Distance in Anti-Matroids*

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May 9, 1997

Abstract

The concept of distance in matroids and geometric lattices is a familiar one. This paper examines the problem of defining a metric over anti-matroids and their corresponding lower semi-modular lattices and some ensuing consequences. One of the most intriguing results is that distance need not be "local" in an anti-matriod. The distance between two points may be affected by changes that are far removed from either point.

1 Anti-Matroids

An **anti-matroid** consists of a set of elements, points, or other phenomena which we generically call its *universe*, denoted by $\mathbf{U}$. Individual elements of $\mathbf{U}$ are denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. By $2^{\mathbf{U}}$, we mean the powerset on $\mathbf{U}$, or collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters: X, Y, Z. There must also be a closure operator, φ, that satisfies the usual three closure axioms:

$X \subseteq X.\varphi$
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$
$X.\varphi.\varphi = X.\varphi^2 = X.\varphi$

To these we postulate an anti-exchange axiom

If $p, q \notin X.\varphi$ then $p \in (X \cup \{q\}).\varphi$ implies $q \notin (X \cup \{p\}).\varphi$.

*Research supported in part by DOE grant DE-FG05-95ER25254.

Closure systems satisfying an exchange axiom are called matroids in general, or vector spaces when there is an underlying field. *C.f.* [26] [17] [25]. Anti-matroids which satisfy the anti-exchange axiom have received less attention, *c.f.* [7] [15] [11]. And frequently they have been studied under different names, such as *alignments* [10], *convex geometries* [9], or *closure spaces* [21]. The latter name emphasizes the central role of the closure operator in anti-matroids. It is not difficult to show that the anti-exchange axiom is equivalent to the property

$$X.\varphi = Y.\varphi \quad \text{implies} \quad (X \cap Y).\varphi = X.\varphi = Y.\varphi \tag{1}$$

which has been called the *unique generation property* [21].

Anti-matroids are far more abundant than one might expect. For example, there exist at least 202 distinct anti-matroids comprised of 5 elements. More generally, for an n element universe $\mathbf{U}$, $n \geq 10$, it can be shown that there exist more than n^n distinct, non-isomorphic anti-matroids over $\mathbf{U}$ [20].

By the **generator** (or basis) of X, denoted $X.\beta$, we mean a minimal set Y such that $Y.\varphi = X.\varphi$. And, because φ is uniquely generated (1), this set is unique. Another important property of uniquely generated closure operators [9] [21] is

Lemma 1.1 *Let X be closed. $X - \{p\}$ is closed if and only if $p \in X.\beta$.*

The subsets of an anti-matroid $(\mathbf{U}, \varphi)$ can be partially ordered by $\leq_\varphi$ where,

$$X \leq_\varphi Y \qquad \text{if and only if} \qquad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi \tag{2}$$

This is a partial order on *all* the subsets of $\mathbf{U}$, not just its closed subsets. It is possible to show that this partial ordering of $2^{\mathbf{U}}$ is, in fact, a well structured lattice, $\mathcal{L}$, called the **closure lattice** of $\mathbf{U}$ [21]. Figure 1 illustrates a representative closure lattice.

The regularity of structure suggested by this figure really exists, *c.f.* [21]. The collection of **closed** subsets, for which $X = X.\varphi$, forms a lower semi-modular sublattice $\overline{\mathcal{L}} = [\emptyset, abcde]$, denoted in this figure by bolder characters and solid lines denoting the covering relationships. The generators, c, d, bc, cd and e, are connected to the corresponding closed sets that they generate by dashed lines. It can be shown that each of the lattice intervals $[X.\varphi, X.gen]$ is a boolean lattice. In the case of the 16 subsets comprising $[abcde, e]$ we just suggest them with few representative elements and a dashed outline. The dotted lines denote covering relationships between elements in different boolean intervals. As expressed in the following theorem, these covering relationships do indeed echo those of the closed subgraph sublattice. The following theorem, from [21], has been called the "Fundamental Structure Theorem".

Theorem 1.2 *Let $X.\varphi \leq_\varphi Y.\varphi$ and let $X \in [X.\varphi, X.\beta]$. There exists a unique $Y \in [Y.\varphi, Y.\beta]$ such that $X \leq_\varphi Y$, where Y is minimal wrt. $\leq_\varphi$ (maximal wrt. $\subseteq$). Moreover $Y = X \cup \Delta$ where $\Delta = Y.\varphi - X.\varphi$ and $Y = Y.\varphi - \delta$ where $\delta = X.\varphi - X$.*

A classical way of regarding the properties of distance and separation is in terms of the

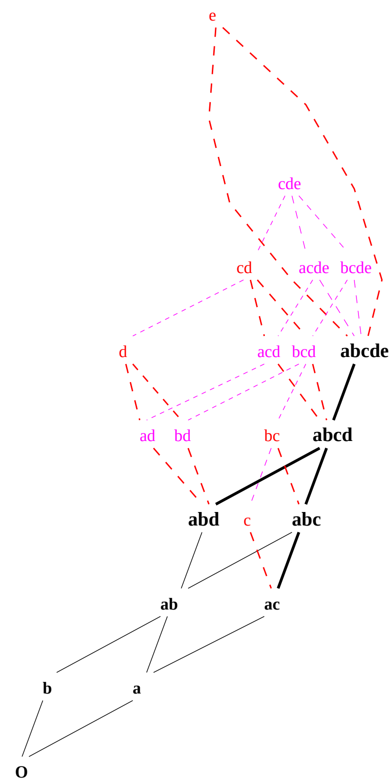


Figure 1: A Closure Lattice, $\mathcal{L}$

topology of the space and its open sets. With a discrete space, $\mathbf{U}$, we use closure spaces in much the same way, as the underlying structure by which to develop a distance concept δ.

