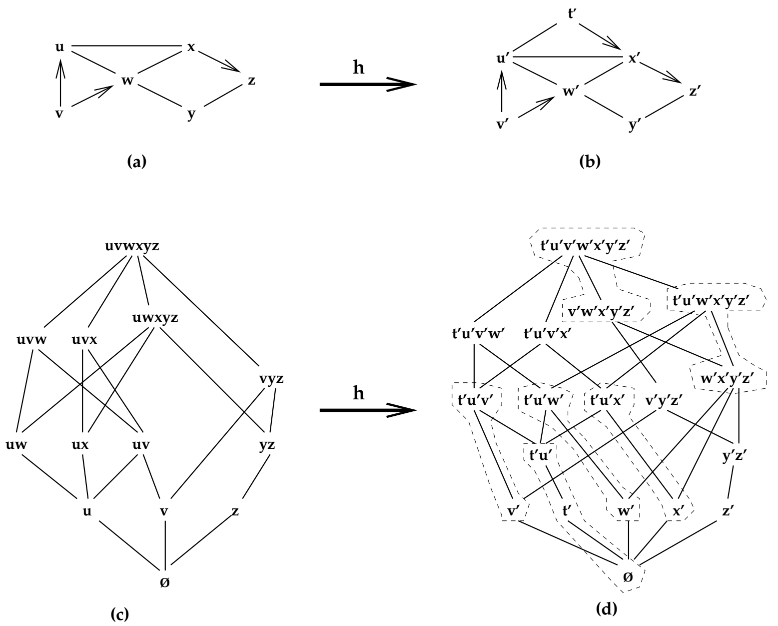

Figure 9. An ill-behaved transformation $U \xrightarrow{h} U'$ of Figure 2.

throughout. For example, why do the sets $\{t'u'v'w'\}$ and $\{t'u'v'x'\}$ suddenly appear. Figure 8 illustrates a regularity that Figure 9 does not. We will develop this theme in the following section.

5.1. Continuous transformations

In high school we are told that a "continuous" function, $f(x)$, is one whose graph can be drawn without lifting one's pencil from the paper. The more precise definition encountered in real analysis is quite analogous to the definition that follows.[11] A discrete transformation, $U \xrightarrow{f} U'$, is said to be *continuous* if, for all $Y \subseteq U$,

$$Y.\varphi.f \subseteq Y.f.\varphi'$$

This is the traditional definition of continuity for functions on discrete spaces [18,19,21,36,37]. Yet, this short equation conveys little intuitive sense of its import. The transformations f and g of figures 7 and 8 are continuous; they are "smooth". The " ill-behaved" transformation $U \xrightarrow{h} U'$ of Figure 9 is not continuous because $\{x\}.\varphi.f = \{uwx\}.f = \{u'w'x'\} \not\subseteq \{x'\} = \{x\}.f.\varphi'$. We say that h is *discontinuous*. Continuity takes on additional importance when viewed as a function on knowledge lattices. It effectively asserts that, if a learning transformation is continuous, it only expands the knowledge units of an individuals experiential awareness. That is, if $K = Y.\varphi$ then $K.f \subseteq Y.f.\varphi_{\rho'} = K'$.

[11] A real function $y = f(x)$ is said to be continuous if, for any open set O_y containing y, there exists an open set O_x containing x such that $f(O_x) \subseteq O_y = O_{f(x)}$, or using suffix notation $x.O.f \subseteq y.f.O'$.

Before considering more fully what comprises continuous transformations in a cognitive context, it can be valuable to examine the purely formal characteristics of continuity. For example, a sequence of continuous transformations will, itself, be continuous, or ...

Proposition 5.1. *Let $(U, \varphi) \xrightarrow{f} (U', \varphi')$, $(U', \varphi') \xrightarrow{g} (U'', \varphi'')$ be monotone transformations. If both f and g are continuous, then so is $U \xrightarrow{f.g} U''$.*

Proof. We have $X.\varphi.f \subseteq X.f.\varphi'$ for any $X \in U$ and $Y.\varphi'.g \subseteq Y.g.\varphi''$ for any $Y \in U'$. Consequently, as g is monotone, $X.\varphi.f.g \subseteq X.f.\varphi'.g \subseteq X.f.g.\varphi''$. Thus, $f{\cdot}g$ is continuous. $\square$

If a portion of a closed knowledge unit, K, maps onto a closed knowledge unit, K', under a continuous transformation, f, then the entire knowledge unit maps onto K', or ...

