Evaluating the binary partition function when $N = 2^n$ *

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Abstract

We present a linear algorithm to count the number of binary partitions of 2^n. It is also shown how such binary partitions are related to closure spaces on n elements, thereby giving a lower bound on their enumeration as well.

1 Background

A *binary partition* of the integer N is a sequence of non-negative integers $< a_n, \cdots, a_0 >$, such that

$$a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0 = N. \tag{1}$$

The number of such sequences, denoted *b(N)*, is called the *binary partition function.* Both the function and its evaluation have been well investigated. It is described in Sloane's Handbook, [13]. A short history of the binary partition function can be found in [1], in which Churchhouse describes his calculation of *b(N)* on an early Atlas computer. Our method of evaluation improves on his only because we restrict ourselves to the special case in which $N = 2^n$. Consequently, we must first address the issue: "why consider such a special case?".

The concept of uniquely generated closure spaces has begun to be studied as a common thread emerging in computer applications, in graphs, and in discrete geometries. Briefly, a closure operator φ is said to be *uniquely generated* if in addition to the customary closure axioms[1]

$$X \subseteq X.\varphi$$

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[1]We will denote closure operators using a suffix notation.

$$\begin{aligned} X &\subseteq Y \text{ implies } X.\varphi \subseteq Y.\varphi \\ X.\varphi.\varphi &= X.\varphi^2 = X.\varphi \end{aligned} \tag{2}$$

we add a fourth which distinguishes this closure concept from more familiar topological closure,

$$X.\varphi = Y.\varphi \text{ implies } (X \cap Y).\varphi = X.\varphi = Y.\varphi \tag{3}$$

Closure operators satisfying (3) above are uniquely generated in the sense that for any set Z, there exists a unique minimal set $X \subseteq Z$, called its *generator*[2] and denoted $Z.gen$, such that $X.\varphi = Z.\varphi$. Such a closure operator acting on a set, or universe, of elements, $\mathbf{U}$, is said to be a *closure space* $(\mathbf{U}, \varphi)$, as in [7]. Readily, a subset X will be *closed* if $X.\varphi = X$.[3] The importance of uniquely generated closure spaces lies in the fact that in discrete systems they play a role that is in many respects analogous to the vector spaces of classical mathematics. We establish this parallel in the next paragraph.

A closure operator σ, satisfying the three closure axioms of (2), together with the Steinitz-MacLane *exchange* property

$$\text{if } y \notin X.\sigma \text{ then } y \in (X \cup \{x\}).\sigma \text{ implies } x \in (X \cup \{y\}).\sigma \tag{4}$$

can be shown to be the closure operator of a matroid, $\mathcal{M}$ [14]. Similarly, a closure φ satisfying the three closure axioms and the *anti-exchange* property

$$\text{if } x, y \notin X.\varphi \text{ then } y \in (X \cup \{x\}).\varphi \text{ implies } x \notin (X \cup \{y\}).\varphi \tag{5}$$

is the closure operator of an anti-matroid, $\mathcal{A}$ [3]. It can be shown [8] [12] that a closure operator is uniquely generated if and only if it satisfies the anti-exchange property (5). A matroid, $\mathcal{M}$, is a set system that generalizes the independent sets of a linear algebra. The closure of these sets, commonly called its *spanning* operator, is a *vector space.* Uniquely generated closure spaces, therefore, are the analogs of vector spaces, but with respect to anti-matroids. From now on, we will simply call them *closure spaces.*

Closure operators are fairly common, although they frequently have other names, for example "convexity". The convex hull of a discrete set is an uniquely generated closure. A theory of convex geometries is developed in [5]. Convexity in graphs has been examined in [11] [6]. The "lower ideals", or "down sets" of a partially ordered set are closed. In concurrent computing, the concept of a "transaction" is a simple closure operator. Algorithmic closure, in particular that of greedy algorithms is found in [9], which introduces the term "greedoid", a secial kind of anti-matroid.

[2] Readily, if X_1 and X_2 were distinct minimal generators of $Z.\varphi$, then because $X_1.\varphi = X_2.\varphi = Z.\varphi$, we must have, by (3), $(X_1 \cap X_2).\varphi = Z.\varphi$ contradicting minimality.

[3] The family $\mathcal{C}$ of closed sets is closed under intersection, and this characterization is equivalent to (2), *c.f.* [4].

