

Application and Analysis of Burnside's Theorem

A Senior Comprehensive Project

by

Kristen Walcott
Allegheny College
Meadville, PA

December 7, 2004

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Project Advisor: Dr. Tamara J. Lakins

I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

Kristen R. Walcott

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Abstract

Some counting problems are simple enough to solve by observation, but many require a more sophisticated approach. Burnside's Theorem is a result of group theory that is often used to calculate the number of nonequivalent arrangements of colorings of objects in a set under a group of permutations. In this project, we will discuss the notion of an arbitrary group acting on a set, the analysis and several applications of Burnside's Theorem, and a generalization of the theorem.

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1 Group Actions

Simple counting techniques can be used to determine that there are $3^6 = 729$ ways to color the faces of a cube with three colors. However, not all of these colorings are unique since one can rotate a particular coloring of the faces of a cube to obtain another coloring. For example, all of the 6 colorings in which one face is white and all the others are red are equivalent. To determine the number of unique colorings, one could try to compare all of the 729 possibilities. Worse still, consider the combinatorial difficulty of performing a similar computation on an icosahedron. Clearly, a comparison of all possible colorings is not the way to go. In this project, a more sophisticated approach for calculating the number of distinct colorings or patterns will be considered. This approach is called Burnside's Theorem.

In order to understand the proof and application of Burnside's Theorem, consideration must first be given to several aspects of group theory, in particular, the notion of a group acting on a set.

Definition 1.1 ([6]). If X is a set and G is a group, then G **acts** on X if there is a function $\tau : G \times X \rightarrow X$, denoted by $\tau(g, x) = gx$ for $g \in G$ and $x \in X$, such that

- i) $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$;
- ii) $1x = x$ for all $x \in X$ where 1 is the identity in G .

We also call X a G -**set** if G acts on X .

Example 1.2. As an example, consider G as the group $D_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ of symmetries of an equilateral triangle, where ρ_i corresponds to rotating the triangle counterclockwise $\frac{i2\pi}{3}$ radians through the center point C and μ_i corresponds to flipping the triangle on the perpendicular bisector m_i of side s_i . In Figure 1, we show the triangle with vertices $1, 2, 3$ and label the sides s_1, s_2, s_3 , the perpendicular bisectors m_1, m_2, m_3 , the center point C , and the midpoints P_i of the sides s_i .

We let

$$X = \{1, 2, 3, s_1, s_2, s_3, m_1, m_2, m_3, C, P_1, P_2, P_3\}.$$

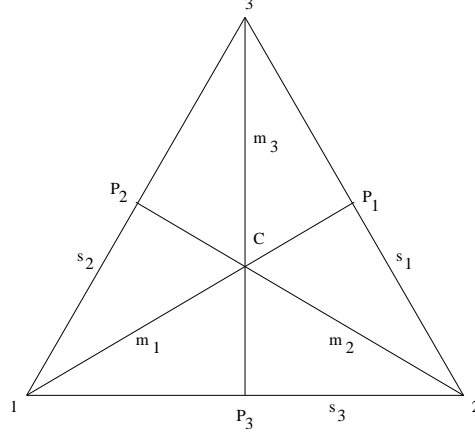


Figure 1: X

	1	2	3	s_1	s_2	s_3	m_1	m_2	m_3	C	P_1	P_2	P_3
ρ_0	1	2	3	s_1	s_2	s_3	m_1	m_2	m_3	C	P_1	P_2	P_3
ρ_1	2	3	1	s_2	s_3	s_1	m_2	m_3	m_1	C	P_2	P_3	P_1
ρ_2	3	1	2	s_3	s_1	s_2	m_3	m_1	m_2	C	P_3	P_1	P_2
μ_1	1	3	2	s_1	s_3	s_2	m_1	m_3	m_2	C	P_1	P_3	P_2
μ_2	3	2	1	s_3	s_2	s_1	m_3	m_2	m_1	C	P_3	P_2	P_1
μ_3	2	1	3	s_2	s_1	s_3	m_2	m_1	m_3	C	P_2	P_1	P_3

Table 1: The action of D_3 on X

Then X can be regarded as a D_3 -set because D_3 acts on X . Table 1 describes completely the action of D_3 on X .

