Cycle Matroids

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Abstract

A matroid whose elements are cycles is quite different from the better known "cycle matroid" of the matroid literature. In this paper, we show that every cycle system, $\mathcal{C}$, comprised of cycles together with a cycle composition operator, $\circ$, is a matroid, $\mathcal{M}$, or generalized vector space. In addition, the set of cycles under composition, $\circ$, would be a commutative group, except that $\circ$ turns out to be only a partial binary operator. We establish that $\mathcal{C}$ is a Brandt semigroup, but endowed with two-sided identity and inverses.

The interplay between graph, matroid, and group properties makes cycle matroids an interesting new mathematical object with possible application as a model of biological information.

1 Introduction

This paper is about cycle structures, such as Figure 1 (ignore the stringy tendrils). These are 3-dimensional structures. The figure is a 2-dimensional rendition of a protein polymer found in the membrane surrounding the nucleus of every cell of our bodies [1]. It is customary to project such cycle structures into a 2-dimensions so that we can more easily visualize and describe them. These projections become ordinary undirected graphs. Consequently, this paper will involve a mixture of graph theory, group theory, matroid theory. We combine these three to define a class of mathematical objects which appear to be quite novel, and relatively unstudied.

A **matroid** is a collection of sets, some of which are regarded as "independent". If every maximal, independent collection has the same cardinality, r, then this system of independent collections is called a matroid of rank r. There is an abundance of literature on matroids, of which [3, 15, 20, 22] is only a sample. The term "cycle matroid" of a graph $G = (N, E)$ is well-known. It is frequently used as a simple introduction to basic matroid concepts. In that introductory example, a

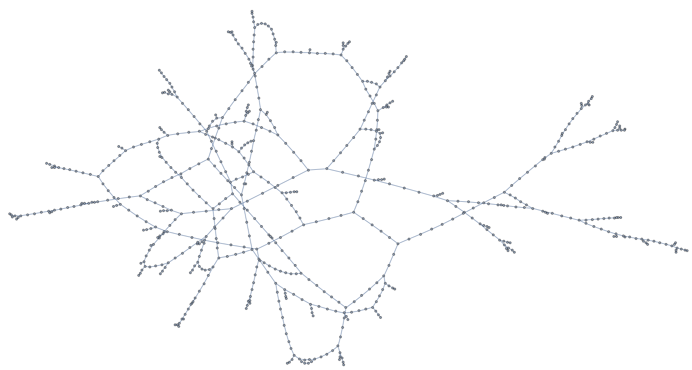


Figure 1: A membrane polymer controling protein transport across the neucleus wall.

set of edges $X \subseteq E$ is said to be "independent" if it contains no cycles. A maximal, independent set is thus a spanning tree of G, with cardinality $n-1$. So this matroid has rank $r = n - 1$. Any set of edges containing a cycle is a dependent set. This notion of dependence and independence yields a clear, intuitive example of the matroid concept.

However, it is not at all what we mean by a "cycle matroid"!

The elements of graph based matroids are the edges of the graph. The elements of our "cycle matroids" are the cycles themselves. We use nodes and edges only to help describe the individual cycles.

In Section 2, we define the notion of cycle composition, together with its properties and that of independent sets of cycles. In particular, we show that cycle composition, $\circ$, is an associative operator. Then in Section 3 we show that a cycle system, $\mathcal{C}$, is a non-trivial matroid of rank r, that is, all sets of independent cycles have the same cardinality. We examine three distinct cycle systems, all of rank $r = 3$ in Section 4. Two have the same adjacency relationships, yet one is planar and the other is not. The third system, $\mathcal{C}_\gamma$, demonstrates that the composition operator, $\circ$, is only a partial binary operator. Nevertheless, we establish when associativity can be assumed, and demonstrate that every cycle system is a Brandt semigroup. Finally, in Section 5 we examine the flats of a cycle matroid, and their semilattice structure when partially ordered by inclusion.

2 Cycle System Basics

Let $G = (N, E)$ denote a finite, undirected **graph** consisting of a set N of nodes and a set E of edges $\{x, y\}, x, y \in N$. [1] Many authors believe that the relationships between the nodes of a graph, or a network, can constitute a mathematical model of information.

A **path**, $\rho(x, z)$ of length n is a sequence $< y_1, y_2, \ldots, y_n >$ of nodes such that $y_1 = x, y_n = z$ and for $1 \leq i \leq n$, $\{y_i, y_{i+1}\} \in E$. If $x = z$, we say $\rho(x, z)$ is **closed**, or an n-**cycle** which we denote by C.[2] If $y_i \neq y_k$ for all $1 \leq i, k \leq n$, we say the path (or cycle) is **simple**. In this paper we assume all paths/cycles are simple; but may occasionally say it again for emphasis. We call the sequence $< y_0, y_1, \ldots, y_n >$ an **enumeration** of C. Readily, any of the nodes y_i in the cycle can serve as the initial node $x = z$ of an enumeration. Let $G = (N, E)$ and let $X \subseteq N$. By the **subgraph** on X denoted $[X] \subseteq G$, we mean the graph (X, E_X) where $E_X = \{\{x, y\} \in E, x, y \in X\}$.

In the following pages we will denote a cycle C_i by $\bar{C}_i$ if we want to emphasize the edge structure, and by $\dot{C}_i$ if we want to emphasize its nodes, or by just C_i if we are only identifying the cycle. Similarly, we will use the same notation $\dot{\rho}(x, z)$ and $\bar{\rho}(x, z)$ to denote the nodes and edges of a path. By the **length** of a cycle $C_i =< y_1, \ldots, y_{n+1} >, y_{n+1} = y_1$ denoted $|C_i|$ we mean $|\dot{C}_i| = |\bar{C}_i| = n$.

By a **cycle system** $\mathcal{C} = \{C_1, \ldots, C_n\}$, we simply mean a collection of cycles, each of length ≥ 3. Figure 2 illustrates a small cycle system, which we will denote as $\mathcal{C}_\alpha$. It consists of 3 cycles, $\dot{C}_1 =< abckji >$, $\dot{C}_2 =< cdek >$ and $\dot{C}_3 =<$

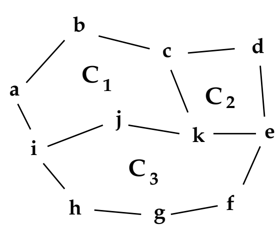


Figure 2: A small representative cycle system, $\mathcal{C}_\alpha$.