The subsets of a closure space can be partially ordered to create a lattice [12], with many interesting properties. Of most importance is the observation that for any set $Z \subseteq \mathbf{U}$ the cardinality of $\{ X \mid X.\varphi = Z.\varphi \}$ must be a power of 2. Thus any uniquely generated closure operator φ partitions the subsets of $\mathbf{U}$ into a disjoint collection of subsets, each containing a single closed set and each consisting of 2^k subsets. Let a_k denote the number of collections with 2^k subsets. The sequence $< a^n, a^{n-1}, a^{n-2}, \cdots, a^2, a^1, a^0 >= 2^n$ is thus a compact description of a closure space $(\mathbf{U}, \varphi)$, where $|\mathbf{U}| = n$. Moreover, it is shown in [12] that for every such binary partition of 2^n there exists at least one closure space with that property. Consequently, the enumeration of binary partitions of 2^n becomes a lower bound on the enumeration of closure spaces over n elements.

2 Counting Partitions

Let $\mathbf{P}^n$ denote the set $\{\pi_i =< a_n, \cdots, a_k, \cdots a_0 >\}$ of all binary partitions of 2^n. Several characteristics of $\mathbf{P}^n$ are readily apparent. First, $a_n \neq 0$ if and only if $a_k = 0$ for all $0 \leq k < n$. Second, since the right hand side is even and all terms $a_k \cdot 2^k$, $k > 0$ must be even, the coefficient a_0 must be even. Third, if $< \cdots, a_k, a_{k-1}, \cdots >$ is a partition of $\mathbf{P}^n$, then $< \cdots, a_k - 1, a_{k-1} + 2, \cdots >$ must be as well. And fourth, if $< a_n, \cdots, a_k, \cdots, a_o >$ is a partition in $\mathbf{P}^n$ then $< a_n, \cdots, a_k, \cdots, a_0, 0 >$ is a partition in $\mathbf{P}^{n+1}$.

With these observations, it is not difficult to write a process which generates all partitions in lexicographic order. Doing so, and displaying each partition, generates the following enumerations of $\mathbf{P}^3$ and $\mathbf{P}^4$. It is quite easy to verify by inspection that each sequence is a

n = 3

1	0	0	0
0	2	0	0
0	1	2	0
0	1	1	2
0	1	0	4
0	0	4	0
0	0	3	2
0	0	2	4
0	0	1	6
0	0	0	8

n = 4

1	0	0	0	0	0	0	2	1	6
0	2	0	0	0	0	0	2	0	8
0	1	2	0	0	0	0	1	6	0
0	1	1	2	0	0	0	1	5	2
0	1	1	1	2	0	0	1	4	4
0	1	1	0	4	0	0	1	3	6
0	1	0	4	0	0	0	1	2	8
0	1	0	3	2	0	0	1	1	10
0	1	0	2	4	0	0	1	0	12
0	1	0	1	6	0	0	0	8	0
0	1	0	0	8	0	0	0	7	2
0	0	4	0	0	0	0	0	6	4
0	0	3	2	0	0	0	0	5	6
0	0	3	1	2	0	0	0	4	8
0	0	3	0	4	0	0	0	3	10
0	0	2	4	0	0	0	0	2	12
0	0	2	3	2	0	0	0	1	14
0	0	2	2	4	0	0	0	0	16

Figure 1: $\mathbf{P}^3$ and $\mathbf{P}^4$

partition of 2^n. And because they are in lexicographic order, one can verify that all possible partitions have been generated.

Because $< a_{n-1}, \cdots, a_0 > \in \mathbf{P}^{n-1}$ implies $< a_{n-1}, \cdots, a_0, 0 > \in \mathbf{P}^n$, it follows that

$$b(2^n) = b(2^{n-1}) + p_n \tag{6}$$

where p_n denotes the number of partitions $\pi_i \in \mathbf{P}^n$ in which $a_0 \neq 0$. We say such partitions are *normal* because they correspond to closure spaces in which the empty set is closed.

In the lexicographic order of $\mathbf{P}^n$, if $\pi_i^n = < a_n, \cdots, a_2, a_1, 0 > \in \mathbf{P}^n, a_1 \neq 0$, then there must follow the sequence $S_{a_1}^n$ of partitions, $< a_n, \cdots, a_2, a_1 - 1, 2 >$, $< a_n, \cdots, a_2, a_1 - 2, 4 >$, $\cdots$, $< a_n, \cdots, a_2, 0, 2a_1 >$. There are two such sequences in $\mathbf{P}^3$; $< 0, 1, 2, 0 >$ followed by $< 0, 1, 1, 2 >$ and $< 0, 1, 0, 4 >$, and $< 0, 0, 4, 0 >$ followed by $< 0, 0, 3, 2 >$, $< 0, 0, 2, 4 >$, $< 0, 0, 1, 6 >$, and $< 0, 0, 0, 8 >$. In $\mathbf{P}^4$ there are 6 such subsequences because there are 6 normal partitions in $\mathbf{P}^3$; the last consists of 8 normal partitions following $< 0, 0, 0, 8, 0 >$. Once this pattern is perceived the counting process becomes evident. In Figure 2 we reinforce this pattern by showing just the first 8 and the last 34 (of 202) partitions in $\mathbf{P}^5$.