Next, let's consider a similar example, the group D_4 , the symmetries of the square.

Example 1.3 ([2], Example 3.5.8). Consider G as the group $D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ of symmetries of a square. In Figure 2, we show the square with vertices 1, 2, 3, 4 and label the sides s_1, s_2, s_3, s_4 , the diagonals d_1, d_2 , the horizontal axes m_1, m_2 , the center point C , and the midpoints P_i of the sides s_i .

Note that ρ_i corresponds to rotating the square counterclockwise $\frac{i\pi}{2}$ radians through C , μ_i corresponds to flipping the square on the axis m_i , and δ_i corresponds to flipping the square on the diagonal d_i .

We let

$$X = \{1, 2, 3, 4, s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}.$$

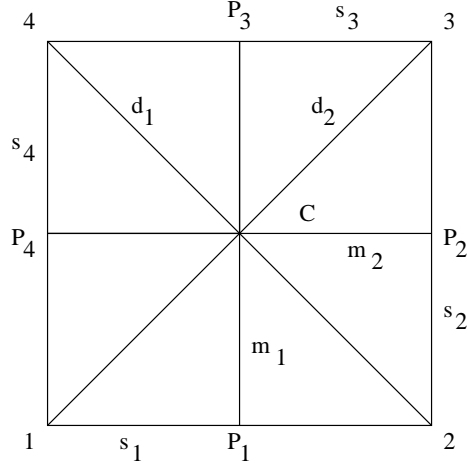


Figure 2: X

	1	2	3	4	s_1	s_2	s_3	s_4	m_1	m_2	d_1	d_2	C	P_1	P_2	P_3	P_4
ρ_0	1	2	3	4	s_1	s_2	s_3	s_4	m_1	m_2	d_1	d_2	C	P_1	P_2	P_3	P_4
ρ_1	2	3	4	1	s_2	s_3	s_4	s_1	m_2	m_1	d_2	d_1	C	P_2	P_3	P_4	P_1
ρ_2	3	4	1	2	s_3	s_4	s_1	s_2	m_1	m_2	d_1	d_2	C	P_3	P_4	P_1	P_2
ρ_3	4	1	2	3	s_4	s_1	s_2	s_3	m_2	m_1	d_2	d_1	C	P_4	P_1	P_2	P_3
μ_1	2	1	4	3	s_1	s_4	s_3	s_2	m_1	m_2	d_2	d_1	C	P_1	P_4	P_3	P_2
μ_2	4	3	2	1	s_3	s_2	s_1	s_4	m_1	m_2	d_2	d_1	C	P_3	P_2	P_1	P_4
δ_1	3	2	1	4	s_2	s_1	s_4	s_3	m_2	m_1	d_1	d_2	C	P_2	P_1	P_4	P_3
δ_2	1	4	3	2	s_4	s_3	s_2	s_1	m_2	m_1	d_1	d_2	C	P_4	P_3	P_2	P_1

Table 2: The action of D_4 on X

Then X can be regarded as a D_4 -set because D_4 acts on X . Table 2 describes completely the action of D_4 on X .

Note that a G -set is not necessarily a group. A G -set is only a set X which may or may not have the properties of a group. A binary operation associated with the set is not necessary. The fact that X is a G -set only means that a group G acts on the set. For example, consider the group D_3 and the D_3 -set, $X = \{1, 2, 3, s_1, s_2, s_3, m_1, m_2, m_3, C, P_1, P_2, P_3\}$. X is not a group.

When a group G acts on a set X , several situations can be observed. For example, given $x_1, x_2 \in X$ and $g \in G$, $gx_1 = gx_2$, or given $x \in X$ and $g_1, g_2 \in G$, $g_1x = g_2x$. At first glance, it would seem that $x_1 = x_2$ and $g_1 = g_2$ respectively, but let's consider these situations more carefully. First consider if $gx_1 = gx_2$.