2 Distance

By the **upper set** over X, denoted $X.\uparrow$, we mean the set of all closed sets W such that $X \subseteq W$. By definition, this collection $\uparrow$ of subsets has a minimal element; it is $X.\varphi$. In Figure 1, $\{b\}.\uparrow = \{b, ab, abc, abd, abcd, abcde\}$ and $\{c\}.\uparrow = \{ac, abc, abcd, abcde\}$. One may think of $X.\uparrow$ as being the set of all closed sets of $\mathbf{U}$ in which X participates.

If one wants to define a metric on an anti-matroid $(\mathbf{U}, \varphi)$ it should be with respect to the closure operator, φ. By the **distance**, $\delta(X, Z)$, between two sets X and Z we mean the relative number of closed sets that separate X and Z; that contain one or the other, but not both. More specifically, we will let

$$\delta(X,Z) = |X.\varphi.\uparrow| + |Z.\varphi.\uparrow| - 2 \cdot |(X.\varphi \vee Z.\varphi).\uparrow| \tag{3}$$

We should observe that δ is really defined only on the lower semi-modular sublattice of closed subsets, because all subsets with the same closure are considered to be equivalent, hence of distance zero from each other.

Lemma 2.1 $(X \cup Z).\varphi = X.\varphi \vee Z.\varphi$

Proof: $X.\varphi \subseteq (X \cup Z).\varphi$ so $X.\varphi \leq_\varphi (X \cup Z).\varphi$ because the sublattice of closed sets is partially ordered by inclusion. Similarly, $Z.\varphi \leq_\varphi (X \cup Z).\varphi$. Thus $X.\varphi \vee Z.\varphi \leq_\varphi (X \cup Z).\varphi$.
Let W denote $X.\varphi \vee Z.\varphi$. By Theorem 1.2, we know there exists X' in the lattice interval $[W, W.\beta]$ such that $X \subseteq X'$. Similarly, there exists $Z', Z \subseteq Z'$ in this same interval. Because the interval is a boolean algebra, $X' \cup Z' \in [Z, Z.\beta]$. Moreover, $X \cup Z \subseteq X' \cup Z'$ so $(X \cup Z).\varphi \leq_\varphi X.\varphi \vee Z.\varphi$. $\Box$

Now, we may replace the $\vee$ operator of the last term with an easier to manage $\cup$ operator, so that (3) becomes

$$\delta(X,Z) = |X.\varphi.\uparrow| + |Z.\varphi.\uparrow| - 2 \cdot |(X \cup Z).\varphi.\uparrow| \tag{4}$$

A non-negative symmetric function δ satisfying the triangle inequality is called a *metric* provided $\delta(x,y) = 0$ iff $x = y$. Weaker distance functions such as ours are called *pseudo-metrics*. Kelley has observed that "the closure of a set X in a pseudo-metric space must be the set of all points which are of zero distance from X" ([14] p.120).

Lemma 2.2 *The integer function, $\delta(X,Z)$ defined by*

$$\delta(X,Z) = |X.\varphi.\uparrow| + |Z.\varphi.\uparrow| - 2 \cdot |(X \cup Z).\varphi.\uparrow|$$

is a pseudometric over $(\mathbf{U}, \varphi)$.

Proof: Readily, $\delta(X,Z) = \delta(Z,X)$, and if $X.\varphi = Z.\varphi$ then $\delta(X,Z) = 0$; so in particular $\delta(X,X) = 0$. Consequently, it remains only to prove the triangle inequality; that $\delta(X,Z) \leq \delta(X,Y) + \delta(Y,Z), \forall X,Y,Z$. To simplify our expressions we may assume that X, Y, Z are all closed, and use the form $\delta(X,Z) = |X.\varphi.\uparrow| + |Z.\varphi.\uparrow| - 2 \cdot |(X \cup Z).\varphi.\uparrow|$. So we need show that
$|X.\uparrow| + |Z.\uparrow| - 2 \cdot |(X \cup Z).\varphi.\uparrow| \leq |X.\uparrow| + |Y.\uparrow| - 2 \cdot |(X \cup Y).\varphi.\uparrow| + |Y.\uparrow| + |Z.\uparrow| - 2 \cdot |(Y \cup Z).\varphi.\uparrow|$
or, after collecting and cancelling terms, $|Y.\uparrow| - |(X \cup Y).\varphi.\uparrow| \geq |(Y \cup Z).\varphi.\uparrow| - |(X \cup Z).\varphi.\uparrow|$.
The left-hand expression counts those closed sets W such that $Y \subseteq W$, but $X \not\subseteq W$; while the right expression counts the closed sets W' such that $Y \cup Z \subseteq W'$ and $X \cup Z \not\subseteq W'$. Readily, if W' is

δ_L	a	b	c	d	e
a	0	3	3	4	6
b		0	4	3	5
c			0	3	3
d				0	2
e					0

Table 1: Distances between singleton elements of Figure 1

counted in the second sum, $Y \subseteq W'$, and $X \not\subseteq W'$; so it was counted in the first sum. Consequently the inequality holds. □

Applying this distance function to the anti-matroid of Figure 1 yields Table 1.

The definition of distance, as given in (3), is somewhat unusual. It can be worthwhile examining other candidates. Many lattice based distance functions employ the height of the lattice element within the lattice. We recall that an atomic, upper semi-modular lattice is said to be *geometric* [6] [12], and that one can show the expression $d_1(x, z) = h(x)+h(z)-2\cdot h(x \wedge z)$, where h denotes height, is a metric on these geometric lattices. Expression (3) was suggested by this definition and the similarity and difference is apparent. This definition seems counter intuitive in anti-matroids because, in the lattice of Figure 1, the subset $\{bd\}$ is closer to $\{bcd\}$ with a different closure than to $\{ad\}$ which has the same closure. Further, all closed points (singleton sets), whose intersection must be $\emptyset$, will be of distance 2 from each other.