$efghijk >$, of lengths 6, 4 and 7 respectively. (We normally elide the commas when enumerating sets if no confusion is possible.)

[1]Since G is undirected, edges between nodes are really just sets $\{x, y\}$, where $|\{x, y\}| = 2$. If E can have sets $\{x, y, z\}$ with cardinality > 2, then G is called a **hypergraph**.

[2]Some authors, *e.g.* [2, 16] make the distinction between directed "cycles" and undirected "circuits. Others, *e.g.* [10], talk of cycles in graphs, but circuits in derived matroids. We follow the usage in [9] and since we only consider undirected graphs/relationships we excusively use the term *cycle* without confusion.

The cycles C_1, C_2 and C_3 have been labled in the figure, but there are more. For example, we can identify $\dot{C}_4 =< abcdekji >$, $\dot{C}_5 =< abckefghi >$, and $\dot{C}_6 =< cdefghijk >$. (Enumerating sets in cyclic order simplifies identifying them in figures. Enumerating sets in alphabetic order simplifies comprehension of set operations. Enumerating cycles by their edges, as in $\bar{C}_3 =< \{ef\}, \{fg\}, \{gh\}, \{hi\}, \{ij\}, \{jk\}, \{ke\} >$ is unambiguous; but tedious. We try to avoid it whenever possible.)

The cycle system $\mathcal{C}_\alpha$ of Figure 2 is also an undirected graph $G = (N, E)$, where $N = \{a, b, \ldots, k\}$. We use $\mathcal{C}$ and cycle terminology when we want to emphasize the roles of the cycles and cycle composition (developed in the following section) and G when we want to explore its traditional graph theoretic properties. A cycle C is a subgraph on $\dot{C}$ in the base graph G.

2.1 Cycle Composition, $\circ$

We observe that the cycle $C_4 =< abcdekji >= \bar{C}_4 = (\bar{C}_1 \cup \bar{C}_2) \sim (\bar{C}_1 \cap \bar{C}_2)$. We say a cycle C_m is the **composition** of C_i and C_k, denoted $C_i \circ C_k$, whenever

$$C_m = C_i \circ C_k = \bar{C}_m = (\bar{C}_i \cup \bar{C}_k) \sim (\bar{C}_i \cap \bar{C}_k) \tag{1}$$

Not only is $C_4 = C_1 \circ C_2$ we have $C_5 =< abckefghi >= (\bar{C}_1 \cup \bar{C}_3) \sim (\bar{C}_1 \cap \bar{C}_3)$ and $C_6 = C_2 \circ C_3 = C_4 \circ C_5 = C_1 \circ C_3 \circ C_4$. Two cycles C_i and C_k are said to be **adjacent** if $\bar{C}_i \cap \bar{C}_k \neq \emptyset$.

The **empty cycle**, $C_\emptyset$ is precisely that, the empty set, or $\dot{C}_\emptyset = \bar{C}_\emptyset = \emptyset$. Readily, for all C_i, $C_i \circ C_\emptyset = C_i = C_\emptyset \circ C_i$, so $C_\emptyset$ serves as the identity element for $\circ$. Moreover, because $(\bar{C}_k \cup \bar{C}_k) \sim (\bar{C}_k \cap \bar{C}_k) = \emptyset$, for all k, we have $C_k \circ C_k = C_\emptyset$.

Since union, $\cup$, and intersection, $\cap$, are symmetric, it seems apparent that $\circ$ is symmetric. However the following trivial lemma makes it evident.

Lemma 2.1 *Let $C_i \circ C_k = C_m$. Then $\{x, y\} \in C_m$ if and only if $\{x, y\} \in \bar{C}_i \cup \bar{C}_k$ and $\{x, y\} \notin \bar{C}_i \cap \bar{C}_k$.*

Proof: Evident from the definition of $\circ$ in (1). □

Proposition 2.2 *For all i, k, $C_i \circ C_k = C_k \circ C_i$.*

Proof: One simply applies the lemma 2.1 to both sides of the equation. □

Proposition 2.3 *For all i, k, m, $C_i \circ (C_k \circ C_m) = (C_i \circ C_k) \circ C_m$.*

Proof: Let $\{x, y\} \in \bar{C}_i \circ (\bar{C}_k \circ \bar{C}_m)$. Then, $\{x, y\} \in \bar{C}_i \cup (\bar{C}_k \cup \bar{C}_m) = (\bar{C}_i \cup \bar{C}_k) \cup \bar{C}_m)$ and $\{x, y\} \notin \bar{C}_i \cap (\bar{C}_k \cap \bar{C}_m)$. By lemma 2.1, $\{x, y\} \notin \bar{C}_i \cap (\bar{C}_k \cap \bar{C}_m)$ implies $\{x, y\} \notin (\bar{C}_i \cap \bar{C}_k) \cap \bar{C}_m)$, thus $\{x, y\} \in (\bar{C}_i \circ \bar{C}_k) \circ \bar{C}_m$. □

The lemma is not really necessary to prove Prop. 2.3, but it helps to keep track whether $\{x, y\}$ is has been eliminated because it is in one or more of the intersections.

Table 1 completely details the composition operator for the 8 cycles of $\mathcal{C}_\alpha$.

$\circ$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	$C_\emptyset$
C_1	$C_\emptyset$	C_4	C_5	C_2	C_3	C_7	C_6	C_1
C_2	C_4	$C_\emptyset$	C_6	C_1	C_7	C_3	C_5	C_2
C_3	C_5	C_6	$C_\emptyset$	C_7	C_1	C_2	C_4	C_3
C_4	C_2	C_1	C_7	$C_\emptyset$	C_6	C_5	C_3	C_4
C_5	C_3	C_7	C_1	C_6	$C_\emptyset$	C_4	C_2	C_5
C_6	C_7	C_3	C_2	C_5	C_4	$C_\emptyset$	C_1	C_6
C_7	C_6	C_5	C_4	C_3	C_2	C_1	$C_\emptyset$	C_7
$C_\emptyset$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	$C_\emptyset$

Table 1: Composition table for the cycle system of $\mathcal{C}_\alpha$ of Figure 2.