Proposition 1.4. *Let X be a G -set with $x_1, x_2 \in X$ and $g \in G$. If $gx_1 = gx_2$, then $x_1 = x_2$.*

Proof. Since $g \in G$ and G is a group, $g^{-1} \in G$. Then

$$\begin{aligned}g^{-1}(gx_1) &= g^{-1}(gx_2), \\(g^{-1}g)x_1 &= (g^{-1}g)x_2 \text{ by Definition 1.1(i),} \\1x_1 &= 1x_2 \text{ because } g^{-1}g = 1 \text{ in } G, \\x_1 &= x_2 \text{ by Definition 1.1(ii).}\end{aligned}$$

□

However, in the second example where $g_1x = g_2x$, we cannot assume that $g_1 = g_2$. This only means that g_1 acts on x and g_2 acts on x in the same way. For example, for $\rho_1, \mu_3 \in D_3$ and $1 \in X = \{1, 2, 3, s_1, s_2, s_3, m_1, m_2, m_3, C, P_1, P_2, P_3\}$ from Example 1.2, $\rho_1 1 = 2 = \mu_3 1$, but $\rho_1 \neq \mu_3$.

By definition, in a group action, $1x = x$ for all $x \in X$, so each element in the G -set is left fixed by the identity of G . However, the fact that every element of a G -set is left fixed by the same element g of G does not imply that g is the identity element of G . For example, let $Y = \{C\} \subseteq X$, and let $G = D_3$ from Example 1.2. We know that G acts on Y because of the following remark.

Remark 1.5. Let the group G act on a set X . Then G acts on a set $Y \subseteq X$ if Y is closed under the action of G on X . This can also be expressed by saying that for every $g \in G$ and $y \in Y$, $gy \in Y$.

Every element of D_3 fixes every element of Y , so Y is closed under the action of G . In other words, every element of D_3 sends C to itself. However, there is only one identity of D_3 , ρ_0 .

We will now consider a slightly more complicated example demonstrating Remark 1.5.

Example 1.6 ([3], Exercise 29.13). As in Example 1.3, let

$X = \{1, 2, 3, 4, s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}$. We will show that D_4 acts on $Y = \{m_1, m_2\} \subseteq X$.

In order to show that D_4 acts on Y , we only need to show that Y is closed under the action of D_4 on X . The subset Y is closed under the action if for every $g \in D_4$, $gy \in Y$ for all $y \in Y$. This can easily be seen by observation. In Table 2, $\rho_0 m_1 = \rho_2 m_1 = \mu_1 m_1 = \mu_2 m_1 = m_1$. The remaining elements of D_4 map m_1 to m_2 : $\rho_1 m_1 = \rho_3 m_1 = \delta_1 m_1 = \delta_2 m_1 = m_2$. Similarly, $\rho_0 m_2 = \rho_2 m_2 = \mu_1 m_2 = \mu_2 m_2 = m_2$ and the remaining elements of D_4 map m_2 to m_1 : $\rho_1 m_2 = \rho_3 m_2 = \delta_1 m_2 = \delta_2 m_2 = m_1$. Thus, Y is closed under the action of D_4 on X . Therefore, D_4 acts on Y .

From this example, we see that if G acts on X , G also acts on subsets of X that are closed under the action of G on X . However, G not only acts on individual subsets of X , but also on the powerset $P(X)$.

Proposition 1.7. *If a group G acts on a set X , then G acts on the powerset $P(X)$.*

Proof. Assume G acts on X . Given $g \in G$ and $Y \in P(X)$, define $gY = \{gy \mid y \in Y\}$. Let $\phi : G \times P(X) \rightarrow P(X)$ be defined as $\phi(g, Y) = gY$ for all $g \in G$ and $Y \in P(X)$. We want to show that ϕ defines an action of G on $P(X)$. Let $g, h \in G$ and $Z \in P(X)$. Then $(gh)Z = \{(gh)z \mid z \in Z\}$. Since Z is a subset of X and G acts on X , $(gh)z = g(hz)$ for all $z \in Z$ by Definition 1.1(i). Then

$$\begin{aligned} (gh)Z &= \{(gh)z \mid z \in Z\} \\ &= \{g(hz) \mid z \in Z\} \\ &= g(hZ) \end{aligned}$$

Next, since Z is a subset of X and G acts on X , $1z = z$ for all $z \in Z$ by Definition 1.1(ii). Then

$$\begin{aligned} 1Z &= \{1z \mid z \in Z\} \\ &= \{z \mid z \in Z\} \\ &= Z \end{aligned}$$

Thus ϕ is a group action and G acts on $P(X)$. □

Therefore, when D_4 acts on X , D_4 also acts on the powerset of X . The same is the case for D_3 or any other group acting on some set X .