Another distance candidate could be

$$d_2(x, z) = h(x \vee z) - h(x \wedge z)$$

This can be shown to be a metric whenever the sublattice of closed sets is modular. But, when this sublattice is only lower semi-modular[1] as in Figure 4, shown on a later page, it fails to satisfy the triangle inequality. In that anti-matroid, $d_2(a, e) = h(a \vee e) - h(a \wedge e) = h(acde) - h(0) = 4$. But, $d_2(a, c) + d(c, e) = h(ac) - h(0) + h(ce) - h(0) = 2$. As yet, we have found no other reasonable candidate formulae to be pseudometrics in anti-matroids.

Let $d(X, Z)$ denote any arbitrary, symmetric, positive function that is bounded over the power set of a discrete space $\mathbf{U}$. Let M denote the maximum value of d. It is easy to see that d' defined $d'(X, Z) = d(X, Z) + M$ when $X \neq Z$ must satisfy the triangle property. Consequently, in a discrete space, for a real (or integer) valued function to just satisfy the three traditional metric axioms:

$$\delta(X, X) = 0$$

[1] As shown in [19] [8] the sublattice of sets, closed with respect to a uniquely generated closure operator, must be lower semi-modular.

$$\delta(X, Y) = \delta(Y, X)$$
$$\delta(X, Z) \le \delta(X, Y) + \delta(Y, Z)$$

is only a minimal requirement. We will say that an integer valued function, δ is an **efficient pseudometric** if δ satisfies the three metric axioms, and in addition,

$$\delta(X, Z) = \delta(X, Y) + \delta(Y, Z), \quad \text{for some} \quad X, Y, Z$$
$$\delta(X, Y) = 1 \qquad\qquad\qquad \text{for some} \quad X, Y.$$

We observe that $\delta(X, Z)$ as shown in Table 1 conforms to the first of these properties, since $\delta(a, e) = \delta(a, c) + \delta(c, e)$; but not apparently to the second. However, δ is not defined only on singleton elements, but on all 32 subsets of $\mathbf{U}$. Table 1 shows just a fraction of these distances. In particular, the reader can verify that $\delta(\{abcde\}, \{abcd\}) = 1$. We can show, in general, that:

Lemma 2.3 *In any non-trivial anti-matroid* (with more than 2 closed sets) *there exist X, Y, Z such that $\delta(X, Z) = \delta(X, Y) + \delta(Y, Z)$.*

Proof: Let $Z = \mathbf{U}$. If Z covers only one set, let that set be Y and let X be any set covered by Y. (Note that Y is the only closed set covering X, and that X, Y and Z are all closed.) $|Z.\uparrow| = 1$, $|Y.\uparrow| = 2$ and $|X.\uparrow| = 3$ and $X \cup Y = Y$, $X \cup Z = Y \cup Z = Z$. Consequently, $\delta(X, Z) = 3+1-2 = 2$, $\delta(X, Y) = 3 + 2 - 4 = 1$, $\delta(Y, Z) = 2 + 1 - 2 = 1$, so that $\delta(X, Z) = \delta(X, Y) + \delta(Y, Z)$.
So assume Z covers several sets, including Y_1 and Y_2 Since the sublattice of closed sets is lower semi-modular, Y_1 and Y_2 cover $Y_1 \wedge Y_2$. Let Y be Y_1 and let $X = Y_1 \wedge Y_2$. Since $\mathcal{L}$ is meet distributive, Y_1 and Y_2 are the only elements covering X, so $|X.\uparrow| = 4$, $|Y.\uparrow| = 2$, and $|Z.\uparrow| = 1$. Consequently, $\delta(X, Z) = 4 + 1 - 2 = 3$, $\delta(X, Y) = 4 + 2 - 4 = 2$, and $\delta(Y, Z) = 2 + 1 - 1 = 1$. □

Corollary 2.4 *The pseudometric $\delta(X, Z) = |X.\varphi.\uparrow| + |Y.\varphi.\uparrow| - 2 \cdot |(X \cup Y).\varphi.\uparrow|$ is efficient.*

Proof: By Lemma 2.3 there exist X, Y, Z such that $\delta(X, Z) = \delta(X, Y) + \delta(Y, Z)$.
Consider any closed set Z covered by $\mathbf{U}$. Readily, $\delta(Z, \mathbf{U}) = 1$. □

δ is a pseudometric on *all* the subsets of $(\mathbf{U}, \varphi)$, but, if $\emptyset$ is closed and x and y denote singleton elements in $\mathbf{U}$, it is not difficult to show that $\delta(x, y) = 0$ implies $x = y$. Because φ is uniquely generated, x and y cannot have the same closure, lest $(\{x\} \cap \{y\}).\varphi = \emptyset.\varphi = \{x\}.\varphi$. It is not hard to see that if $\emptyset$ is closed, every singleton set $\{x\}$ must be either closed or a generator. In any case, δ is a *metric* on the points of $\mathbf{U}$.

Because $\emptyset.\uparrow = \mathcal{C} = \{$ all closed sets $\}$, $\delta(\emptyset, X)$ provides an upper bound on all distances from X to any other set. In fact,

Lemma 2.5 *Let $(\mathbf{U}, \varphi)$ be any anti-matroid. For all sets X, Y, $\delta(X, Y) \le \delta(\emptyset, \mathbf{U})$, with equality only if $X, Y = \emptyset, \mathbf{U}$.*

Proof: We may assume that X, Y, and $\emptyset$ are all closed, because δ is invariant with respect to closure equivalence classes. Next, we let $\mathcal{C}$ denote the collection of **all** closed sets.