The following proposition is used so often in the following sections that the term "theorem" seems appropriaate

Proposition 2.4 (Exchange theorem) *If $C_m = C_i \circ C_k$ then $C_k = C_i \circ C_m$.*

Proof: Let $C_m = C_i \circ C_k$, then

$$\begin{aligned} C_k &= C_\emptyset \circ C_k \\ &= (C_i \circ C_i) \circ C_k \\ &= C_i \circ (C_i \circ C_k) \\ &= C_i \circ C_m. \ \Box \end{aligned}$$

Proposition 2.5 *If $C_i \neq C_k$ where $i \neq k$ then $C_i \circ C_k \neq C_\emptyset$.*

Proof: Suppose $C_i \circ C_k = C_\emptyset$, where $i \neq k$, then $C_k = C_\emptyset \circ C_k = (C_i \circ C_i) \circ C_k = C_i \circ (C_i \circ C_k) = C_i \circ C_\emptyset = C_i$, or $C_i = C_k$ contradicting the condition. $\Box$

Proposition 2.6 *If $C_i \circ C_k = C_i \circ C_m$ then $C_k = C_m$.*

Proof: Let $C_i \circ C_k = C_s = C_i \circ C_m$. By Prop. 2.4, $C_k = C_i \circ C_s = C_i \circ (C_i \circ C_m) = C_\emptyset \circ C_m = C_m$. $\Box$

The consequence of Proposition 2.6 is that C_i defines a permutation on the cycles $\{C_1, C_2, \ldots C_n\}$ of $\mathcal{C}$, since each composition $C_i \circ C_k$ is a unique element of $\mathcal{C}$. The rows (or columns) of Table 1 illustrate this.

Figure 3 provides a slightly larger cycle system, $\mathcal{C}_\beta = \{C_1, C_2, C_3, C_4, C_5\}$.

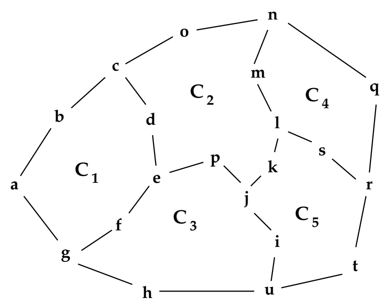


Figure 3: A cycle system $\mathcal{C}_\beta$ on 21 elements with 25 edges.

2.2 Basic Cycles

A set $\mathcal{S} = \{C_i\}$ of non-empty cycles is said to be **dependent** if there exists $C_m \in \mathcal{S}$ such that $C_m = C_i \circ \ldots \circ C_k$ where $C_i, \ldots, C_k \in \mathcal{S}$. If $\mathcal{S}$ is not dependent, it is said to be **independent**. Any cycle can be a member of an independent set.

A maximal independent set of cycles $\mathcal{B}$ is said to be a **basis** for the system $\mathcal{C}$. For any cycle $C_k \in \mathcal{C}$, either $C_k \in \mathcal{B}$ or $C_k = C_i \circ \ldots \circ C_j$, where $C_i, \ldots C_j \in \mathcal{B}$. A system may have many bases. $\mathcal{B}_1 = \{C_1, C_2, C_3\}$ is one basis for the system $\mathcal{C}_\alpha$ of Figure 2; $\mathcal{B}_2 = \{C_1, C_2, C_6\}$ is another. If one "draws" the graph G as in Figures 2 or 3 (they need not be planar) then the evident simple cycles without cross connections constitute one basis for G. This simplifies reasoning about $\mathcal{C}$. For example $\mathcal{B}_\beta = \{C_1, C_2, C_3, C_4, C_5\}$ is a basis for Figure 3, as can be exhaustively verified. (Note: we have not yet established that the cardinality of all basis sets must be the same. However, propositions 2.4 and 2.6 suggest that this must be so.)

2.3 Lexicographic Labelling

We have been denoting the cycles in a graph G, or cycle system $\mathcal{C}$, by the labels $C_1, C_2, \ldots, C_i, \ldots$ where the subscripts are integer. Clearly any index set could be used. Given any initial basis set of r cycles, we will arbitrarily label (denote) them by $C_1, C_2, \ldots C_r$. Then $C_1 \circ C_2 = C_{r+1}$, The label C_{r+2} is assigned to $C_1 \circ C_3$ and $C_1 \circ C_4 = C_{r+3}$. The cycle $C_1 \circ C_r$ must be labeled C_{2r}. Now $C_2 \circ C_3$ becomes C_{2r+1} *etc*. This is a lexicographic labelling.

Thus if $r = 3$ as in $\mathcal{C}_\alpha$ of Figure 2, $C_6 = C_2 \circ C_3$ regardless of the actual position of these cycles C_2 and C_3 in the graph. Consequently Table 1 can serve a the composition table for every cycle system with 3 basis cycles.

The more interesting cycle system of Figure 3 has the lexicographic labelling as follows.