Similarly, assume that G acts on a set of n elements, and let C be an arbitrary set. Then G also acts on the set of all n -tuples formed from the elements of C .

Proposition 1.8 ([6]). *Note that this statement corrects an error in the source.*

If a group G acts on a set $X = \{1, \dots, n\}$, and if C is an arbitrary set, then G acts on the set C^n of all n -tuples by

$$\tau(c_1, \dots, c_n) = (c_{\tau^{-1}1}, c_{\tau^{-1}2}, \dots, c_{\tau^{-1}n}) \text{ for all } \tau \in G.$$

Proof. Think of an element of C^n as a function $f : X \rightarrow C$, where f is identified with the n -tuple, $(f(1), f(2), \dots, f(n))$. Given $\tau \in G$ and $f \in C^n$, define τf to be the function in C^n defined by $(\tau f)(i) = f(\tau^{-1}i)$, for all $i \in X$. Note that $\tau^{-1}i \in X$ because G acts on X .

We will show that G acts on the set C^n . Given $f : X \rightarrow C$ and $\sigma, \tau \in G$, we first show that $(\sigma\tau)f = \sigma(\tau f)$; i.e., for all $i \in X$, $((\sigma\tau)f)(i) = (\sigma(\tau f))(i)$. Given $i \in X$,

$$\begin{aligned} ((\sigma\tau)f)(i) &= f((\sigma\tau)^{-1}i) \text{ by the above definition} \\ &= f((\tau^{-1}\sigma^{-1})i) \text{ by [[7], Theorem 3.4]} \\ &= f(\tau^{-1}(\sigma^{-1}i)) \text{ because } G \text{ acts on } X \\ &= (\tau f)(\sigma^{-1}i) \text{ by the above definition} \\ &= (\sigma(\tau f))(i) \text{ again by the above definition.} \end{aligned}$$

Thus, $((\sigma\tau)f)(i) = (\sigma(\tau f))(i)$.

Next, given $f : X \rightarrow C$ we show that $(1f)(i) = f(i)$ for all $i \in X$ where 1 is the identity

element of G . Given $i \in X$,

$$\begin{aligned}(1f)(i) &= f(1^{-1}i) \text{ by the above definition} \\ &= f(1i) \\ &= f(i) \text{ because } G \text{ acts on } X\end{aligned}$$

Therefore, since both conditions for a group action are met according to Definition 1.1, G acts on the set C^n . \square

A group acting on a set is very much like the concept of a permutation group. The permutation group (also known as a symmetric group), S_X for a set X , is a group under composition of functions.

Definition 1.9 ([7]). Let X be a nonempty set. Then S_X is the set of all permutations of X , where a permutation of X is a one-to-one onto function mapping $X \rightarrow X$. Then (S_X, \circ) , the **symmetric group on X** , is a group formed from S_X under the binary operation \circ , function composition.

A group action on a set is closely related to permutation groups. In fact, a group acting on a set immediately gives rise to a homomorphism mapping the group to the permutation group of the set and vice versa.

Theorem 1.10 ([6]). A group G acts on a set X iff there exists a homomorphism $\tau : G \rightarrow S_X$.

Proof. (\Rightarrow) Let G be a group that acts on a set X . Then there exists a function $\varphi : G \times X \rightarrow X$ denoted $\varphi(g, x) = gx$ that satisfies Definition 1.1. We now want to create a function that represents a permutation of X . Fix $g \in G$. Define the function $\varphi_g : X \rightarrow X$ by $\varphi_g(x) = \varphi(g, x) = gx$ for all $x \in X$.