Now, we observe that for any set X,

$$\delta(X, \emptyset) = |X.\uparrow| + |\mathcal{C}| - 2 \cdot |X.\uparrow| = |\mathcal{C}| - |X.\uparrow|$$

or $|X.\uparrow| = |\mathcal{C}| - \delta(X, \emptyset)$.
Now, let X and Y be any closed sets.

$$\begin{aligned} \delta(X,Y) &= |X.\uparrow| + |Y.\uparrow| - 2 \cdot |(X \cup Y).\uparrow| \\ &= |\mathcal{C}| - \delta(X,\emptyset) + |\mathcal{C}| - \delta(Y,\emptyset) - 2 \cdot (|\mathcal{C}| - \delta(X \cup Y, \emptyset)) \\ &= 2 \cdot \delta(X \cup Y, \emptyset) - [\delta(X,\emptyset) + \delta(Y,\emptyset)] \\ &\leq 2 \cdot \delta(X \cup Y, \emptyset) - \delta(X,Y) \qquad \text{(by triangle inequality)} \end{aligned}$$

Hence, $\delta(X,Y) \leq \delta(X \cup Y, \emptyset)$, with equality only if $X = \emptyset$ or $Y = \emptyset$.
Since $\delta(X \cup Y, \emptyset)$ denotes the number of closed sets that do *not* contain $X \cup Y$, it has a unique maximum when $X \cup Y = \mathbf{U}$. So

$$\delta(X,Y) \leq \delta(X \cup Y, \emptyset) \leq \delta(\mathbf{U}, \emptyset)$$

with equality if and only if $X = \emptyset, Y = \mathbf{U}$. $\square$

A **diameter** of an anti-matroid $(\mathbf{U}, \varphi)$ is a pair of non-empty sets, X and Y, such that $\delta(X,Y)$ is maximal. We let d denote the value of this diameter and interchangably use the term *diameter* to mean both the pair of sets and the distance. We require the sets defining the diameter to be non-empty, because otherwise by Lemma 2.5, the maximal $\delta(X,Y)$ is always $\delta(\emptyset, \mathbf{U})$ which equals the total number of closed sets minus one.

3 Anti-Matroids Induced by Ideals in Partial Orders

If $\mathbf{U}$ is any partially ordered set, then there are three natural closure transformations:

$Y.\varphi_L = \{x | x \leq y, \quad y \in Y\}$
$Y.\varphi_R = \{z | y \leq z, \quad y \in Y\}$
$Y.\varphi_C = \{x | y_1 \leq x \leq y_2, \quad y_1, y_2 \in Y\}$.

In [21] these are called *path closures*. The first two are *ideal* operators, the latter is an *interval* operator.[2] The closure lattice of Figure 1 is that generated by considering the partially ordered set of Figure 2 and φ_L as the closure operator. Verifying that the lattice of Figure 1 is generated by the partial order (2) and closure operator φ_L on Figure 2 can be valuable.

We may instead employ the right ideal closure operator, φ_R. This gives rise the the lattice structured anti-matroid shown in Figure 3

[2] In [19], the author called this interval operator a *convex hull* operator; in [9] [10], it is called an *order convex* operator. The C subscript suggests "convex".

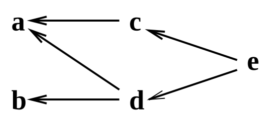


Figure 2: Partially ordered set giving rise to Figure 1

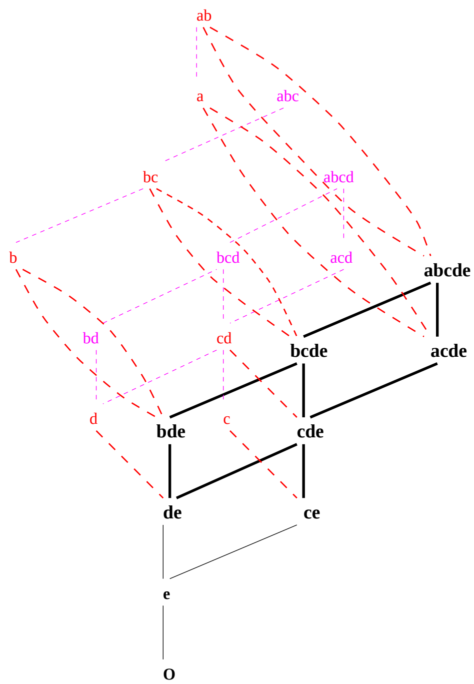


Figure 3: Closure lattice, $\mathcal{L}_R = (\mathbf{U}, \varphi_R)$

Even though the anti-matroids of figures 1 and 3 were generated by the same underlying graph (Figure 2), they are quite different. (Because anti-matroids and distances can be generated with respect to multiple closure operators, we commonly subscript lattice designators and the distance designator with respect to the underlying closure operator, *e.g.* $\mathcal{L}_L, \mathcal{L}_R, \delta_L, \delta_R$.) But, when we apply the distance function to the anti-matroid $\{abcdef\}, \varphi_R)$ of Figure 3 we obtain Table 2.

The values of Table 2 are identical to those of Table 1, even though the anti-matroids are not at all similar. This is not a coincidence. But to establish this curious result we need two preliminary lemmas based on the anti-chains (mutually incomparable elements) in G.

δ_R	a	b	c	d	e
a	0	3	3	4	6
b		0	4	3	5
c			0	3	3
d				0	2
e					0

Table 2: Distances between individual points of Figure 2 given φ_R.