Proposition 5.2. *Let $(U, \varphi) \xrightarrow{f} (U', \varphi')$ be monotone, continuous and let $Y.f = Y'$ be closed. Then, $Y.\varphi.f = Y'$.*

Proof. Let $Y.f$ be closed in U'. Because f is continuous $Y.\varphi.f \subseteq Y.f.\varphi' = Y.f$, since $Y.f$ is closed. By monotonicity, $Y.f \subseteq Y.\varphi.f$, so $Y.\varphi.f = Y.f$. $\square$

If X is a generator of a closed knowledge unit K, then $X.f$ is a generator of $K.f.\varphi'$, or ...

Proposition 5.3. *Let $(U, \varphi) \xrightarrow{f} (U', \varphi')$ be monotone. Then, f is continuous if and only if $X.\varphi = Y.\varphi$ implies $X.f.\varphi' = Y.f.\varphi'$.*

Proof. Let f be continuous, and let $X.\varphi = Y.\varphi$. By monotonicity and continuity, $X.f \subseteq X.\varphi.f = Y.\varphi.f \subseteq Y.f.\varphi'$. Similarly, $Y.f \subseteq X.f.\varphi'$. Since $Y.f.\varphi'$ is the smallest closed set containing $X.f$ and $X.f.\varphi'$ is the smallest closed set containing $Y.f$, $X.f.\varphi' = Y.f.\varphi'$.
Conversely, assume f is not continuous. So there exists Y with $Y.\varphi.f \not\subseteq Y.f.\varphi'$ There exists $X \in Y.\varphi^{-1}$. $X.f \subseteq X.\varphi.f = Y.\varphi.f \not\subseteq Y.f.\varphi'$, so $X.f.\varphi' \neq Y.f.\varphi'$, contradicting the condition. $\square$

Corollary 5.4. *If $(U, \varphi) \xrightarrow{f} (U', \varphi')$ is a monotone, continuous transformation and X generates K ($X.\varphi = K$), then $X.f$ generates $K.f.\varphi'$.*

Note that, even though f is monotone and continuous, and K is closed with respect to φ, $K.f$ need not be closed with respect to φ'. However, by Corollary 5.4, $K.f$ must be a generating set of $K.f.\varphi'$.

Continuous transformations are very well-behaved with other demonstrable properties, *cf.* [21]. It is our conjecture that a continuous transformation of a human's experiential network (as exemplified by ρ) corresponds to our "natural" reaction to new experience and stimuli. It is an, almost automatic, response to novel experiences. It is what Vygotsky would consider *development.* As a corollary, we conjecture that a proper role of education is to create experiential connections that lead to a *discontinuous* restructuring of our knowledge structure, regardless of its actual formation. Vygotsky called this *neoformation* as distinct from *development*. We believe that properly directed education can change connections between knowledge units in a discontinuous manner that would not occur naturally. However, for now, let us focus on the nature of those transformations that occur continuously, or "naturally".

5.2. Visual continuity

Persistence is important in visual perception. It is a form of continuity with respect to time. Let us assume that a line L_{t_0} such as shown in Figure 5 has been identified within the visual cortex, (and possibly "enlarged" to include adjacent cells/pixels to account for "jitter"). Let L_{t_1} denote the same closed line as identified some nano-seconds later. If $L_{t_1} \subseteq L_{t_0}$ then φ^* is continuous and L can be regarded a persistent. Note that, since φ^* is a closure operator, if the image is persistent and thus continuous, then all the preceding propositions are true here. In particular, the set X of edge detection cells that generated L_{t_0} must still generate L_{t_1}.

5.3. Small incremental change

As is evident from Figures 8 and 9, the key to continuous learning is not just exposure to new experience, but how that new experience is integrated with other related experience. It has been suggested that new experience, new stimuli, is integrated into our memory, or knowledge structure, as we sleep. Apparently this occurs through the creation of new axons and synaptic connections [2, 6]. Some researchers believe that the elimination of connections may be equally important as creating new ones [38]. The following three propositions present necessary and sufficient conditions for creating, or deleting, a single link in ρ to be continuous.

It was shown in [25] that, if a discontinuity exists, it will manifest itself at a single experiential event, or...

Proposition 5.5. *If there exists Y such that $Y.\varphi.f \not\subseteq Y.f.\varphi'$, then there exists a singleton set $\{y\} \subseteq Y.\rho$ such that $\{y\}.\varphi.f \not\subseteq \{y\}.f.\varphi'$.*

This makes testing for continuity viable.

