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The Interior of a Network

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***Abstract*—In his study of social network systems, Granovetter [9], made the distinction between strong and weak ties between individuals. We show that these weak ties constitute a well-defined "interior", $\mathcal{I}$, that is part of every network, $\mathcal{N}$. From $\mathcal{I}$, which is an algebraic matroid, one can rigorously define network properties such as "similarity" or "betweenness" whose implementation is scalable.**

I. Introduction

In his study of social network systems, Granovetter [9], made the distinction between strong and weak ties between individuals. This paper can be regarded as a mathematical study involving the weak ties of network, which we call its "interior", $\mathcal{I}$. It is known that this interior of a network preserves many of the important structural properties, such as shortest paths and betweenness centers.

Figure 1 illustrates a representative, but rather small, network, $\mathcal{N}_\alpha$, which will serve as a running example. In Section II-A we present an algorithmic process that extracts the "interior" of this, or any network, $\mathcal{N}$.

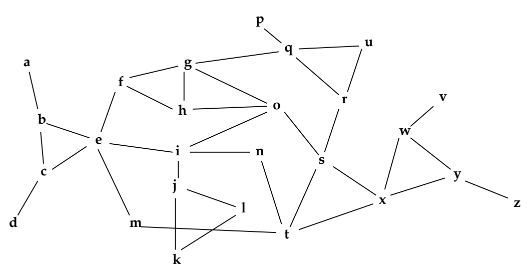

Fig. 1. A small network, $\mathcal{N}_\alpha$, on 26 elements.

This interior will consist of a collection, $\mathcal{I}$, of cycles. It can be shown that $\mathcal{I}$ is a matroid with well defined rank, r [22].

In Section IV, we will develop a concept of network similarity based on each networks interior. Similarity of this form is a true equivalence relation.

Finally, in Section V, we reflect on the impact of this "interior" notion. In particular, we consider the value of growing a series of similar networks given a specific interior, $\mathcal{I}$.

II. Elements, Links and Cycles

Let $\mathcal{R}$ be a symmetric relation on some universe $\mathcal{U}$ of elements. That is, for all $x, y \in \mathcal{U}$, if $x\mathcal{R}y$ then $y\mathcal{R}x$. So, $\mathcal{R}$ can be regarded as simply a collection of subsets $\{x, y\} \in \mathcal{R} \subseteq \mathcal{U}$.

We wish to model the structure of $\mathcal{R}$ on $\mathcal{U}$. Presumably the elements of $\mathcal{U}$ are real objects such as people, atoms, organizations, each with their own structure. We call them **elements** in our model, and designate them with lower case letters, such as x, y, and z. If $x\mathcal{R}y$, or equivalently $\{x, y\} \in \mathcal{R}$, we say a **link** exists between x and y.

Let $\{x, y\} \in \mathcal{R}$ be any link. The elements, x, y are said to be **neighbors**. Let $X \subseteq \mathcal{U}$ be any set of elements. By the **neighborhood** of a element x, we mean the set $\{x\}.\eta = \{y : \exists y \in \mathcal{U}$ where $\{x, y\} \in \mathcal{R}\}$. By natural extension the neighborhood of a set $X \subseteq N$, $X.\eta = \cup_{x \in X}\{x\}.\eta\}$.[1] In Figure 1, $\{b\}.\eta = \{a, b, c, e\}$, $\{x, w, y\}.\eta = \{s, t, v, w, x, y, z\}$.

A. Closure and Interior Operators

An arbitrary operator, φ, is said to be a **closure** operator if φ is: *expansive* ($Y \subseteq Y.\varphi$), *monotone* ($X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$), and *idempotent* ($Y.\varphi.\varphi = Y.\varphi$). On the other hand, if φ is *contractive*, that is $Y.\varphi \subseteq Y$, φ, then φ is said to be an **interior** operator and denoted by ω instead of φ.