Lemma 3.1 *Let $G = (\mathbf{U}, \leq)$ be any partial order (or acyclic graph). The anti-matroids $(\mathbf{U}, \varphi_L)$ and $(\mathbf{U}, \varphi_R)$ determined by the path closures φ_L and φ_R on G have precisely the same number of closed sets.*

Proof: Let Z be any closed set in $(\mathbf{U}, \varphi_L)$. We claim its generator $Z.\beta$ is an anti-chain in G; because, if $x, z \in Z.\beta$ and $x \leq z$ then $Z.\beta$ is not minimal.
Clearly, the generators of closed sets with respect to φ_R are also anti-chains. Consequently, the number of closed sets in $(\mathbf{U}, \varphi_L)$ and $(\mathbf{U}, \varphi_R)$ is the number of distinct anti-chains in G, and these serve as the common generators of both. □

Lemma 3.2 *Let $G = (\mathbf{U}, \leq)$ be any partial order (or acyclic graph), and let $\{x_1, \ldots, x_m, z_1, \ldots, z_n\}$ be an anti-chain. Let $\uparrow_x$ denote all subsets, closed w.r.t φ_L that contain $\{x_1, \ldots, x_m\}$ but not $\{z_1, \ldots, z_n\}$, and let $\uparrow_z$ denote all subsets, closed w.r.t φ_R that contain $\{z_1, \ldots, z_n\}$ but not $\{x_1, \ldots, x_m\}$, then $|\uparrow_x| = |\uparrow_z|$.*

Proof: Let X be a closed set w.r.t φ_L, with $x_i \in X, z_j \notin X$. $\bar{X} = \mathbf{U} - X$ is closed w.r.t φ_R because if there exists an edge $(a, b) \in E, a \in \bar{X}, b \in X$, then X was not closed w.r.t. φ_L. Hence, $z_j \in \bar{X}$ establishing the 1-1 correspondence.
(Readily, this lemma can be generalized because the elements x_i need not be themselves an anti-chain, only incomparable with any z_j, and conversely.) □

Theorem 3.3 *Let $G = (\mathbf{U}, E)$ be an acyclic graph (partial order), giving rise to two distinct closure spaces $(\mathbf{U}, \varphi_L)$ and $(\mathbf{U}, \varphi_R)$. For any singleton elements, $w, z \in \mathbf{U}$, $\delta_L(w, z) = \delta_R(w, z)$.*

Proof: We must show that $|w.\varphi_L.\uparrow| + |z.\varphi_L.\uparrow| - 2 \cdot |\{wz\}.\varphi_L.\uparrow| = |w.\varphi_R.\uparrow| + |z.\varphi_R.\uparrow| - 2 \cdot |\{wz\}.\varphi_R.\uparrow|, \forall w, z \in \mathbf{U}$.
Let $n = |\mathbf{U}|$ be fixed and run an induction on the number of edges of G.
If $|E| = 0$, then every point $x \in \mathbf{U}$ is isolated and every subset $X \subseteq \mathbf{U}$ is closed. Since each point x is in half the subsets, and each doubleton $\{xy\}$ is in half of these:

$$\begin{aligned} \delta_L(w, z) &= 2^{n-1} + 2^{n-1} - 2 \cdot 2^{n-2} \\ &= 2^{n-1} = \delta_R(w, z) \end{aligned}$$

Let $|E| = 1$ and let $E = \{(x, y)\}$. All sets closed w.r.t φ_L that contain y must contain x as well. So using the reasoning above and observing that of the 2^{n-1} subsets containing w (or z) $\neq x, y$, half will contain y, and of those half (or 2^{n-3}) will not contain x, and so not be closed, we have:

$$
\begin{aligned}
\delta_L(x, y) &= 2^{n-1} + 2^{n-1} - 2 \cdot 2^{n-2} \\
&= 2^{n-2} \\
\delta_L(x, z) &= 2^{n-1} + (2^{n-1} - 2^{n-3}) - 2 \cdot 2^{n-2} \\
&= 2^{n-1} - 2^{n-3} \\
\delta_L(w, y) &= (2^{n-1} - 2^{n-3}) + 2^{n-2} - 2 \cdot 2^{n-3} \\
&= 2^{n-1} - 2^{n-3} \\
\delta_L(w, z) &= (2^{n-1} - 2^{n-3}) + (2^{n-1} - 2^{n-3}) - 2 \cdot (2^{n-2} - 2^{n-4}) \\
&= 2^{n-1} - 2^{n-3}
\end{aligned}
$$

Similar reasoning will yield identical values for δ_R.
Finally, assume the induction hypothesis that $\delta_L(w, z) = \delta_R(w, z)$, and let the edge (x, y) be added to E, whence we will talk of $\delta'_L(w, z)$ and $\delta'_R(w, z)$.

Case 1: $\delta'_L(x, y) = \delta'_R(x, y)$ In $\delta_L(x, y) = |x.\varphi_L.\uparrow| + |y.\varphi_L.\uparrow| - 2 \cdot |\{xy\}.\varphi_L.\uparrow|$ we observe that neither the first nor third terms will be changed since any closed set containing x remains closed. But, the second term may be decremented since formerly closed subsets not containing x that contain y will not longer be closed.
In $\delta_R(x, y) = |x.\varphi_R.\uparrow| + |y.\varphi_R.\uparrow| - 2 \cdot |\{xy\}.\varphi_R.\uparrow|$ we employ the same analysis to observe that only the first term changes — to be decremented by the number of formerly closed subsets of x not containing y.
But, by Lemma 3.2 there is a 1-1 correspondence between these two sets, so both expression are decremented equally.

Case 2: $\delta'_L(x, z) = \delta'_R(x, z)$ We consider subcases involving the relationship of x to z in G:

$x \leq z$ **in** G**:** then every δ_L-closed set containing z contains x. Consequently, $\delta'_L(x, z) = |x.\varphi_L.\uparrow| + |z.\varphi_L.\uparrow| - 2 \cdot |\{xz\}.\varphi_L.\uparrow|$ is unchanged. Similarly, every δ_R-closed set containing x also contains z, so $\delta'_R(x, z)$ is also unchanged.