C_1	$C_{12} = C_2 \circ C_5$	$C_{23} = C_2 \circ C_3 \circ C_5$
C_2	$C_{13} = C_3 \circ C_4$	$C_{24} = C_2 \circ C_4 \circ C_5$
C_3	$C_{14} = C_3 \circ C_5$	$C_{25} = C_3 \circ C_4 \circ C_5$
C_4	$C_{15} = C_4 \circ C_5$	$C_{26} = C_1 \circ C_2 \circ C_3 \circ C_4$
C_5	$C_{16} = C_1 \circ C_2 \circ C_3$	$C_{27} = C_1 \circ C_2 \circ C_3 \circ C_5$
$C_6 = C_1 \circ C_2$	$C_{17} = C_1 \circ C_2 \circ C_4$	$C_{28} = C_1 \circ C_2 \circ C_4 \circ C_5$
$C_7 = C_1 \circ C_3$	$C_{18} = C_1 \circ C_2 \circ C_5$	$C_{29} = C_1 \circ C_3 \circ C_4 \circ C_5$
$C_8 = C_1 \circ C_4$	$C_{19} = C_1 \circ C_3 \circ C_4$	$C_{30} = C_2 \circ C_3 \circ C_4 \circ C_5$
$C_9 = C_1 \circ C_5$	$C_{20} = C_1 \circ C_3 \circ C_5$	$C_{31} = C_1 \circ C_2 \circ C_3 \circ C_4 \circ C_5$
$C_{10} = C_2 \circ C_3$	$C_{21} = C_1 \circ C_4 \circ C_5$	
$C_{11} = C_2 \circ C_4$	$C_{22} = C_2 \circ C_3 \circ C_4$	

This provides a standard factorization for all cycles; so for example $C_{10} \circ C_{19} = (C_2 \circ C_3) \circ (C_1 \circ C_3 \circ C_4) = (C_1 \circ C_2 \circ C_4) = C_{17}$.

3 Cycle Matroids

A matroid is a generalized vector space. Neel and Neudauer [15] is a gentle introduction to matroids. But, beware of misleading terminology. For example, based on the notion of "cycle matroids" mentioned in Section 1, a "circuit" refers to a minimal dependent set; it need have not geometric meaning.

Let $Y = \{C_1, \ldots, C_n\}$ be a set of cycles. By the **span** of Y, denoted $Y.\sigma$, we mean the set of all cycles $\{C_m\}$ such that $C_m = C_i \circ \ldots \circ C_k$, where $C_i, \ldots C_k, \in Y$.[3] Readily, $Y \subseteq Y.\sigma$.

An arbitrary operator, φ, is said to be a **closure** operator if φ is

expansive, $Y \subseteq Y.\varphi$

monotone, $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$ and

idempotent, $Y.\varphi.\varphi = Y.\varphi$.

Readily, $Y \subseteq Y.\varphi$.

Proposition 3.1 *The spanning operator, σ is a closure operator over sets Y of cycles.*

Proof: Readily, σ is expansive and monotone.
Let Y be a set of cycles $\{C_i\}$. Suppose $C_m \in Y.\sigma.\sigma$ implying that there exists some sequence $1 \leq i \leq k$ such that

$$C_m = C_1 \circ \ldots \circ C_i \circ \ldots \circ C_k \quad (2)$$

[3]In graph theory, the term "span" usually refers to a tree whose nodes include all $y \in N$. Since a tree has no cycles, it has no connection to our usage which is taken from the notion of spanning vector spaces. Another commone term is the "graphic matroid". These are matroids derived from graphs; cycle matroids are derived from cycle composition and graphs are just used to visualize them.

where $C_i \in Y.\sigma$, $1 \leq i \leq k$. Hence $C_i = C_{i_1} \circ \ldots \circ C_{i_n}$ where $C_{i_j} \in Y$.
Thus, substituting into the sequence (2) for each i, we get $C_m = (C_{1_1} \circ \ldots \circ C_{1_n}) \circ (C_{2_1} \circ \ldots \circ C_{2_n}) \circ \ldots \circ (C_{k_1} \circ \ldots \circ C_{k_n})$ implying $C_m \in Y.\sigma$. □

A closure system is said to be a **matroid** if it satisfies the Steinitz-MacLane **exchange axiom** [8, 13, 20], that is:

if $x, y \notin Y.\varphi$ and $y \in (Y \cup x).\varphi$ then $x \in (Y \cup y).\varphi$.

If φ satisfies the *anti-exchange* axiom [17], that is:

if $x, y \notin Y.\varphi$ and $y \in (Y \cup x).\varphi$ then $x \notin (Y \cup y).\varphi$

then the system is called an *antimatroid* [5, 11, 12, 14, 19].

Proposition 3.2 *Let $\mathcal{C}$ be a cycle system and let σ be the spanning operator. The system $(\mathcal{C}, \sigma)$ satisfies the Steinitz-Maclane exchange axioms and is thus a matroid.*

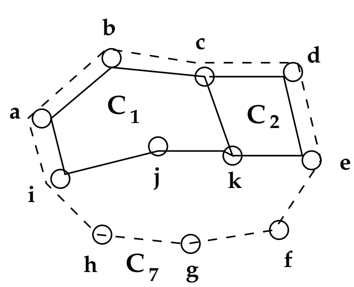
Proof: By Prop. 3.1, σ is a closure operator.
Let $C_i, C_k \not\subseteq Y.\sigma$ where $Y = \{\ldots, C_j, \ldots\}$. Suppose $C_k \in (Y \cup C_i).\sigma$ implying that $C_k = C_i \circ (\ldots C_j \ldots) = C_i \circ C_m$ where $C_m \in Y.\sigma$. Consequently, by Prop. 2.4, $C_i = C_k \circ C_m$ or $C_i \in (Y \cup C_k).\varphi$. □

Since $\mathcal{C}$ is a matroid, the cardinality of every maximal independent set is fixed and this number, r, is the **rank** of the system. Since any cycle system $\mathcal{C}$ constitutes a matroid, it satisfies the following fundamental basis exchange theorem [20, 21].

Proposition 3.3 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be any two bases of $\mathcal{C}$, and let $C_i \in \mathcal{B}_1$. Then there exists $C_k \in \mathcal{B}_2$ such that $(\mathcal{B}_1 \sim C_i) \cup C_k$ is a basis of $\mathcal{C}$.*

Matroids and vector spaces are more often characterized by this ability to arbitrarily exchange basis elements.