The following two propositions characterize continuous transformations that add, or delete, edges/relationships within a symmetric network. In both Propositions 5.6 and 5.7, we assume that $U' = U$ and that f is the identity function on $\mathcal{L}_\rho$, and that $y' = \{y\}.f$ denotes the same node, but within the new structure of $\mathcal{L}_{\rho'}$. In the statement of these propositions, we use the term $x.\eta$. By $Y.\eta$, which we call the *neighborhood* of Y, we mean the set $Y.\eta = Y.\rho - Y$, that is, the immediate neighbors of Y with respect to ρ.[12]

In Proposition 5.6 we show that new links can be continuously created between two experiential events x and z if there already exists a reasonably close relationship. Granovetter [13], and many other sociologists have observed this phenomenon.

Proposition 5.6. *Let $U \xrightarrow{f} U'$ be the identity transformation. If f adds an edge (x', z') to create a network ρ', it will be continuous at x if and only if for all $y \in x.\eta$, if $x \in y.\varphi$ then $z \in y.\rho$.*

[12] Note that the η operator is normally neither expansive nor monotone.

Proof. Assume that $\exists y \in x.\eta, x \in y.\varphi$, but $z \notin y.\eta$. Since $x \in y.\varphi$, $x.\eta \subseteq y.\rho$. But, because $z \notin y.\eta$, $x'.\eta' \not\subseteq y'.\rho'$ and $y.\varphi.f \not\subseteq y.f.\varphi'$.
Conversely, assume f is discontinuous. First, we observe that $x.\varphi.f \subseteq x.f.\varphi'$ since the addition of an edge (x', z') cannot reduce the closure $x'.\varphi'$. So, f must be discontinuous at $y \in x.\eta$; that is, $\exists w \in y.\varphi$ such that $w' \notin y'.\varphi'$ because $w'.\eta \not\subseteq y'.\rho'$. Readily, $w' = x'$ (or z'). After adding the edge (x', z'), $x'.\eta' \not\subseteq y'.\rho'$ only if $z' \notin y'.\eta$, that is, $z \notin y.\eta$. □

We say that f is "discontinuous at x" even though the actual *discontinuity* may occur at $y \in \{x\}.\eta \subseteq \{x\}.\rho$ as noted in Proposition 5.5. This slight abuse of terminology allows us to focus on the structure surrounding the node x before (x', z') is created. Note that, for f to be continuous, f must be continuous at both x and z.

Next we show that a link between two experiential events x and z can be continuously deleted if they are not too closely connected, or ...

Proposition 5.7. *Let $U \xrightarrow{f} U'$ be the identity transformation. If f deletes an edge (x, z) from ρ', it will be discontinuous at x if and only if either*
(a) $z \in x.\varphi$ and $z.\varphi \neq x.\varphi$ or
(b) there exists $y \in x.\varphi$, with $z \in y.\eta$.

Proof. Suppose (a), $z \in x.\varphi$. Since (x, z) is being deleted $z' \notin x'.\eta'$. Consequently, $\{x\}.\varphi.f \not\subseteq \{x'\}.f.\varphi'$.
The last conjunct $x.\varphi \neq z.\varphi$ of condition (a) covers the special case illustrated in Figure 10. The closure structure is the same whether the edge (x, z) exists or not.
Suppose (b) that $\exists y \in x.\varphi$ and $z \in y.\eta$. $\{y'\} \subseteq x.\varphi.f$, but $z' \notin x'.\eta'$ implies that $y'.\eta' \not\subseteq x'.\eta'$, hence, $y' \notin x'.\varphi' = x.f.\varphi'$. Now, $\{x\}.\varphi.f \not\subseteq \{x\}.f.\varphi'$, and f is discontinuous.
Conversely, suppose f is not continuous at x. Then, by Prop. 5.5, either (1) $\{x\}.\varphi.f \not\subseteq \{x\}.f.\varphi'$ or (2) for some $y \in \{x\}.\eta$, $\{y\}.\varphi.f \not\subseteq \{y\}.f.\varphi'$.
Assume the former, then there exists some $w \in \{x\}.\varphi$ such that $w' = w.f \notin \{x\}.f.\varphi'$. Since (x, z) is the only edge being deleted, w must be z.
Now assume the latter. If $y \in \{x\}.\varphi$, then $y.\eta \subseteq x.\rho$. If $z \notin y.\eta$, then $\{y\}.\varphi.f \subseteq \{y\}.f.\varphi'$; but f is assumed to be discontinuous, so $z \in y.\eta$. □