Literature regarding closure operators is abundant, $e.g.$ [2], [13], [21]. A familiar example is the convex hull operator of geometry [5]. Contractive operators have been far less studied. Exceptions are [3], [10], [11], for whom any contractive operator is treated as a "choice function" of social theory. If one considers an arbitrary plane figure, then one could choose a minimal circle circumscribing the figure to be its *closure*, and a maximal inscribed circle to be its *interior*.

To develop the following interior operator, we begin by using a **neighborhood closure**, φ_η, on sets, X, of elements, defined $X.\varphi_\eta = \{y | \{y\}.\eta \subseteq X.\eta\}$. It is not difficult to show that φ_η satisfies the three closure axioms. In social networks, a neighborhood can be regarded as the element's "social horizon". If anyone's social horizon, $\{x\}.\eta$, is contained in that of another, $\{y\}.\eta$, that person (or organization) is indeed "closed" and contributes little to understanding the structure of the social community.

Let $\mathcal{N}$ be a network $(\mathcal{U}, \mathcal{R})$, with the neighborhood operator η. Suppose $z \in \{y\}.\varphi_\eta$, implying that $\{z\}.\eta \subseteq \{y\}.\eta$. Readily, the element z does not contribute much to the structure of $\mathcal{N}$; we say z is **reduced into** by y. Consequently, z, together with its incident links, can be removed from $\mathcal{N}$ with little loss of information. This monotone reduction step, call it ω_z,

[1] We denote set valued operators using suffix notation; for scalar valued functions we use more traditional prefix notation.

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is somewhat analogous to grinding off the high spots on a solid body until it is reduced to a perfect sphere.

If ω_z is iterated until there are no elements y such that $z \in \{y\}.\varphi_\eta$ then $\mathcal{N}$ is **irreducible**. We call this irreducible sub-network, I, in which all singleton subsets $\{y\}$ are closed, the **interior** of $\mathcal{N}$. The following computer procedure, *reduce* implements the interior operator, ω.

```
while there exist reducible elements {
   for_each y in N {
      get {y}.nbhd
      for_each z in ({y}.nbhd - {y}) {
         if ({z}.nbhd contained_in {y}.nbhd {
            remove z from network } } } }
```

(Set operators, such as `contained_in` are features of our C++ implementation [14].) It repeatedly sweeps through all elements $y \in \mathcal{U}$, deleting any elements $z_i \in \{y\}.\varphi_\eta$, together with all links incident to z_i, until no such z remain in N. That is, $\omega = \omega_{z_1} \cdot \omega_{z_2} \cdot \ldots \cdot \omega_{z_n}$. Since each ω_{z_i} is monotone ω is as well. The process terminates when every singleton subset $\{y\} \subseteq \mathcal{I}$ is closed. At this point, ω is readily idempotent

In Figure 2, the elements a and d have been deleted because $\{a\}.\eta = \{ab\} \subset \{abce\} = \{b\}.\eta$ and $\{d\}.\eta = \{cd\} \subset \{bcde\} = \{c\}.\eta$. Then on a successive sweep through

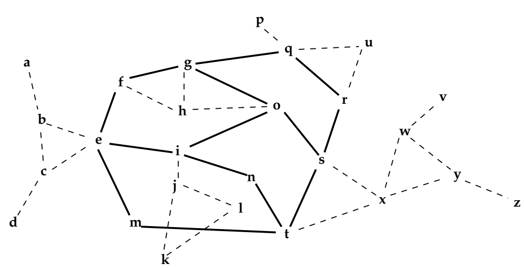

Fig. 2. The interior of Figure 1 shown as solid bold links.

the remaining elements, b and c can be removed because $\{b\}.\eta = \{c\}.\eta \subset \{bcefim\} = \{e\}.\eta$. The other elements (denoted by dashed links) have been similarly removed. The remaining elements and links (denoted by bolder solid lines) constitute the interior of Figure 1. Many correspond to the weak ties identified by Granovetter.

Technically, this interior operator, ω, is of order $O(n^2)$ because we can construct examples where ω removes only one element on each sweep through $\mathcal{U}$. However, it is rather effective in practice, reducing large networks $|\mathcal{U}| \geq 1,000$, with at worst 6 iterative sweeps of $\mathcal{U}$.