We want to show that φ_g is one-to-one and onto. If for every $y \in X$, there exists some $x \in X$ such that $\varphi_g(x) = y$, then φ_g is onto. Since G is a group and $g \in G$, $g^{-1} \in G$. Given $y \in X$, let $x = g^{-1}y$. Then $\varphi_g(x) = \varphi_g(g^{-1}y) = \varphi(g, g^{-1}y) = g(g^{-1}y) = (gg^{-1})y$ by Definition 1.1(i). Then $(gg^{-1})y = 1y$ where 1 is the identity in G , and $1y = y$ by Definition 1.1(ii). Hence, φ_g is

onto. Now we want to show that φ_g is also a one-to-one function. Given $x_1, x_2 \in X$, show that if $\varphi_g(x_1) = \varphi_g(x_2)$, then $x_1 = x_2$.

$$\text{Assume } \varphi_g(x_1) = \varphi_g(x_2).$$

$$\text{Then } \varphi(g, x_1) = \varphi(g, x_2),$$

$$\text{so } gx_1 = gx_2.$$

By Proposition 1.4, $x_1 = x_2$. Hence, φ_g is a one-to-one function, and φ_g is therefore a bijection. Then $\varphi_g \in S_X$ for all $g \in G$.

Next, define a function $\tau : G \rightarrow S_X$ by $\tau(a) = \varphi_a$ for all $a \in G$. We want to show that τ is a homomorphism. That is, given $a, b \in G$, we must show $\tau_a \circ \tau_b = \tau_{ab}$. Let $x \in X$. Then

$$\begin{aligned} (\varphi_a \circ \varphi_b)(x) &= \varphi_a(\varphi_b(x)) \\ &= \varphi_a(\varphi(b, x)) \\ &= \varphi_a(bx) \\ &= \varphi(a, bx) \\ &= a(bx) \\ &= (ab)x, \quad \text{by Definition 1.1(i)} \\ &= \varphi_{ab}(x). \end{aligned}$$

Thus, $\varphi_a \circ \varphi_b$ is the same element in S_X as φ_{ab} . For $a, b \in G$, $\tau(a)\tau(b) = \varphi_a \circ \varphi_b = \varphi_{ab} = \tau(ab)$. Therefore τ is a homomorphism.

(\Leftarrow) Now let $\tau : G \rightarrow S_X$ be a homomorphism.

Given $a \in G$, $\tau(a) \in S_X$. Then $\tau(a)$ is a bijection, say ρ_a , mapping X onto itself. Since τ is a homomorphism, for all $a, b \in G$, $\tau(a)\tau(b) = \tau(ab)$. Because $\tau(a)\tau(b) = \rho_a \circ \rho_b$ and $\tau(ab) = \rho_{ab}$, $\rho_a \circ \rho_b = \rho_{ab}$.

Now define a function $h : G \times X \rightarrow X$ denoted by $h(g, x) = gx$ for all $g \in G$ and $x \in X$. Given $g \in G$ and $x \in X$, let $gx = \rho_g(x)$. We will now show that h defines an action of G on X .

Let $a, b \in G$ and $x \in X$. Then

$$\begin{aligned}(ab)x &= \rho_{ab}(x) \\ &= (\rho_a \circ \rho_b)(x) \\ &= \rho_a(\rho_b(x)) \\ &= \rho_a(bx) \\ &= a(bx).\end{aligned}$$

Next, let 1 be the identity element of G . We know that every group homomorphism must map the identity to itself. Then for $x \in X$,

$$\begin{aligned}1x &= \rho_1(x) \\ &= x.\end{aligned}$$

Therefore, h is a group action of G on X . □

Thus, we can see that if X is a G -set, every element of G acts as a permutation of X . There exists a function $\tau : G \rightarrow S_X$ defined by $\tau(g) = \phi_g$ for every $g \in G$ where ϕ_g is a permutation in the set S_X .

The proof of Cayley's Theorem, named in honor of Arthur Cayley, involves the notion of a group acting on itself. It shows that every group can be thought of as a subgroup of some symmetric group.