Figure 10 illustrates a special case which forces the last conjunct $x.\varphi \neq z.\varphi$ in condition (a). Addition, or deletion, of the dashed edge (x, z) can make no change

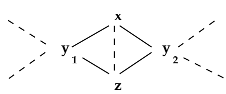

Figure 10. Two points where $x.\varphi = z.\varphi$.

in the closed set structure whatever, in particular, deletion will not create a discontinuity even though $z \in \{x\}.\varphi$.

If f and g are both continuous single edge additions or deletions, then, by Proposition 5.1, their composition $f{\cdot}g$ is as well. It would be mathematically satisfying, if, conversely, every continuous restructuring of ρ could be decomposed into primitive single edge transformations; but in [25], it is shown that this need not be true.

5.4. Continuous evolution

We have suggested that continuous transformation of our network of experiences is analogous to "natural" learning. The author has conducted an experiment that seems to support this contention [26].

We first generated *random networks* on n elements by creating m symmetric relationships, $y = x.\rho$. Then, we permuted the network by adding and deleting randomly chosen relationships, *provided* the transformation would be continuous, as defined by Propositions 5.6 and 5.7. In this way the density, m/n, of the relationship, ρ, was kept constant. Depending on the original size, n, of the network, this perturbation was iterated between 50 to 1,000 times. In all cases, the organizational structure of the network was markedly increased, as shown by a significantly greater number of embedded triangles, a traditional measure of network structure [13, 35], and a decrease in the length of embedded chordless cycles. The details of these experiments can be found in [26].

It is our contention that these random processes mimic, to some extent, the neural processes that occur during sleep [2,6], including the conjecture [38] that neural connections must be broken, or weakened, as well as created, or strengthened to maintain a constant neuronal density. The increase in organizational structure can only be due to the continuity constraint because all else is random. Note that a simple increase in the total number of neuronal relationships, ρ, such as will occur in the mental growth of young children must also increase the number of embedded triangles and decrease the length of chordless cycles.

6. Large experiential networks

Networks such as Figure 2 abound in the human existence. Their granularity can range from the neural networks of our mind to social networks of our international friends and associates. On the other hand, as we pointed out at the end of Section 3, lattices such as Figure 6 have no reality. They are purely formal mathematical constructs created to understand network structure and behavior. How else could we have seen the distinction between the transformations g (Figure 8) and h (Figure 9), both of which simply add a new element t' to an existing network.

Using these closed set lattices to understand the behavior of the underlying network structure only works so long as we consider small, unrealistic, experiential networks and knowledge units. As the size of a network grows, the size of its closure lattice tends to explode. They are just too big for visual comprehension.

Relational networks can also become too large to be retained by a finite neural network. There must be both mathematical and psychological tools to cope with large networks.

6.1. Relative closure

One effective tool is the idea of using a *window* to restrict our view to just a portion of the entire network and its lattice. This is sometimes called "focusing one's attention". Let $(\mathbf{U}, \varphi)$ be any closure space and let $W \subseteq \mathbf{U}$. By the *restriction* of φ to W, denoted $\varphi|_W$, we mean $X.\varphi|_W = X.\varphi \cap W$ for all $X \subseteq W, X \cap W \neq \emptyset$. This is also known as *relative closure*. Proposition 6.1(a) tells us that, if X is closed in U, it will still be closed in the smaller window W. Proposition 6.1(b) and (c) give us sufficient conditions to assume that what we see in the smaller window, W, can be extrapolated to the larger universe, U.

Proposition 6.1.
(a) X closed with respect to φ implies $X \cap W$ is closed with respect to $\varphi|_W$.
(b) If W is closed with respect to φ and $X \subseteq W$, then X is closed with respect to $\varphi|_W$ implies X is closed with respect to φ.
(c) If X is closed with respect to $\varphi|_W$ and $(X.\varphi - X) \cap (\boldsymbol{U} - W) = \emptyset$, then X is closed with respect to φ.