This "exchange" property is illustrated by the two examples in Figure 4. $\mathcal{B}_1 =$



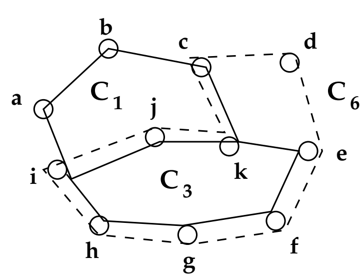


Figure 4: Alternate bases for $\mathcal{C}_\alpha$ shown in Figure 2

$\{C_1, C_2, C_3\}$ is one basis for $\mathcal{C}_\alpha$ If we remove C_3 from $\mathcal{B}_1$ then it can be replaced with $C_7 = C_1 \circ C_2 \circ C_3$ to yield $\mathcal{B}_2 = \{C_1, C_2, C_7\}$ shown as the left-most figure.[4] If C_2 is removed from $\mathcal{B}_1$ then $\{C_1, C_3, C_6\}$ can be a basis $\mathcal{B}_3$. Using $\mathcal{B}_2$, $C_3 = C_1 \circ C_2 \circ C_7$. With $\mathcal{B}_3$, $C_2 = C_3 \circ C_6$.

4 Three Cycle Systems

In this section we compare 3 different cycle systems, each with rank 3. It will be useful to use adjacency to compare the basis sets of these systems. Recall, that two cycles, C_i, C_k, were said to be "adjacent" if $\bar{C}_i \cap \bar{C}_k \neq \emptyset$. Thus, for any system $\mathcal{C}$ of cycles we can construct an **adjacency graph**, $A_{\mathcal{C}}$, where $N = \{C_1, \ldots, C_n\}$ and $\{C_i, C_k\} \in E$ if C_i is adjacent to C_k. (In graph theory, the adjacency graph of such a graph is called its **dual** graph.) For this section we will only construct an adjacency graph, $A_{\mathcal{B}}$, with respect to a basis set. Readily, for the basis $\mathcal{B}_\alpha$ of Figure 2, the adjacency graph $A_{\mathcal{B}}$ is that of Figure 5.

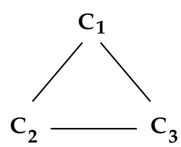


Figure 5: Adjacency graph, $A_{\mathcal{B}}$ for $\mathcal{C}_\alpha$ shown in Figure 2.

Now, consider the $K_{3,3}$ graph of Figure 6 with a basis of 3 cycles shown to

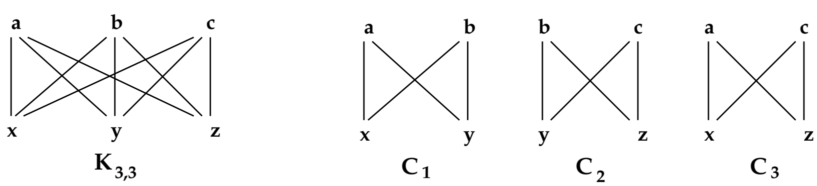


Figure 6: The non-planar system $K_{3,3}$.

the right. The $K_{3,3}$ graph of Figure 6 is non-planar[5]; but since cycle systems are embedded within 3-space, this is not an issue. It is not hard to see that the adjacency graph of this basis $\mathcal{B}$ of $K_{3,3}$ is identical to that of $\mathcal{C}_\alpha$ shown above as Figure 5.

The cycle $C_1 \circ C_2 \circ \ldots \circ C_r$, for all $C_i \in \mathcal{B}$, sometimes regarded as the "boundary" of $\mathcal{C}$ with respect to $\mathcal{B}$, is often of interest. Figure 7 visually compares

[4] $\{C_1, C_2, C_5\}$, $\{C_1, C_2, C_6\}$ would also be bases. $\{C_1, C_2, C_4\}$ can not be a basis because $C_4 = C_1 \circ C_2$.

[5] Kuratowski's theorem states that a graph G is planar if and only if he has no subgraphs homeomorphic to $K_{3,3}$ or K_5 [2, 10].

the "interior" structure of Figures 2 and 6 with respect to their "boundary" cycles

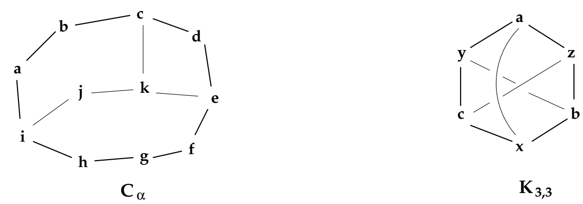


Figure 7: Figures 2 and 6 redrawn with respect to their "boundary".

(emboldened). They are clearly different, even though their bases have precisely the same adjacency structure.

The cycle system $\mathcal{C}_\gamma$ shown in Figure 8 is obviously quite different from either $\mathcal{C}_\alpha$ or $K_{3,3}$. So too, is the adjacency graph $A_\mathcal{B}$ of this basis $\mathcal{B}_\gamma = \{C_1, C_2, C_3\}$

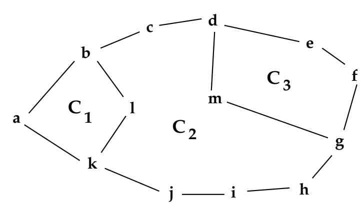


Figure 8: A different cycle system, $\mathcal{C}_\gamma$, with rank, $r = 3$.

shown in Figure 9.

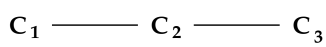


Figure 9: Adjacency graph of $\mathcal{B}_\gamma$.

However, if we consider the alternate basis $\mathcal{B}_\gamma = \{C_1, C_2, C_6\}$ where $C_6 = C_2 \circ C_3$ as shown in Figure 10, we observe that its adjacency graph, shown as Figure 11, is isomorphic to the basic adjacency graph of $\mathcal{C}_\alpha$ and $K_{3,3}$; but quite unlike $A_\mathcal{B}$ for the first basis of $\mathcal{C}_\gamma$. Evidently, the adjacency structure of different basis sets of any given cycle system need not be consistent. And clearly, cycle systems (or cycle matroids) cannot be distinguished with regard to the structure of the basis sets.

4.1 Partial Binary Operators

The cycle system $\mathcal{C}_\gamma$ of Figure 8 raises an even more serious question. By the definition of cycle composition (1), $C_1 \circ C_3 =< ablk > \cup < defgm >$. It is not

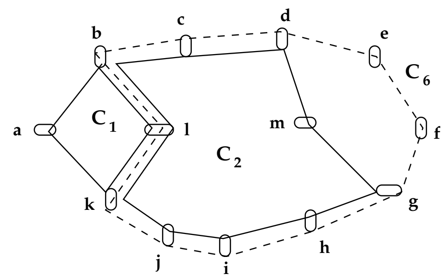


Figure 10: Cycles in a different basis for $\mathcal{C}_\gamma$ shown in Figure 8.