$z \leq x$ **in** G**:** This case follows the argument above.

x **is incomparable with** z**:** For this case, we must further consider the relationship of y to z in G.
If y is incomparable to z, or $y \leq z$ we use Lemma 3.2 to ensure that $\delta_L(x, z)$ and $\delta_R(x, z)$ are decremented identically to obtain $\delta'_L(x, z) = \delta'_R(x, z)$.
So assume $y \leq z$. Consider the expression $\delta_L(x, z) = |x.\varphi_L.\uparrow| + |z.\varphi_L.\uparrow| - 2 \cdot |\{xz\}.\varphi_L.\uparrow|$. Every closed set containing z contains y, but only the second term need be decremented by those subsets that do not also contain x. In $\delta_R(x, z) = |x.\varphi_R.\uparrow| + |z.\varphi_R.\uparrow| - 2 \cdot |\{xz\}.\varphi_R.\uparrow|$ only the first term $|x.\varphi_R.\uparrow|$ need be decremented by the number of sets not containing y. Again we use Lemma 3.2 to establish that these are the same.

The full proof requires that we individually demonstrate that $\delta'_L(y,z) = \delta'_R(y,z), \cdots \delta'_L(w,z) = \delta'_L(w,z)$ on a tedious case by case basis, using similar arguments. $\Box$

It is important to note the restrictions implicit in this theorem. First, it applies only to those closure operators on $\mathbf{U}$ which are ideal path operators, φ_L or φ_R. Second, only the distances between singleton elements of $\mathbf{U}$ are invariant; distances between arbitrary subsets need not be equal. For example, it is easy to calculate that $\delta_L(\{cd\}, \mathbf{U}) = 1$ in $(\mathbf{U}, \varphi_L)$ of Figure 1, but $\delta_R(\{cd\}, \mathbf{U}) = 3$ in $(\mathbf{U}, \varphi_R)$ in Figure 3.

Finally, we may employ the path closure operator, φ_C, or convex closure. If we calculate the distances between individual points with respect to this closure, we get those shown in Table 3. There are many more closed sets in the lattice $\mathcal{L}_C$ than in $\mathcal{L}_L$ or $\mathcal{L}_R$, so the

δ_C	a	b	c	d	e
a	0	11	11	12	14
b		0	12	11	13
c			0	11	11
d				0	10
e					0

Table 3: Distances between individual elements of Figure 2 given φ_C.

distance are apparently greater. We discuss path closure, along with convex geometries in the following section.

4 Atomic Anti-matroids

If φ is an ideal closure on a poset, as in the preceding section, then $\overline{\mathcal{L}}$, its sublattice of closed subsets, must be distributive. And, if $\overline{\mathcal{L}}$ is distributive, there exists a poset $(\mathbf{U}, \leq)$ such that φ_L will yield an isomorphic closed set structure over $(\mathbf{U}, \leq)$ [24]. For the closure operators we normally encounter in applications, $\overline{\mathcal{L}}$ will be either distributive or **atomic**, in the sense of [1] [22]. That is, the atoms A will consist of those elements covering the 0-element, and $\forall X \in \mathcal{L}, X \neq 0, X = \bigvee\{a_i | a_i \leq X, a_i \in A\}$.[3] We emphasize that the closure lattice $\mathcal{L}_{(\mathbf{U}, \leq_\varphi)}$ will be neither distributive nor atomic, only its sublattice of closed subsets, $\overline{\mathcal{L}}$.

An anti-matroid in which every singleton point is closed must be atomic. We will call them **atomic anti-matroids**.[4] Convex path closure, φ_C, on acyclic graphs is atomic. So

[3] H. Crapo and G.-C. Rota call this a "point lattice" in [5].

[4] One can have closure lattices in which $\overline{\mathcal{L}}$ is atomic, yet every singleton point need not be closed. This will be the case if $\emptyset$ is not closed.

are convex geometries and monophonic closure on chordal graphs. Convex geometries, in which the closure operator is the *convex hull*, of the designated points generate an abundance of anti-matroids. We will denote closure with respect to a convex hull operator in a geometry G by φ_G. Figure 4(a) illustrates a small convex geometry on 6 points. The entire space

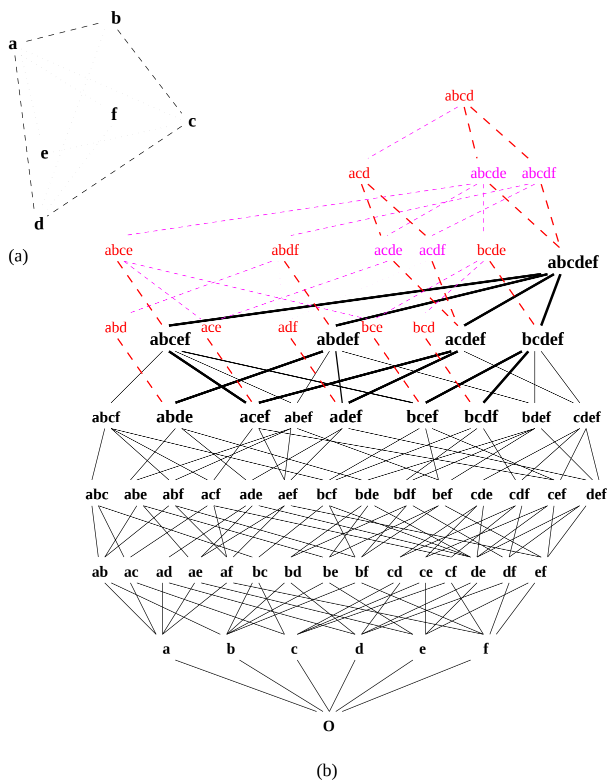


Figure 4: Closure lattice, $\mathcal{L}_G$, based on φ_G

$\{abcdef\}$ is generated by its *extreme points* $\{abcd\}$.[5] The lattice $\mathcal{L}_G$ of Figure 4(b) illustrates this geometric anti-matroid $(\{abcdef\}, \varphi_G)$. Because all singleton and doubleton sets, and most triples, must be closed, the lower half is extremely busy. Consequently, we have emphasized with larger type and bolder lines those closed sets with **non-trivial** generators, *i.e.* $X.\beta \neq X.\varphi$. These constitute the "interesting" portion of the anti-matroid.