Proof. (a) If X is closed with respect to φ, then $X.\varphi = X$ and $(X \cap W).\varphi|_W \subseteq X.\varphi|_w \cap W.\varphi|_W = X.\varphi \cap W \cap W = X \cap W$.
(b) X closed with respect to $\varphi|_W$ implies $X.\varphi \cap W = X$. But $X \subseteq W$ and W closed imply $X.\varphi \subseteq W$, so $X.\varphi \cap W = X.\varphi = X$.
(c) If X is not closed with respect to φ, then $(X.\varphi - X) = \Delta$. Let X be closed with respect to $\varphi|_W$ so $\emptyset = X.\varphi|_W - X = (X.\varphi \cap W) - X = (X.\varphi - X) \cap W$. Consequently, $\Delta \subseteq \mathbf{U} - W$, and $(X.\varphi - X) \cap (\mathbf{U} - W) \neq \emptyset$.

□

Thus, subject to reasonable constraints, we can focus in on smaller regions without distortion.

6.2. Reduction and chordless cycles

Relative closure supports close examination of local structure. Often we are more interested in its global understanding, which can be facilitated by collapsing much of the local detail and retaining only a "broad brush" representation of its structure. We can "reduce" large networks by iteratively deleting any node z for which $z \in \{y\}.\varphi$. It is not hard to show that removing z (and all its connections) will be continuous. Hence, by Proposition 5.1, the entire transformation $(U, \varphi) \xrightarrow{\omega} (U', \varphi')$ is continuous. The actual computer process, ω, is presented in some detail in [24, 25, 27].

The extent of the reduction can be quite variable, depending on the structure of the network. Tightly structured networks (with many triangles) seem to show the largest degree of reduction. In Figure 11, we have the 65 node reduction of Newman's 379 node collaboration network that we had illustrated as Figure 1. In the node symbols, $< n >$, of this figure, n denotes the number of original elements that were reduced to this single element. Each such cluster can be reconstructed,

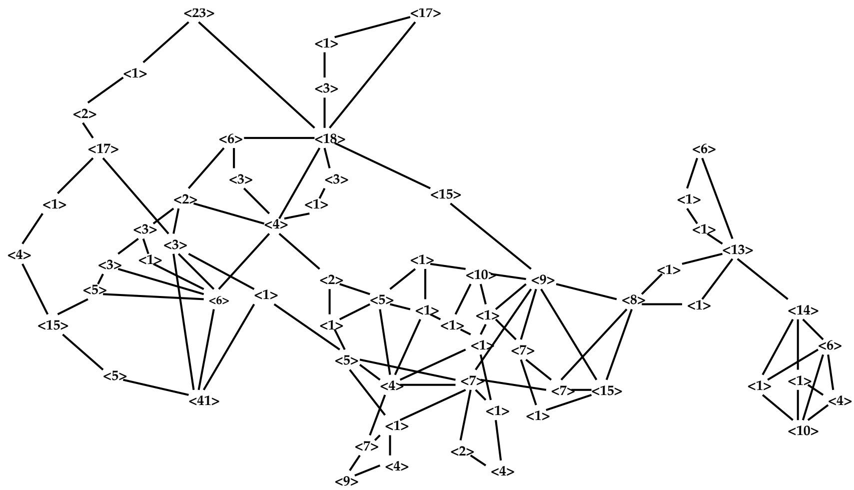

Figure 11. The reduced 65 node version of Newman's collaboration network.

if desired. This reduction process is fast and efficient.[13] In [25], we illustrate the reduction of a 1421 node network depicting relationships between Norwegian boards of directors to just 103 nodes. This kind of reductive performance seems representative.

Reduction is a useful tool for understanding networks of all kinds. The reduced network has a number of interesting properties [27]. For example,

Proposition 6.2. *Let* $(U.\rho) \xrightarrow{\omega} (U', \rho')$. *There exists a path from* x *to* z *in* (U, ρ) *if and only if there exists a path from* $x' = x.\omega$ *to* $z' = z.\omega$ *in* (U', ρ').

Proof. This is a rather long, tedious proof. See [27]. □

Perhaps equally important, it has been shown that all remaining nodes in the reduced network lie on a chordless cycle of $k \geq 4$ nodes, or a unique path joining two such cycles. A cycle $< x_0, x_1, \ldots x_k = x_0 >$ where $x_i \; \rho \; x_{i+1}$ is *chordless* if there exists no edge, or relationship, $x_i \; \rho \; x_{i \pm m}$ where $m \geq 2$. These chordless cycles represent the somewhat tenuous connections that exist between the more tightly related clusters typically seen in social networks. Granovetter [13] speaks of the strength of "weak connections". In Figure 11, the cluster of individuals denoted by $< 15 >$ is part of a chordless cycle that connects $< 18 >$ and $< 9 >$.