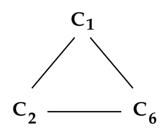


Figure 11: Adjacency graph $A_\mathcal{B}$ for second basis of $\mathcal{C}_\gamma$.

a cycle!

There are two ways of resolving this question. One can define a "cycle" to be the disjoint union of one, or more, "simple cycles". Or one can accept the fact that cycle composition is not a true binary operator; but rather a **partial binary operator** [6, 24]. We prefer the latter.

A partial binary operator over a domain X need not be well-defined for all $x, y \in X$. Such partial binary operators exist in graph theory. Edge concatenation of two edges, (u, v) and (x, y),' to form a longer path by transitive closure is well-defined only if $v = x$.

If cycle composition is not everywhere well-defined, then it throws into question our earlier propositions. Clearly, Proposition 2.3 must be modified to read "*For all i, k, m, if $C_k \circ C_m$ and $C_i \circ C_m$ are well-defined cycles, then* ... ". Surprisingly, careful reading will show that propositions 2.4 through 2.6 are still correct as written. In particular we re-examine the proof of Proposition 2.4.

Proof: Let $C_m = C_i \circ C_k$, (assumes $C_i \circ C_k$ is a well-formed cycle C_m) then

C_k	$= C_\emptyset \circ C_k$	composition with $C_\emptyset$ always yields a cycle
	$= (C_i \circ C_i) \circ C_k$	composition defines $C_\emptyset$
	$= C_i \circ (C_i \circ C_k)$	a cycle by previous assumption
	$= C_i \circ C_m.$	$\square$

So the proof of Proposition 2.4 is still valid as written. Similarly, in Propositions 2.5 and 2.6, the necessary existance assumptions assure associativity. But, this is not always the case, we need a more general result.

Proposition 4.1 *Let $X = \{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{B}$ be a connected set (in the adjacency graph, $A_{\mathcal{B}}$). The product $C_k = C_1 \circ C_2 \circ \ldots \circ C_n$ is well-formed cycle.*

Proof: If $n = 2$, since C_1, C_2 are connected, $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$, so $C_1 \circ C_2$ is well-formed. We assume the proposition for sets of $n - 1$ cycles.
Now let $|X| = n$. Since X is connected, C_n is connected to $C_k = C_1 \circ C_2 \circ \ldots \circ C_{n-1}$, which by assumption is a well-formed cycle C_k. Hence $\bar{C}_k \cap \bar{C}_n \neq \emptyset$, so $C_k \circ C_n = (C_1 \circ C_2 \circ \ldots \circ C_{n-1}) \circ C_n$ is a well-formed cycle. □

This proposition that the composition of all cycles is a connected set is well-defined is an important one, because individual compositions within the set need not be.

Proposition3.1 implicitly assumes associativity. Zhang ([24], Prop. 2.5) shows the same closure result with respect to a partial operator by using the fact that the intersection of closed sets must be closed. There is no need to duplicate his approach.

4.2 Partial Semigroups

Partial binary operators have not been extensively studied; but there is some literature regarding them. If a binary operator, $\cdot$, is partial, the system $(S, \cdot)$ is called a **groupoid** [Wikipedia]. But, a groupoid $(S, \cdot)$ may have more structure. In particular, there are **Brandt groupoids** [4, 6]. A partial groupoid is a Brandt groupoid, B, if it satisfies ([6], p.99):

(B1) If $a \cdot b = c$ $(a, b, c \in B)$ then each of the three elements a, b, c is uniquely determined by the other two.

(B2) Let a, b, c be elements of B

(i) If $a \cdot b$ and $b \cdot c$ are defined, so are $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$ and these are equal.

(ii) If $a{\cdot}b$ and $(a{\cdot}b){\cdot}c$ are defined, so are $b{\cdot}c$ and $a{\cdot}(b{\cdot}c)$, and $a{\cdot}(b{\cdot}c) = (a{\cdot}b){\cdot}c$.

(iii) If $b{\cdot}c$ and $a{\cdot}(b{\cdot}c)$ are defined, so are $a{\cdot}b$ and $(a{\cdot}b){\cdot}c$, and $(a{\cdot}b){\cdot}c = a{\cdot}(b{\cdot}c)$.

(B3) To each element a in B there correspond unique elements e, f and a' of B such that $e \cdot a = a \cdot f = a$ and $a \cdot a' = f$.

(B4) If $e^2 = e$ and $f^2 = f$ $(e, f$ in $B)$, there exists an element a in B such that $e \cdot a = a \cdot f = a$.

Proposition 4.2 *A cycle structure $\mathcal{C}$ with the partial binary operator $\circ$ (as defined by (1)) is a Brandt groupoid.*

Proof: (B1) is established by Prop. 2.4.
(B2) (i) Suppose $C_a \circ C_b$ and $C_b \circ C_C$ are defined, the C_a and C_b are adjacent in $A_{\mathcal{C}}$ as are

C_b and C_C. Consequently C_C is adjacent to $C_a \cup C_b$, so $(C_a \circ C_b) \circ C_C$ is well-defined. Parts (ii) and (iii) are similar.
(B3) & (B4) Since $C_\emptyset$ serves as a two-sided identity, and $C_a \circ C_a = C_\emptyset$ for all C_a, these are trivially satisfied. □

Clifford and Preston then establish that a Brandt groupoid can be made into a Brandt semigroup by merely adding a zero element, 0, such that

$a \circ b = a \cdot b$ if $a \cdot b$ is defined in B, else 0

$a \circ 0 = 0 \circ a = 0 \circ 0 = 0.$

(The zero element, 0, should not be confused with $C_\emptyset$ which is an identity element, and would be denoted by 1 in their notation.) Readily, any disjoint union of cycles can be regarded as a zero element, so that if $C_i \circ C_k$ is not a cycle, but rather $C_i \cup C_k$, we can say $C_i \cup C_k = 0$. This would make any cycle system a Brandt semigroup. Clifford and Preston ([6], p.102) go on to prove that the following 3 conditions on a semigroup, S, are equivalent

(i) S is a Brandt semigroup,
(ii) S is a completely 0-simple inverse semigroup,
(iii) S is isomorphic to a (regular) Rees $I \times I$ matrix semigroup over a group with zero G^0 and with the $I \times I$ identity matrix Δ as sandwich matrix.