						n = 5						
1	0	0	0	0	0		0	1	1	1	2	0
0	2	0	0	0	0		0	1	1	1	1	2
0	1	2	0	0	0		0	1	1	1	0	4
0	1	1	2	0	0		0	1	1	0	4	0
		⋮								⋮		
0	0	0	2	1	22		0	0	0	0	16	0
0	0	0	2	0	24		0	0	0	0	15	2
0	0	0	1	14	0		0	0	0	0	14	4
0	0	0	1	13	2		0	0	0	0	13	6
0	0	0	1	12	4		0	0	0	0	12	8
0	0	0	1	11	6		0	0	0	0	11	10
0	0	0	1	10	8		0	0	0	0	10	12
0	0	0	1	9	10		0	0	0	0	9	14
0	0	0	1	8	12		0	0	0	0	8	16
0	0	0	1	7	14		0	0	0	0	7	18
0	0	0	1	6	16		0	0	0	0	6	20
0	0	0	1	5	18		0	0	0	0	5	22
0	0	0	1	4	20		0	0	0	0	4	24
0	0	0	1	3	22		0	0	0	0	3	26
0	0	0	1	2	24		0	0	0	0	2	28
0	0	0	1	1	26		0	0	0	0	1	30
0	0	0	1	0	28		0	0	0	0	0	32

Figure 2: First 8 and last 34 partitions of $\mathbf{P}^5$

Notice that these subsequences of normal partitions (with $a_0 \neq 0$) were generated by the three normal partitions $< 0, 1, 1, 1, 2 >$, $< 0, 0, 0, 1, 14 >$, and $< 0, 0, 0, 0, 16 >$ of $\mathbf{P}^4$.

The length of a sequence $S_{a_1}^n$ is a_1. Hence, each normal partition $\pi_i^{n-1} \in \mathbf{P}^{n-1}$ gives rise to a subsequence of $a_1^n = a_0^{n-1}$ normal partitions in $\mathbf{P}^n$. If one carefully keeps track of all normal permutations in $\mathbf{P}^{n-1}$, then one can use the mechanism above to generate all

```
n =  1      2      3       4      5

                           2   ⟹2 4
                    2  ⟹   4   ⟹2 4 6 8
                           2   ⟹2 4
                           4   ⟹2 4 6 8
                           6   ⟹2 4 6 8 10 12
             2  ⟹4  ⟹      8   ⟹2 4 6 8 10 12 14 16
                           2   ⟹2 4
                    2  ⟹   4   ⟹2 4 6 8
                           2   ⟹2 4
                           4   ⟹2 4 6 8
                           6   ⟹2 4 6 8 10 12
                    4  ⟹   8   ⟹2 4 6 8 10 12 14 16
                           2   ⟹2 4
                           4   ⟹2 4 6 8
                           6   ⟹2 4 6 8 10 12
                           8   ⟹2 4 6 8 10 12 14 16
                          10   ⟹2 4 6 8 10 12 14 16 18 20
                    6  ⟹12     ⟹2 4 6 8 10 12 14 16 18 20 22 24
                           2   ⟹2 4
                           4   ⟹2 4 6 8
                           6   ⟹2 4 6 8 10 12
                           8   ⟹2 4 6 8 10 12 14 16
                          10   ⟹2 4 6 8 10 12 14 16 18 20
                          12   ⟹2 4 6 8 10 12 14 16 18 20 22 24
                          14   ⟹2 4 6 8 10 12 14 16 18 20 22 24 26 28
     2  ⟹4  ⟹8  ⟹16            ⟹2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
```

Figure 3: a_0 coefficient in sequences S_k^n of normal partitions

normal partitions in $\mathbf{P}^n$. This is illustrated in Figure 3 in which subsequences S_k^n of normal partitions are enumerated (by showing only the a_0 value) in vertical columns for $n = 1$ through 4, and horizontally (to conserve space) for $n = 5$. For $n = 1$ through 4, each entry a_0^n in S_i^n denotes to its right (with $\Rightarrow$) the *last* entry $< 2a_1, 0, \cdots, a_{n+1} >$ in the sequence $S_{a_0}^{n+1}$ that it generates.