Chordless cycles are easily broken by continuous transformation (Proposition 5.7) and difficult to create (Proposition 5.6). Reduction reveals the presence of chordless cycles, but does not create them. For this reason, it is conjectured that, in knowledge networks, these chordless k-cycles can result from discontinuous learning processes just as tight clusters and triangles may be the result of natural, continuous development processes.

[13] Worst case performance is $O(n^2)$ where n denotes the size of the network, but expected performance is $O(n)$, which has been observed in practice.

While the reduction code was written to simplify large social networks, the process may mimic that of "natural abstraction" in cognitive development. Clusters of experiential input can be replaced by a single descriptive node. Verbalization and the role of *signs* is important to Vygotsky. Initially, he thinks, memory is the recall of external events by means of specific attributes, as in a "grandmother has a soft lap". Or we might say, by the recall of generators. Later in childhood, words are used to bind together "whole groups of visual things connected by visual ties". Finally, toward the end of childhood, the interfunctional relations involving memory reverse their direction. *For the young child, to think means to recall; but for the adolescent, to recall means to think*" ([5] pp. 50-51, italics his). The process of "self talk", frequently observed in young children when engaged in a problem, may be a manifestation of a similar on-going abstraction process where the original cluster of experiential stimuli is being replaced by a verbal representative. So long as the abstraction process is continuous, it is, by Proposition 6.2, faithful to reality.

7. Summary

Even though we have repeatedly shown parallels with concepts propounded by Vygotsky, we are not proposing any particular model of knowledge in this paper. Mathematics is only a language by which various ideas may be expressed and tested. We could as well have chosen to illustrate some of these concepts using the theories of Jean Piaget who describes a process of *assimilation* which is the "integration of external elements into evolving or completed structures of an organism" ([29] p.706). This he contrasts with *accommodation*, in which the individual restructures his existing understanding of the world. This latter could be interpreted in the light of continuous *reduction* and *abstraction* described in Section 6.2, or it could be understood as a more discontinuous process. Similarly, both the issue of *peripheral-origins* vs. *central-origins* explored by Spelke, Breinlinger, Macomber and Jacobson [34], and the conjecture of Case regarding *central conceptual structures* [3], could be accommodated within our closed set approach. It favors none of them. But, it can provide an internal structure that could be used, in turn, to test the consistency of these models.

Our real goal has been to explore whether properties of closure operators and closed set systems can be relevant to modeling cognitive processes. We have presented ρ as an *experiential* operator. We have considered closed sets as units of *knowledge* that can be characterized by their generators and partially ordered to form a *knowledge lattice*. We have couched *learning* in terms of transformations.

Proposition 5.7 provides necessary and sufficient conditions for a specific kind of transformation which removes a link in a relationship to be continuous. It seems to be a widely held contention that *learning* involves the acquisition of more experiences and more data. In early childhood when our neural capabilities are growing this would seem so. But, even at an early age, children appear to be condensing raw stimuli into abstract identifiable concepts. In the process of *learning*, deletion seems to be as valuable as addition. In many forms of autism, it is the inability to delete and control an overload of raw sensory images that is problematic.

We, and others, have said that discontinuous transformations may be essential to our cognitive development. But, aside from Figure 9, we have otherwise said very little about discontinuity. That is largely because discontinuity is difficult to handle mathematically. Besides acknowledging its existence through example, there is little one can prove about a discontinuous function, or transformation. On the other hand, we have shown through the reduction process, ω, that certain forms of major abstract restructuring are quite possible through continuous transformation. So it is not clear to what extent discontinuity is a necessary feature of cognitive development, or an artifact of external circumstance created by the learning environment. Vygotsky cites an example of $neoformation$ arising from a rather disruptive external force [31]. This would be discontinuous.

We believe that we have made a strong case for regarding the cognitive process as being an, as yet unknown, closure operator; and demonstrated that such closure processes can actually be implemented in a neural network setting. Moreover, dynamic changes to our cognitive awareness can be modeled in terms of continuous, and discontinuous, processes. It will certainly take further refinement, including consideration of multiple experiential relationships, and considerable experimental testing to validate this claim.

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John L. Pfaltz, Department of Computer Science, University of Virginia, 85 Engineer's Way, Charlottesville, VA 22904, USA
e-mail: jlp@virginia.edu