Using the betweeness concept, the Girvan-Newman algorithm, [8] extracts much the same internal structure of a network. The major difference is that our reduction process preserves the very links that they would delete on the basis of high "betweenness". Because it requires a breadth-first search from every node in the network, their algorithm must be at least $O(n^2)$.

One can show [16] that this interior operator, ω, as embodied by the pseudo code above, yields a unique (up to isomorphism) sub-network regardless of the order in which the elements $y \in \mathcal{U}$ are visited by ω, or the order in which elements $z \in \{y\}.\varphi_\eta$ are deleted. That is:

Proposition 2.1: Let $\mathcal{I} = \mathcal{N}.\omega$ and $\mathcal{I}' = \mathcal{N}.\omega'$ be two irreducible subsets of a finite network $\mathcal{N}$, then $\mathcal{I} \cong \mathcal{I}'$.

In the proof of this theorem one encounters certain triangles that force isomorphism instead of equality. Triangles seem to abound in social networks; though not necessarily in networks in general [4]. This is a consequence of *triadic closure*[2] in which two individuals x and z who have a common friend y tend to either be friends, or become friends, themselves.

Proposition 2.2 below asserts that even though triangles may exist in $\mathcal{I}$, all elements belong to a cycle of length ≥ 4.

Proposition 2.2: Let $\mathcal{N}$ be a finite network with $\mathcal{I} = \mathcal{N}.\omega$ being its irreducible interior. If $y \in \mathcal{I}$ is not an isolated point then either

(1) there exists a k-cycle C, $k \geq 4$ such that $y \in C$, or

(2) there exist k-cycles C_1, C_2 each of length ≥ 4 with $x \in C_1$ $z \in C_2$ and y lies on a path from x to z.

III. Cycle Systems

We have informally used the idea of cycles above. Now we will be more careful. A **cycle**, C_i is a closed sequence of $n \geq 4$ links, $<\{y_1, y_2\}, \{y_2, y_3\}, \ldots \{y_n, y_1\}>$ where $y_{i-1} \neq y_i$. C_i is said to have **length** n. While the irreducible interior, $\mathcal{I}$, extracted by the process ω (of Section IIa,) can contain triangles, Proposition 2.2 demonstrates that every element y_k is part of at least one cycle of length ≥ 4, so this restriction on the cycles in a cycle system $\mathcal{C}$ is appropriate. Figure 3, consisting of 4 obvious cycles, $C_1 = <abckja>$, $C_2 = <cdmlkc>$, $C_3 = <defgmd>$ and $C_4 = <ghijklmg>$, is an example. This cycle system has been drawn on a planar surface, but

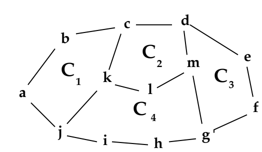

Fig. 3. A small cycle system, $\mathcal{C}$, on 4 basic cycles.

cycle systems exist in, at least, 3-space. In particular, cycles never "bound" a volume; think of a hula hoop.

A cycle is really a sequence of links, as in $\bar{C}_1 = <\{a,b\}, \{b,c\}, \{c,k\}, \{k,j\}, \{j,a\}>$, however we often just enumerate its constituent elements, as $\dot{C}_1 = <abckja>$ above. We use the notation $\bar{C}_i$ to mean the links comprising the cycle C_i and $\dot{C}_i$ to mean its constitutent elements.

We observe that there is a cycle $\dot{C}_5 = <abcdmlkja> = (\bar{C}_1 \cup \bar{C}_2) \sim (\bar{C}_1 \cap \bar{C}_2)$ in $\mathcal{C}_\alpha$, Figure 3. We will say a cycle C_m is the **composition** of C_i and C_k, denoted $C_i \circ C_k$, whenever

$$C_m = C_i \circ C_k = \bar{C}_m = (\bar{C}_i \cup \bar{C}_k) \sim (\bar{C}_i \cap \bar{C}_k) \quad (1)$$

[2]In spite of the the name, "triadic closure" is not a mathematical closure operator. For example, it is not idempotent. Indeed, social networks tend to become ever more triangle "dense" over time.