We will not pursue this rather interesting characterization of cycle systems further.

5 Flats and Subgroups and Semilattices

In matroid theory, a **flat** is a closed subset of the matroid $\mathcal{M}$, that is, for any $X \subseteq \mathcal{M}$, $X.\sigma$ is a flat [20, 21]. If for some basis $\mathcal{B}$, $\mathcal{B} \subseteq X$ then $X.\sigma = \mathcal{M}$. More often, $X \subset \mathcal{B}$ and $X.\sigma$ is a subspace of $\mathcal{M}$. In group theory, a subset $H \subseteq G$ is a subgroup if 1) $e \in H$, and 2) if $x, y \in H$ then $x \cdot y \in H$, [23]. In a cycle space the two are closely connected.

Consider the description of the cycle system $\mathcal{C}_\gamma$ represented by Figure 8 and Table 1. Because $C_i \circ C_i = C_\emptyset$ the identity cycle, for all i, every individual cycle constitutes a subgroup of order 2. Consequently, there can be no non-trivial, primary subgroups (generated by a single element). But, there are non-trivial subgroups. One can verify that $\{C_\emptyset, C_1, C_2, C_4\}$, $\{C_\emptyset, C_2, C_3, C_6\}$ and $\{C_\emptyset, C_4, C_5, C_7\}$ are all subgroups of order 4. None are basis sets. Each are non-empty flats in the matroid. Readily,

Proposition 5.1 *For any cycle matroid $\mathcal{M}$, if $X = \{C_i, C_k\}$ and $C_i \circ C_k = C_m$, then $\{C_\emptyset, C_i, C_k, C_m\}$ is a flat.*

Proof: Follows directly from Prop. 2.4. □

The set X in the proposition above is called a **generator** of the flat if X is an independent set.

Let X be a generator, and let $|X| = n$. We will say $X.\sigma$ is an n-**flat**. A flat is said to be **complete** if for all $C_i, C_k \in X.\sigma$, $C_i \circ C_k = C_m \in X.\sigma$. The flats of Proposition 5.1 are complete 2-flats.

There are six complete 1-flats in the cycle system, C_γ. They are $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, $\{0, 4\}$, $\{0, 6\}$, $\{0, 7\}$. In this system, $\{0, 5\}$ is not a flat because $C_5 = C_1 \circ C_3$ is not a cycle. (Here we are just using the subscript k of C_k to denote the cycle.) There are 4 complete 2-flats in $\mathcal{C}_\gamma$. They are: $\{0, (1, 2), 4\}$, $\{0, (1, 6), 7\}$, $\{0, (2, 3), 6\}$, and $\{0, (3, 4), 7\}$. Here we have denoted two generators by parenthesis $(\ldots)$; but any two non-empty cycles must be independent, and thus a generating pair. There can be only one 3-flat, since any generating set of 3 independent cycles must be a basis. It is customary to partially order the flats of a matroid to form a lattice [7, 17]. In the case of $\mathcal{C}_\gamma$, the lattice of complete flats is that of Figure 12

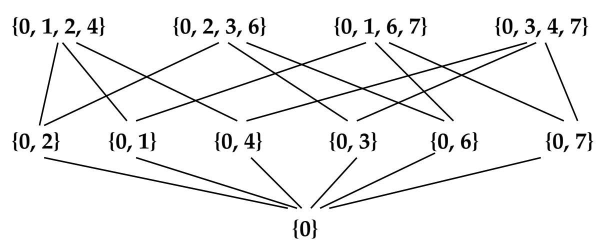


Figure 12: Semilattice of complete flats in $\mathcal{C}_\gamma$.

A more extensive example are the flats of $\mathcal{C}_\beta$ which are shown as Figure 13 which we will enumerate using the lexicographic labelling of Section 2.3. Five of the cycles are of the form $C_i \circ C_k = C_i \cup C_k$. These are (again just using the subscript form) $8 = 1 \circ 4$, $9 = 1 \circ 5$, $13 = 3 \circ 4$, $19 = 1 \circ 3 \circ 4$ and $21 = 1 \circ 4 \circ 5$. Neither these, nor any set containing one of them can be a complete flat. There are 7 complete 2-flats with 4 cycles each, they are: $\{0, (1, 2), 6\}$, $\{0, (1, 3), 7\}$, $\{0, (2, 3), 10\}$, $\{0, (2, 4), 11\}$, $\{0, (2, 5), 12\}$, $\{0, (3, 5), 14\}$ and $\{0, (4, 5), 15\}$.
There are only 3 complete 3-flats with 8 cycles in each: $\{0, (1, 2, 3), 6, 7, 10, 16\}$ $\{0, (2, 3, 5), 10, 12, 14, 23\}$ and $\{0, (2, 4, 5), 11, 12, 15, 24\}$. Again, a generating set $X \subset \mathcal{B}$ is enclosed in parentheses; but any independent subset could serve as a generator.[6]

Proposition 5.2 *A flat is complete if and only if its generator X is a complete subgraph of the adjacency graph $G_{\mathcal{C}}$.*

Proof: Let $C_i, C_k \in X$. C_i is not adjacent to C_k in $[X] \subseteq G_{\mathcal{C}}$ iff $C_i \circ C_k$ is not a cycle

[6]With anti-matroid closure, the generators are unique [12].