Observe in this figure, that when $n = 3$, all 6 partitions with $a_0 \neq 0$ are enumerated in just two subsequences S_2^3 and S_4^3, which were generated by the two normal partitions in $\mathbf{P}^2$. With $n = 4$ the 26 normal partitions of $\mathbf{P}^4$ are enumerated in two occurrences of the subsequences S_2^4 and S_4^4, together with single occurrences of S_6^4 and S_8^4, which themselves were generated from the 6 normal partitions of $\mathbf{P}^3$. Fortunately, since all sequences S_k^n have the form 2, 4, $\cdots, k$, we need only keep track of the number of such sequences in $\mathbf{P}^n$, not their actual composition.

Let σ_k^n, k even, denote the *number* of subsequences S_k^n of normal partitions in $\mathbf{P}^n$. Based on Figure 3 we can construct Table 1.

Since every normal partition of $\mathbf{P}^n$ belongs to such a subsequence, we have

$$p_n = \sum_{even\ k}^{2^{n-1}} k \cdot \sigma_k^n \tag{7}$$

n	2	3	4	5	6
p_n	2	6	26	166	1,626
k			σ_k^n		
2	1	1	2	6	26
4		1	2	6	26
6			1	4	20
8			1	4	20
10				2	14
12				2	14
14				1	10
16				1	10
18					6
20					6
22					4
24					4
26					2
28					2
30					1
32					1

Table 1: Counts σ_k^n of subsequences S_k^n of normal partitions in $\mathbf{P}^n$

Using Table 1 and equation (7) one obtains $p_7 = 25,510$, and by (6) $b(2^6) = 1,828$, so $b(2^7) = b(2^6) + p_7 = 27,338$. It only remains to determine $\sigma_k^{n+1}, 2 \leq k \leq 2^n$ given $\sigma_j^n, 2 \leq j \leq 2^{n-1}$.

Since each sequence S_k^{n-1} of normal partitions in $\mathbf{P}^{n-1}$ generates the subsequences $S_2^n, S_4^n, \cdots, S_{2k}^n$ in $\mathbf{P}^n$, one can simply loop over all such subsequences σ_k^{n-1} and increment $\sigma_2^n, \cdots \sigma_{2k}^n$ as in the following code section

```
max_k = 2**(n-1);
for (k=2; k<=max_k; k+=2)
        {
        for (j=2; j<=2*k; j+=2)
                sigma[n][j] += sigma[n-1][k];
        }
```

The $O(k^2)$ behavior of this double loop can become expensive when $k = 2^{n-1}$ becomes large. We observe in Table 1, that the first two values of σ_k^n are determined by

$$\sigma_2^n = \sigma_4^n = p_{n-2} \tag{8}$$

and that subsequent values of σ_k^n can be calculated as

$$\sigma_k^n = \sigma_{k+2}^n = \sigma_{k-2}^n - \sigma_{\lfloor (k+2)/2 \rfloor - 2}^{n-1} \tag{9}$$

for $k = 6, 10, 14, \cdots$.

Putting together (6), (7), (8), and (9) one obtains

Theorem 2.1 *The number, p_n, of distinct partitions of 2^n is given by:*

$$p_n = p_{n-1} + \sum_{even k}^{2^{n-1}} k \cdot \sigma_k^n$$

where $\sigma_k^n = \begin{cases} \sum_{even\ i} \ k \cdot \sigma_i^{n-2} & : \ k = 2, 4 \\ \sigma_{k-2}^n - \sigma_{\lfloor (k+2)/2 \rfloor - 2}^{n-1} & : \ k = (6, 8), (10, 12) \cdots \end{cases}$

The primary advantage of expressing p_n in this manner is that it permits the following counting procedure, which although somewhat more complex, has linear behavior.

```
long    sigma[MAX_N+1][POWER_MAX_N];

long    calculate_p (int n)
        /*
        ** Assumes sigma[n-1, 2**(n-2)] has been previously determined
        ** and globally stored.
        ** This procedure sets up sigma[n, 2**(n-1)], and returns
        ** the number p[n] of normal partitions with a[0] != 0
        */
        {
        int     k, k_calc, max_k;
        long    sum;

        max_k = 2**(n-1);
        switch (n)
          {
          case 1:
                return 1;
          case 2:
                sigma[2][2] = 1;
                break;
          case 3:
                sigma[3][2] = 1;
                sigma[3][4] = 1;
                break;
          default:
                sigma[n][2] = y[n-2];
                sigma[n][4] = y[n-2];

                for (k=6; k<=max_k; k+=4)
                        {
                        k_calc = (k+2)/2 - 2;
                        sigma[n][k]   = sigma[n][k-2] - sigma[n-1][k_calc];
                        sigma[n][k+2] = sigma[n][k-2] - sigma[n-1][k_calc];
```

```
                }
            break;
    }
sum = 0;
for (k=2; k<=max_k; k += 2)
        {
        sum = sum + sigma[n][k]*k;
        }
p[n] = sum;
return sum;
}
```