Besides $C_5 = C_1 \circ C_2$, we have $C_{15} = C_1 \circ C_2 \circ C_3 \circ C_4$ = $(C_1 \circ C_2) \circ C_3 \circ C_4$ = $C_5 \circ C_3 \circ C_4$ = $<abcdefgh\ ija>$.[3] Two cycles C_i and C_k are said to be **adjacent** if $\bar{C}_i \cap \bar{C}_k \neq \emptyset$.

The **empty cycle**, $C_\emptyset$ is precisely that, the empty set, or $\dot{C}_\emptyset = \bar{C}_\emptyset = \emptyset$. Readily, for all C_i, $C_i \circ C_\emptyset = C_i = C_\emptyset \circ C_i$, so $C_\emptyset$ serves as the identity element for $\circ$. Moreover, because $(\bar{C}_k \cup \bar{C}_k) \sim (\bar{C}_k \cap \bar{C}_k) = \emptyset$, for all k, we have $C_k \circ C_k = C_\emptyset$.

Cycle systems have an abelian group structure, and can be shown to be "matroids" ($i.e.$ a kind of vector space lite) with basis sets of cycles satisfying the standard basis exchange axioms.[4] The rank r of a cycle matroid is the cardinality of any maximal, independent set of cycles ($i.e.$ basis). $\mathcal{C}_\alpha$ of Figure 3 has rank, $r = 4$. In fact, it can be shown that all cycle systems, are a form of "algebraic matroid" [18], [19], [20], [22].

Let $\mathcal{C}$ be a cycle system and let $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ be a transformation mapping the cycles of $\mathcal{C}$ onto the cycles of $\mathcal{C}'$. f is said to be a **cycle homomorphism** if $f(C_i \circ C_k) = f(C_i) \circ f(C_k)$. Figure 4 illustrates $\mathcal{C}'$, one homomorphic image of the cycle system $\mathcal{C}$ in Figure 3, that is $\mathcal{C} \xrightarrow{f} \mathcal{C}' = \{C'_1, C'_2, C'_4\}$ where $C_1 \xrightarrow{f} C'_1$, $C_2 \xrightarrow{f} C'_2$, $C_4 \xrightarrow{f} C'_4$, and we can have either

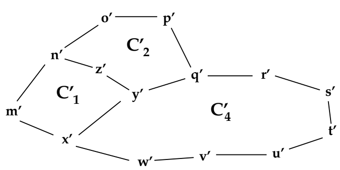

Fig. 4. $\mathcal{C}'$, a homomorphic image of $\mathcal{C}$ in Figure 1.

$C_3 \xrightarrow{f} C'_2$, or $C_3 \xrightarrow{f} C'_4$. Observe that neither elements nor links are involved in cycle homomorphisms, only cycles, their adjacency and composition. This is evident, since C_3, a cycle of length 5, can map to C'_4, of length 10.

If f is one-to-one, it is a cycle **isomorphism**. Readily, the cycle system $\mathcal{C}$ in Figure 3 is isomorphic to the interior, $\mathcal{I}$, of the network of Figure 2. One can regard these as the "same" cycle system.

IV. Network Similarity

Knowledge of a network's interior has a number of applications, such as significantly reducing the cost of locating "betweenness centers" [1], [6], [7]. But, one of its more obvious applications is facilitating a rigorous notion of network "similarity".

Normally in mathematics, "similarity" is an equivalence relation, that is if $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are similar, denoted $\mathcal{N}_\alpha \sim \mathcal{N}_\beta$, then $\mathcal{N}_\beta \sim \mathcal{N}_\alpha$ (symmetry) and $\mathcal{N}_\alpha \sim \mathcal{N}_\beta$ and $\mathcal{N}_\beta \sim \mathcal{N}_\gamma$ implies $\mathcal{N}_\alpha \sim \mathcal{N}_\gamma$ (transitivity).