Theorem 1.11 ([5], **Theorem 2.61**). *Every group G is isomorphic to a subgroup of the symmetric group S_G . In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n .*

Proof. ([5], Theorem 2.61) Let τ be a function $\tau : G \times G \rightarrow G$ defined by $\tau(g, h) = gh$ for all $g, h \in G$. We want to show that G acts on itself. For all $j, k, l \in G$, $(jk)l = j(kl)$ by associativity in G . For all $x \in G$, $1x = x$ where 1 is the identity in G also by definition of a group. Thus, G acts on itself "by left translation." By the proof of Theorem 1.10, this gives rise

to a homomorphism $\phi : G \rightarrow S_G$ where ϕ is defined as $\phi(a) = \tau_a$. We define τ_a by $\tau_a(h) = ah$ when $a, h \in G$ and a is fixed.

It is clear that $\{\phi(g) \mid g \in G\} = \phi(G)$ is a subset of S_G . In fact, $\phi(G)$ is also a subgroup of S_G . Let $H = \phi(G)$. To show that H is a subgroup of S_G , we need to demonstrate that H is closed under multiplication and inverses. If $a, b \in H$, then $a = \phi(c)$ and $b = \phi(d)$ for some $c, d \in G$. Multiplying a and b , we get $ab = \phi(c)\phi(d) = \phi(cd)$ since ϕ is a homomorphism. Thus, $ab = \phi(cd) \in H$, so $\phi(G)$ is closed under multiplication.

Given $a \in H$, $a = \phi(c)$ for some $c \in G$. Because ϕ is a homomorphism, $a^{-1} = \phi(c^{-1}) \in H$ since $c^{-1} \in G$. Then $\phi(G)$ is closed under inverses. Thus, $\phi(G)$ is a subgroup of S_G .

Next, we want to show that the function $\phi : G \rightarrow H$ is a bijection. Since given $x \in H$, x has the form $\phi(z)$ for some $z \in G$. Thus, ϕ is onto. Now ϕ is one-to-one if for $a, b \in G$ where $\phi(a) = \phi(b)$, $a = b$. So let $a, b \in G$ and assume $\phi(a) = \phi(b)$. By definition, $\phi(a) = \phi(b)$ implies $\tau_a = \tau_b$, and hence $\tau_a(x) = \tau_b(x)$ for all $x \in G$. In particular, when $x = 1$, this gives $a = b$ as desired. Thus, ϕ is one-to-one. Then $\phi : G \rightarrow \phi(G)$ is a bijection and therefore, G is isomorphic to $\phi(G)$, a subgroup of the symmetric group S_G .

If G is finite with n elements, then we can represent G by $\{1, 2, \dots, n\}$, and we accordingly denote (S_G, \circ) by S_n . Thus, if $|G| = n$, then G is isomorphic to a subgroup of S_n . \square

Next we will consider the generalization of Theorem 1.11. We see that the group G acts on the left cosets in G , which gives rise to a homomorphism.

Theorem 1.12 ([5], Theorem 2.62). *Let G be a group, and let H be a subgroup of G having finite index n . Then there exists a homomorphism $\phi : G \rightarrow S_n$ with $\ker(\phi) \subseteq H$.*

Proof. ([5], Theorem 2.62) Note that although H may not be a normal subgroup, we are denoting the family of all the left cosets of H in G by G/H . We define a function $\tau : G \times G/H \rightarrow G/H$ denoted by $\tau(g, xH) = g(xH)$ for all $g \in G$ and $xH \in G/H$. Let $xH \in G/H$. Then $x \in G$, and since $g \in G$, $gx \in G$. This implies that $(gx)H \in G/H$. Thus, we can define $g(xH) = (gx)H$.

We want to show that G acts on G/H via τ . For all $j, k \in G$ and $xH \in G/H$,

$$\begin{aligned}
(jk)xH &= ((jk)x)H \text{ by the above definition,} \\
&= (jkx)H \text{ since } j, k, x \in G, \\
&= (j(kx))H \text{ by associativity in } G, \\
&= j(kxH) \text{ because } kx \in G, kxH \in G/H, \text{ allowing the definition to again be applied.}
\end{aligned}$$

Denote 1 as the identity in G . For all $xH \in G/H$,

$$\begin{aligned}
1(xH) &= (1x)H \text{ by definition,} \\
&= xH \text{ because 1 is the identity in } G.
\end{aligned}$$

Thus, G acts on G/H . By Theorem 1.10, this gives rise to a homomorphism $\phi : G \rightarrow S_{G/H}$ denoted by $\phi(g) = \tau_g$ for all $g \in G$. We define τ_g by $\tau_g(h) = gh$ when $g, h \in G$ and g is fixed. Since $|G/H| = n$, $S_{G/H} \cong S_n$. Thus, we can assume $\phi : G \rightarrow S_n$.