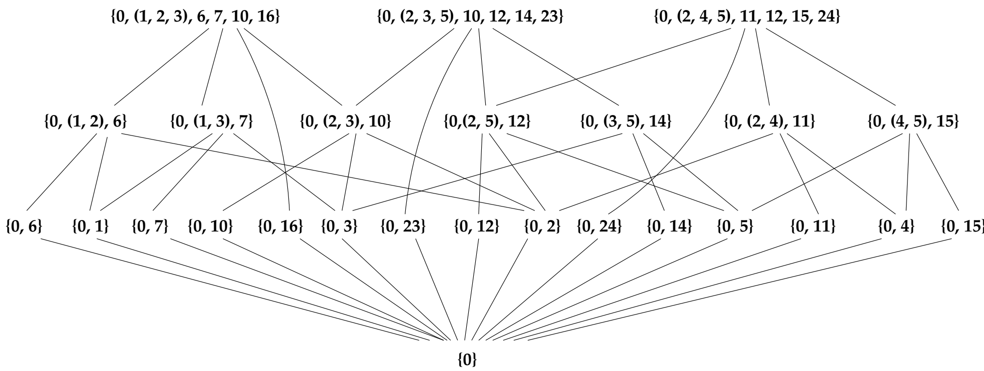


Figure 13: Semilattice of complete flats in $\mathcal{C}_\beta$ of Figure 3

iff the flat is not complete □

Proposition 5.3 *A collection* $H = \{C_i, \cdots, C_k\}$ *is a subgroup of* $\mathcal{C}$ *(viewed as a group) if and only if* $H.\sigma$ *is a complete flat in* $\mathcal{C}$ *(viewed as a matroid).*

Proof: Readily, $C_i, C_j \in H$ implies $C_i \circ C_j \in H \subseteq H.\sigma$.
Conversely, $C_m \in H.\sigma$ imples $C_m = X \circ$ for some minimal $X \subset H$. Hence, $H.\sigma$ is a group. □

It is impossible to create a planar graph consisting of 4 cycles such that there is at least one edge common to each pair. But, cycle systems exist in 3-dimensional space, so one can construct systems with n adjacent independent cycles for all n. For such "complete" systems, the lattice of flats is a complete meet lattice whose greatest element is $\mathcal{C}$, or $\mathcal{M}$, itself.

The idea of a cycle matroid was instigated by considering the cycles in the protein polymers, such as Figure 1, that form membrane proteins. It has been thought that such cyclic structures might constitute a molecular representation of information in non-neural creatures [18]. Whether this conjecture is true, or not, it is clear that this kind of cyclic structure, with its matrix like properties, *could* well be a primitive mechanism for information storage. In any case, they are interesting mathematical objects.

References

[1] Ali A Sobhi Afshar. *Systemic modeling of biomolecular interaction networks*. Dissertation, Johns Hopkins Univ., Oct. 2016.

[2] Geir Agnarsson and Raymond Greenlaw. *Graph Theory: Modeling, Applications and Algorithms*. Prentice Hall, Upper Saddle River, NJ, 2007.

[3] Joseph E. Bonin, James G. Oxley, and Brigitte Servatius, editors. *Matroid Theory*. Contemporary Mathematics, #197. Amer. Math. Soc., Providence, RI, 1995.

[4] Heinrich Brandt. Über eine Verallgemeinerung des gruppenbegriffes. *Math. Ann.*, 96:360–366, 1927.

[5] Vašek Chvátal. Antimatroids, Betweenness, Convexity . In William Clark, László, and Jens Vygen, editors, *Research Trends in Combinatorial Optimization*, pages 57–64. Springer, 2009.

[6] A. H. Clifford and G. B. Preston. *A Algebraic Theory of Semigroups*, volume 1. Amer. Math. Soc., 1961.

[7] Mark S. Granovetter. The Strength of Weak Ties. *Amer. J. of Sociology*, 78(6):1360–1380, 1973.

[8] George Gratzer. *General Lattice Theory*. Academic Press, 1978.

[9] Jonathan L. Gross and Thomas W. Tucker. *Topological Graph Theory*. Dover Publ., Mineola, NY, 1987. (republished 2012).

[10] Frank Harary. *Graph Theory*. Addison-Wesley, 1969.

[11] Robert E. Jamison. Copoints in Antimatroids. *Congressus Numerantium*, 29:535–544, 1980.

[12] Robert E. Jamison and John L. Pfaltz. Closure Spaces that are not Uniquely Generated. *Discrete Appl Math.*, 147:69–79, Feb. 2005. also in Ordinal and Symbolic Data Analysis, OSDA 2000, Brussels, Belgium July 2000.

[13] Bernhard Korte and László Lovász. Intersections of Matroids and Antimatroids. *Discrete Math.*, 73:143–157, 1988/89.

[14] Bernhard Korte and Lzszlo Lovasz. Homomorphisms and Ramsey Properties of Antimatroids. *Discrete Applied Mathematics*, 15(2,3):283–290, Nov. 1986.

[15] David L. Neel and Nancy Ann Neudauer. Matroids You Have Known. *Mathematics Magazine*, 82(1):26–41, Feb. 2009.

[16] Oystein Ore. *Theory of Graphs*, volume XXXVIII of *Colloquium Publ.* Amer. Math. Soc., Providence, RI, 1962.

[17] John L. Pfaltz. Closure Lattices. *Discrete Mathematics*, 154:217–236, 1996.

[18] John L. Pfaltz. Human Have a Distributed, Molecular Long-term Memory. In Tom Mitchell, Leon Iasemidis, and Ning Zhong, editors, *2018 Internat. Conf. on Brain Informatics, BI 2018*, pages 1–12 (B203), Arlington, TX, Dec. 2018.

[19] John L. Pfaltz and John E. Karro. Uniform Antimatroid Closure Spaces. In *SIAM Conf. on Discrete Math.*, Toronto, Canada, July 1998.

[20] D.J.A. Welsh. *Matroid Theory*. Academic Press, 1976.

[21] Neil White. *Combinatorial Geometries*. Cambridge University Press, Cambridge, UK, 1987.

[22] Neil White. *Theory of Matroids*. Cambridge University Press, Cambridge, UK, 1987.

[23] Oscar Zariski and Pierre Samuel. *Commutative Algebra*, volume I. Van Nostrand, 1958.

[24] Guo-Qiang Zhang. Closures in Binary Partial Algebras. *Electronic Notes in theoretical Comp. Sci.*, 257:3–18, 2009. doi:10.1016/j.entcs.2009.11.023.