Calculating the inter-element distances in the anti-matroid of Figure 4, we obtain Table 4. We observe that the point b can be moved away from point d an indefinite distance (in the plane) without altering the closed set relationships. The distances, δ_G, of Table 4 will

[5] In the literature of convex geometry [9] [10], the term extreme point is synomonous to generator.

δ_G	a	b	c	d	e	f
a	0	24	26	28	23	25
b		0	26	26	25	23
c			0	26	27	21
d				0	21	25
e					0	22
e						0

Table 4: Distances between individual elements of $\mathcal{L}_G = (\mathbf{U}, \varphi_G)$ of Figure 4

remain unchanged. Jamison, in [13], asserts that

> "If topology is the *rubber* sheet geometry, in which structures can be stretched and deformed because only limiting behavior is important, then alignments [anti-matroids] are *rigid* sheet geometries, basically combinatorial in nature."

The pseudometric of anti-matroids does not reflect Euclidean distance or any variant of it.

Given Lemma 2.5, one would expect that all diameters occur between $\mathbf{U}$ and singleton elements. Sometimes this is true, for example, in the anti-matroid of Figure 3, one can calculate that $\delta(\{e\}, \mathbf{U}) = 7$. It is a diameter. One would wish, however, that the diameter of a anti-matroid would always occur between some pair of its points — between its atomic elements. In the rest of this section we will show that atomicity of a anti-matroid is sufficient to ensure that diameters always exist between pairs of individual points of the space.

Lemma 4.1 *In an atomic anti-matroid, with* $|\mathbf{U}| \geq 2$
(a) $\emptyset$ *is closed,*
(b) if Y *is closed and* $|Y| \leq 2$ *then* $Y.\beta = Y$*,*
(c) if Y *is closed and* $|Y| \geq 2$ *then* Y *covers* $\{X_1, \ldots, X_m\}, m \geq 2$.

Proof:

(a) The $\mathcal{L}_{(U,\varphi)}$ has a zero element, that is covered by its atoms. Since the closed sets form a sub-lattice that is closed under intersection, $\{x\} \cap \{y\} = \emptyset$ must be closed.

(b) Suppose Y has a generator $Y.\beta \neq Y$. $Y.\beta \subseteq Y$, so $Y.\beta$ must be either $\emptyset$ or a singleton element. But these are closed, by (a) above and by atomicity of the anti-matroid.

(c) Let Y be closed, $|Y| \geq 2$. By the reasoning of (b), $m = |Y.\beta| \geq 2$. By Lemma 1.1, $Y - \{y_1\}, \ldots, Y - \{y_m\}, y_i \in Y.\beta$ are all closed. □

Lemma 4.2 *Let $(\mathbf{U}, \varphi)$ be an atomic anti-matroid, and let $Y <_\varphi Z$, where both are non-empty, closed. There exists X incomparable with Y such that $\delta(X, Y) > \delta(Y, Z)$.*

Proof: Since $Y <_\varphi Z$, $Z = Y \vee Z$ and $Z.\uparrow \subset Y.\uparrow$, so $\delta(Y, Z) = |Y.\uparrow| + |Z.\uparrow| - 2 \cdot |Z.\uparrow| = |Y.\uparrow| - |Z.\uparrow|$. Z cannot cover $\emptyset$. Let A_Z denote the set of atoms spanned by Z. Since $\mathcal{L}$ is atomic, there exists some $a_i \in A_Z, a_i \not\leq_\varphi Y$, $a_i \vee Y \leq_\varphi Z$. Let X be any set, $\{a_i\} \leq_\varphi X <_\varphi a_i \vee Y$. Readily X is incomparable with Y, $X \vee Y = Z$ and $Z.\uparrow \subset X.\uparrow$. Consequently

$$\begin{aligned} \delta(X, Y) &= |X.\uparrow| + |Y.\uparrow| - 2 \cdot |(X \vee Y).\uparrow| \\ &> |Y.\uparrow| + |Z.\uparrow| - 2 \cdot |(Y \vee Z).\uparrow| = \delta(Y, Z) \qquad \square \end{aligned}$$

Lemma 4.3 *Let $(\mathbf{U}, \varphi)$ be an atomic anti-matroid and let Y, Z be incomparable closed sets. If Y does not cover $\emptyset$, there exists $X <_\varphi Y$ such that $\delta(X, Z) > \delta(Y, Z)$.*

Proof: Let A_Y and A_Z denote the atoms spanned by Y and Z respectively. Since they are incomparable $A_y \not\subseteq A_z$, there exists an atom $a_x \in A_y - A_z$. Let X be any set, $\{a_x\} \leq_\varphi X <_\varphi Y$. $X \vee Z = Y \vee Z$. Since $Y.\uparrow \subset X.\uparrow$,

$$\begin{aligned} \delta(X, Z) &= |X.\uparrow| + |Z.\uparrow| - 2 \cdot |(X \vee Z).\uparrow| \\ &> |Y.\uparrow| + |Z.\uparrow| - 2 \cdot |(Y \vee Z).\uparrow| \\ &= \delta(Y, Z) \qquad \square \end{aligned}$$

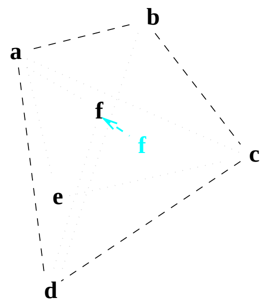
From these we conclude that:

Theorem 4.4 *Let $(\mathbf{U}, \varphi)$ be an atomic anti-matroid. Every diameter is of the form $\delta(\{a\}, \{c\})$. That is, a maximal distance between non-empty sets must be between atoms.*

Proof: Let Y and Z be any two sets. If $Z <_\varphi Y$, by Lemma 4.2 there exists X incomparable with Z such that $\delta(X, Z) > \delta(Y, Z)$.
If Y and Z are incomparable, by Lemma 4.3 there exists $X <_\varphi Y$ such that $\delta(X, Z) > \delta(Y, Z)$
This process can be iterated until both sets are singleton, *i.e.* atoms that cover $\emptyset$. $\square$

5 Non-Local Distance

One of the more surprising properties of this distance function is that it is "non-local". For example, just shifting the position of the point f of Figure 4 yields the distance matrix shown in Figure 5. The distances $\delta(a, b)$, $\delta(b, c)$ and $\delta(d, e)$ have all increased (see Table 4) even though f appears to have no role in those distances.



	a	b	c	d	e	f
a	0	27	27	29	25	23
b		0	24	26	26	24
c			0	26	26	24
d				0	22	26
e					0	22
f						0

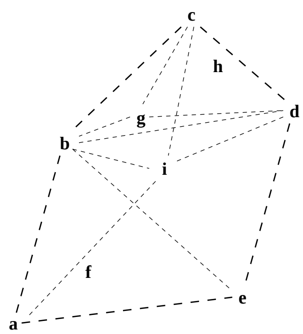
Figure 5: Distances on modified convex geometry using φ_G

One might argue that the space is small and f is really in the neighborhoods of a, b and c. In a larger space with more separated elements the effect should vanish, or at least diminish. Quite the reverse appears to be true. In the convex geometries of Figure 6, moving h closer to g does decrease $\delta(g, h)$ from 99 to 84. But is also decreases the distance $\delta(a, f)$ from 98 to 90. In fact, that small change decreases every distance, but not in any kind of proportional way.

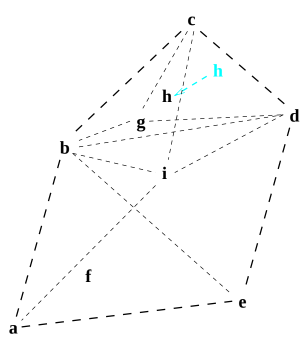
We turn to monophonic closure, φ_M on chordal graphs for a final example of non-local distance in anti-matroids. Recall that $X.\varphi_M$ consists of points on all chordless paths between any two points $x_i, x_j \in X$ [9]. Figure 7(a) illustrates a chordal graph and the corresponding interpoint distances based on φ_M. Removing the single edge (f, g) creates the chordal graph of Figure 7(b) and its associated distances. The distance $\delta(a, b)$ is increased from 17 to 18, even though the edge was removed from a distant portion of the graph.

In a Euclidean world, distance is a function of what is "between" the points. A measure that is affected by non-local events is counter intuitive. Nevertheless it is a metric. We conjecture that all efficient metrics one can define over anti-matroids will exhibit this non-local, global behavior; and that it will be another characteristic that distinguishes anti-matroids from matroids.

Such non-local measurement has received attention with the publication of a biography of the physicist David Bohm [18]. In 1952, Bohm proposed an alternative model of quantum behavior in which the Heisenberg uncertainty principle was explained by non-local events he called "hidden variables" [2] [3]. This approach, refined in [4], had the same predictive value as the wave formulation of the Copenhagen school. But it was not accepted by the physics establishment. J. Robert Oppenheimer issued the well known quote, "if we cannot disprove Bohm, then we must agree to ignore him". Perhaps anti-matroids have a role in a subatomic view of the universe.



	a	b	c	d	e	f	g	h	i
a	0	106	122	119	130	98	117	126	122
b		0	116	119	122	116	101	130	120
c			0	129	116	126	105	96	132
d				0	103	123	130	107	107
e					0	108	133	120	108
f						0	117	124	98
g							0	99	99
h								0	112
i									0



	a	b	c	d	e	f	g	h	i
a	0	99	114	116	122	90	108	120	114
b		0	109	115	115	107	93	123	113
c			0	112	108	122	106	82	132
d				0	96	116	118	110	92
e					0	102	126	114	100
f						0	108	114	100
g							0	84	98
h								0	98
i									0

Figure 6: Two larger convex geometries with interpoint distances

Acknowledgment

The authors would like to acknowledge the contribution of Tom Kitchens, Dept. of Energy, who pointed out the connection with David Bohm's work.

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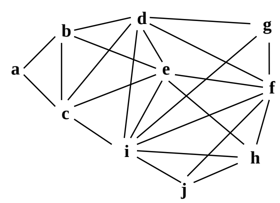
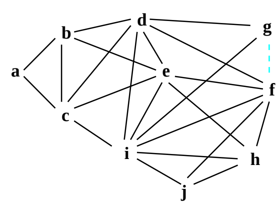


	a	b	c	d	e	f	g	h	i	j
a	0	17	27	43	44	46	28	35	50	27
b		0	20	34	35	47	35	40	49	36
c			0	28	29	45	39	42	39	42
d				0	27	33	35	44	31	48
e					0	32	44	35	30	47
f						0	32	27	26	35
g							0	33	38	29
h								0	33	20
i									0	41
j										0

(a)



	a	b	c	d	e	f	g	h	i	j
a	0	18	30	46	47	44	29	36	53	28
b		0	22	36	37	46	35	42	51	38
c			0	28	31	46	37	46	39	46
d				0	27	36	33	48	29	52
e					0	33	42	39	30	51
f						0	37	24	29	32
g							0	35	36	31
h								0	37	20
i									0	45
j										0

(b)

Figure 7: Two chordal graphs with interpoint distances based on φ_M

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