Next we want to show that $\ker(\phi) \subseteq H$. Let $a \in \ker(\phi)$. By definition of the kernel, $\phi(a) = 1_{S_{G/H}}$. We can then write $\phi(a)H = 1_{S_{G/H}}H = H$. Then

$$\begin{aligned}
\phi(a)H &= \tau_a H \\
&= aH \\
&= H \text{ by the above statement.}
\end{aligned}$$

Thus, $aH = H$, so $a \in H$. Therefore, $\ker(\phi) \subseteq H$. □

Now let's consider some more definitions that will allow us to further discuss properties of group actions.

Definition 1.13 ([5]). If G acts on X and $x \in X$, then the **orbit** of x , denoted by $O(x)$, is the set

$$O(x) = \{gx : g \in G\} \subseteq X.$$

Definition 1.14 ([5]). If G acts on the nonempty set X and $x \in X$, then the **stabilizer** of x , denoted by G_x , is a subset of G defined by

$$G_x = \{g \in G : gx = x\}.$$

We are going to prove that $G_x \leq G$. Given $a, b \in G_x$, $ab \in G_x$ since if $ax = x$ and $bx = x$,

$$\begin{aligned} (ab)x &= a(bx) \text{ by Definition 1.1(i);} \\ &= ax \text{ since } bx = x, \\ &= x \text{ since } ax = x. \end{aligned}$$

Thus, G_x is closed under multiplication.

Next, given $a \in G_x$, we know $ax = x$. Note that a^{-1} is the inverse of a in G . Then since $ax = x$,

$$\begin{aligned} ax &= x \\ a^{-1}(ax) &= a^{-1}x \\ (a^{-1}a)x &= a^{-1}x \text{ by Definition 1.1(i);} \\ x &= a^{-1}x. \end{aligned}$$

Therefore, for every $a \in G_x$, $a^{-1} \in G_x$.

Thus, G_x is a subgroup of G .

Definition 1.15 ([4]). Let a group G act on a set X . Let $g \in G$ and $x \in X$. We say g **fixes** x if $gx = x$, and g **moves** x if $gx \neq x$. Define $\text{Fix}(g)$ as $\{x \in X | gx = x\}$ for $g \in G$ to represent the set of x that g fixes. The number of x that g fixes is represented as $|\text{Fix}(g)|$.

We now have enough information that we can begin looking at a few more concrete examples. Let's first take a more careful look at the group D_3 .

Example 1.16 ([2], based on Exercise 3.5(1-3)). Let

$X = \{1, 2, 3, s_1, s_2, s_3, m_1, m_2, m_3, C, P_1, P_2, P_3\}$ be the D_3 -set of Example 1.2 with action table

in Table 1, and let $G = D_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$.

Let's first find the fixed sets $\text{Fix}(\sigma)$ for each $\sigma \in D_3$, that is $\text{Fix}(\rho_0), \text{Fix}(\rho_1), \dots, \text{Fix}(\mu_3)$. Recall that ρ_i corresponds to rotating the triangle counterclockwise $\frac{i2\pi}{3}$ radians through C , and μ_i corresponds to flipping the triangle on the perpendicular bisector m_i . The fixed set $\text{Fix}(\rho_0) = X$ because ρ_0 leaves the triangle in its original position. Thus, all elements of X stay fixed. Then ρ_1 rotates the triangle from the center $\frac{2\pi}{3}$ radians, so everything changes positions except the center. Thus, $\text{Fix}(\rho_1) = \{C\}$. Continuing in the same fashion, $\text{Fix}(\rho_2) = \{C\}$. Next, μ_1 flips the triangle on the bisector labeled m_1 . Then $\text{Fix}(\mu_1) = \{1, s_1, m_1, C, P_1\}$. Similarly, μ_2 flips the triangle on the m_2 bisector, so $\text{Fix}(\mu_2) = \{2, s_2, m_2, C, P_2\}$. And finally, μ_3 flips the triangle on the m_3 bisector, so $\text{Fix}(\mu_3) = \{3, s_3, m_3, C, P_3\}$.