With this code one can generate the following Table 2 of partitions of 2^n. The values of

n	$b(2^n)$	p_n
3	10	6
4	36	26
5	202	166
6	1,828	1,626
7	27,338	25,510
8	692,004	664,666
9	30,251,722	29,559,718
10	2,320,518,948	2,290,267,226

Table 2: Total $b(2^n)$ and normal p_n partitions of 2^n

$b(2^7)$ and $b(2^8)$ can be verified by enumerating all partitions, using the program of section 1, or by reference to [13].

Readily, $b(2^n)$ must be even because, as observed, a_0 must be even, so every subsequence of normal partitions is even. It is not hard to show that $|\mathbf{P}^n|$ grows super exponentially with respect to n. Based on the expression $log\ b(n) \sim (log\ n)^2/2$ found in [10], Churchhouse [2] gives the asymptotic upper bound $b(n) \sim O(n^{1/2 \cdot log_2 n})$ or

$$b(2^n) = |\mathbf{P}^n| \sim O((2^n)^{n/2}). \tag{10}$$

The nature of this super exponential growth is difficult to intuitively comprehend because, unfortunately, equation (10) is a poor approximation for small values of n. In Table 3, we compare $b(2^n)$ with two lower bounding functions, n^n and $(2^n)^{n/3}$, and the upper bound $(2^n)^{n/2}$ to which it is eventually asymptotic. Besides giving some concrete feeling for the growth of the binary partition function, this table illustrates that a wealth of closure spaces exist for even small n.

n	n^n	$(2^n)^{n/3}$	$b(2^n)$	$(2^n)^{n/2}$
2	$4.000\ 10^0$	$2.519\ 10^0$	$4.000\ 10^0$	$4.000\ 10^0$
3	$2.700\ 10^1$	$8.000\ 10^0$	$1.000\ 10^1$	$2.262\ 10^1$
4	$2.560\ 10^2$	$4.031\ 10^1$	$3.600\ 10^1$	$2.560\ 10^2$
5	$3.125\ 10^3$	$3.225\ 10^2$	$2.020\ 10^2$	$5.792\ 10^3$
6	$4.665\ 10^4$	$4.096\ 10^3$	$1.828\ 10^3$	$2.621\ 10^5$
7	$8.235\ 10^5$	$8.257\ 10^4$	$2.733\ 10^4$	$2.372\ 10^7$
8	$1.677\ 10^7$	$2.642\ 10^6$	$6.920\ 10^5$	$4.294\ 10^9$
9	$3.874\ 10^8$	$1.342\ 10^8$	$3.025\ 10^7$	$1.554\ 10^{12}$
10	$1.001\ 10^{10}$	$1.082\ 10^{10}$	$2.320\ 10^9$	$1.125\ 10^{15}$
11	$2.853\ 10^{11}$	$1.385\ 10^{12}$	$3.163\ 10^{11}$	$1.630\ 10^{18}$
12	$8.916\ 10^{12}$	$2.814\ 10^{14}$	$7.747\ 10^{13}$	$4.722\ 10^{21}$
13	$3.088\ 10^{14}$	$9.078\ 10^{16}$	$3.439\ 10^{16}$	$2.735\ 10^{25}$
14	$1.111\ 10^{16}$	$4.648\ 10^{19}$	$2.789\ 10^{19}$	$3.169\ 10^{29}$
15	$4.379\ 10^{17}$	$3.777\ 10^{22}$	$4.160\ 10^{22}$	$7.343\ 10^{33}$
16	$1.844\ 10^{19}$	$4.874\ 10^{25}$	$4.874\ 10^{26}$	$3.402\ 10^{38}$
17	$8.272\ 10^{20}$	$9.982\ 10^{28}$	$5.888\ 10^{29}$	$3.153\ 10^{43}$

Table 3: $b(2^n)$ compared with upper and lower bounding functions

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