We will say that two networks $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are **similar**, denoted $\mathcal{N}_\alpha \sim \mathcal{N}_\beta$, if they have the "same" interiors, $i.e.$ $\mathcal{I}_\alpha = \mathcal{N}_\alpha.\omega$ and $\mathcal{I}_\beta = \mathcal{N}_\beta.\omega$ are isomorphic. Readily, this definition of similarity yields a true equivalence relation.

Most other statistical or eigenvalue based "similarity" algorithms [15], [23], [24] yield a coeficient c that is more properly regarded as a measure of "closeness", or a "distance" d between the networks. This distance may, or may not, satisfy the triangle inequality although the author has never seen it discussed.

One network that is similar to $\mathcal{N}_\alpha$ (of Figure 1) is shown as $\mathcal{N}_\beta$ in Figure 5 where only those elements of $\mathcal{I}_\beta$ that might

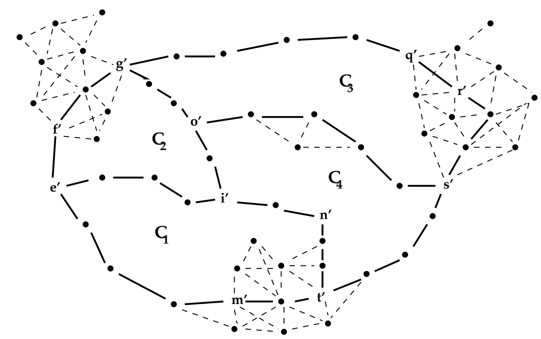

Fig. 5. $\mathcal{N}_\beta$, (with $\mathcal{I}_\beta$) is a network "similar" to $\mathcal{N}_\alpha$.

correspond to similar elements in $\mathcal{I}_\alpha$ have been labeled. Here, it is apparent that most of the interior consists of the weak ties of Granovetter.

But, $\mathcal{N}_\beta$ doesn't look much like $\mathcal{N}_\alpha$. $\mathcal{N}_\beta$ is a larger network with 3 relatively dense clusters of elements that are interconnected by longer paths in $\mathcal{I}_\beta$, and has none of the long, stringy tendrils of Figures 3 and 2. How can these be considered similar?

There actually exist an uncountable number of networks $\mathcal{N}$ that are similar to $\mathcal{N}_\alpha$ because this form of similarity partitions the immense space of all possible networks. Figure 5 is just one of the many such networks. However, it is well known that the conjunction of equivalence relations is itself an equivalence relation. Thus, one may refine this notion of similarity and require that, for $\mathcal{N}_\alpha$ to be similar to $\mathcal{N}_\beta$, they must have isomorphic interiors *and* have the same cardinality; that is, $\mathcal{I}_\alpha \cong \mathcal{I}_\beta$ and $|\mathcal{U}_\alpha| = |\mathcal{U}_\beta|$, or possibly the same density $|\mathcal{R}_\alpha|/|\mathcal{U}_\alpha| = |\mathcal{R}_\beta|/|\mathcal{U}_\beta|$ as well. With these kinds of additional restrictions, one creates a finer partition of the space of all undirected networks, and similar networks will "look" more alike.

Given two irreducible interiors, $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$, we use a standard technique to determine isomorphism. We create two adjacency graphs, $A_\alpha = (V_\alpha, E_\alpha)$ and $A_\beta = (V_\beta, E_\beta)$ whose vertices are cycles C_i with edges $(C_i, C_k) \in E$ whenever C_i is adjacent to C_k, that is $\bar{C}_i \cap \bar{C}_k \neq \emptyset$. There exist a variety of effective procedures for identifying a graph isomorphism, $A_\alpha \xrightarrow{f} A_\beta$, if one exists. Then one need only test if the graph isomorphism is also a cycle isomorphism, that is $C_I \circ C_k = C_m$ implies $f(C_i) \circ f(C_k) = f(C_m)$.

[3] In all, there are 15 non-empty cycles in Figure 3.

[4] These cycle matroids should not be confused with the "cycle matroid" that is often used as an introduction to matroid theory [12] In those elementary graph based matroids the elements are edges (not cycles) with basis sets being any spanning tree.

A. Weak Similarity

Two networks $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are said to be **weakly similar** if $\mathcal{I}_\alpha \xrightarrow{f} \mathcal{I}_\beta$ is a homomorphism. Weak similarity is transitive, that is if $\mathcal{I}_\alpha \xrightarrow{f} \mathcal{I}_\beta$ and $\mathcal{I}_\beta \xrightarrow{g} \mathcal{I}_\gamma$ then $\mathcal{I}_\alpha \xrightarrow{h} \mathcal{I}_\gamma$; but it is not symmetric. We need not have $\mathcal{I}_\gamma \xrightarrow{h} \mathcal{I}_\alpha$ or $\mathcal{I}_\beta \xrightarrow{f} \mathcal{I}_\alpha$.

One can also extend a form of interior based similarity to all pairs, $\mathcal{N}_\alpha, \mathcal{N}_\beta$, of networks. Let $\mathcal{N}_\gamma$ be maximal network such that $\mathcal{I}_\alpha \xrightarrow{f} \mathcal{I}_\gamma$ and $\mathcal{I}_\beta \xrightarrow{g} \mathcal{I}_\gamma$ be cycle homomorphisms. Then $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ can be said to be similar through the greatest common network $\mathcal{N}_\gamma$.

We know of no possible way of implementing weak similarity in practice. Moreover, it is not clear what information would be conveyed if one did know of a greatest common interior. Just because one can define an operation does not mean it can be successfully implemented. The idea has been introduced only for a sense of theoretical "completeness".

V. Discussion

This paper takes a strongly mathematical approach to understanding the weak, that is global, structure of networks. In certain social networks, such as those modeling international relationships this can be most important. While it ignores local neighborhood structure, it can be used to partition large networks into dense neighborhoods in much the same way as the Girvan-Newman procedure. One need only keep track of elements that have been reduced into the remaining element in $\mathcal{I}$, as described in [17].

Does any concept of network similarity make sense? Suppose one defines it in terms of a single value, as in $sim(\mathcal{N}_\alpha, \mathcal{N}_\beta) = 0.683$. What information does this really convey? Is the concept of similarity described in this paper any more valuable? Most pairs of networks are simply "not similar". Yet, if they are similar, the isomorphism $\mathcal{I}_\alpha \xrightarrow{f} \mathcal{I}_\beta$ conveys a great deal of information.

We believe that asking if two networks $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are "similar" is often just an academic exercise. More commonly, we are given a particular network, $\mathcal{N}$, and we seek to understand its structure by generating "similar" networks that have been expanded from its interior, $\mathcal{I}$. Figure 5 is an extreme example. In practice, the generating can be much more targeted. It can force the same number of elements; the same density $|\mathcal{R}|/|\mathcal{U}|$; the same clustering of elements at corresponding locations in the interior. See for example [17]. This opens up the possibility of many "what if" scenarios which change these parameters, or even change the cyclic structure of the interior, $\mathcal{I}$.

Is it actually practical to use this approach with the very large networks one encounters in practice? As noted in Section II-A, the process that reduces a network $\mathcal{N}$ to its irreducible interior $\mathcal{I}$ is theoretically $O(n^2)$, but in practice it is no worse that linear $O(n)$. Similarly, in the test for similarity of Section IV, it is known that determining graph isomorphism is NP in theory. But, again in practice, procedures using heuristic filters can almost always reduce the problem to at worst a linear one. Testing whether a proposed isometry actually is one, is again at worst linear, or $O(n)$. So, using appropriate software, such as described in [14], operations based on the cycle structure of $\mathcal{I}$ are scalable to very large networks.

The interior of a network is a rigorous mathematical object. It is, at least, an algebraic matroid created by a well-defined interior operator. It has the potential for rigorous social network analysis which to this author's knowledge, has yet to be explored.

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