Now let's find the stabilizer subgroups G_x for each $x \in X$, that is, G_1, G_2, \dots, G_{P_3} . By Definition 1.14, $G_x = \{g \in G \mid gx = x\}$ for $x \in X$. We can determine each stabilizer subgroup by looking at Table 1.

$$G_1 = \{\rho_0, \mu_1\} = G_{s_1} = G_{m_1} = G_{P_1}$$

$$G_2 = \{\rho_0, \mu_2\} = G_{s_2} = G_{m_2} = G_{P_2}$$

$$G_3 = \{\rho_0, \mu_3\} = G_{s_3} = G_{m_3} = G_{P_3}$$

$$G_C = G.$$

Finally, let's determine the orbits in X under D_3 . The orbits in X under D_3 are defined as the subset of X , $O(x) = \{gx \mid g \in D_3\}$ by Definition 1.13 for $x \in X$. By this definition, we can see that $O(1)$, for example, equals $\{1, 2, 3\}$ because $\rho_0(1) = 1$, $\rho_1(1) = 2$, $\rho_2(1) = 3$, $\mu_1(1) = 1$, $\mu_2(1) = 3$, and $\mu_3(1) = 2$. Then $O(1) = O(2) = O(3) = \{1, 2, 3\}$. Similarly, $O(s_1) = \{s_1, s_2, s_3\} = O(s_2) = O(s_3)$, $O(m_1) = \{m_1, m_2, m_3\} = O(m_2) = O(m_3)$, $O(C) = \{C\}$, and $O(P_1) = \{P_1, P_2, P_3\} = O(P_2) = O(P_3)$.

As another example of fixes, orbits, and stabilizers, let's consider D_4 again.

Example 1.17 ([2], Exercise 3.5(1-3)). Let

$X = \{1, 2, 3, 4, s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}$ be the D_4 -set of Example 1.3, where the group $G = D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$. The action of D_4 on X can be seen in Table 2.

First, let's find the fixed sets $\text{Fix}(\sigma)$ for each $\sigma \in D_4$, that is $\text{Fix}(\rho_0), \text{Fix}(\rho_1), \dots, \text{Fix}(\delta_2)$. Recall that ρ_i corresponds to rotating the square counterclockwise $\frac{i\pi}{2}$ radians through C , μ_i corresponds to flipping the square on the axis m_i , and δ_i corresponds to flipping the square on the diagonal d_i . The fixed set $\text{Fix}(\rho_0) = X$ because ρ_0 leaves the square in its original position. Thus, all elements of X stay fixed. Then ρ_1 rotates the square $\frac{\pi}{2}$ radians, so everything changes positions except the center. Thus, $\text{Fix}(\rho_1) = \{C\}$. Continuing in the same fashion, $\text{Fix}(\rho_2) = \{m_1, m_2, d_1, d_2, C\}$, and $\text{Fix}(\rho_3) = \{C\}$. Next, μ_1 flips the square on the m_1 axis. Then $\text{Fix}(\mu_1) = \{s_1, s_3, m_1, m_2, C, P_1, P_3\}$. Similarly, μ_2 flips the square on the m_2 axis, so $\text{Fix}(\mu_2) = \{s_2, s_4, m_1, m_2, C, P_2, P_4\}$. Finally, δ_1 flips the square on the diagonal d_1 . Then $\text{Fix}(\delta_1) = \{2, 4, d_1, d_2, C\}$. Likewise, δ_2 flips the square on the diagonal d_2 , and thus, $\text{Fix}(\delta_2) = \{1, 3, d_1, d_2, C\}$.

Now let's find the stabilizer subgroups G_x for each $x \in X$, that is, $G_1, G_2, \dots, G_{P_3}, G_{P_4}$.

The stabilizer of x , denoted by G_x , is the subgroup $G_x = \{g \in G \mid gx = x\}$ by Definition 1.14. By looking at the table, we can determine each stabilizer subgroup.

$$\begin{array}{l}
 G_1 = \{\rho_0, \delta_2\} \\
 G_2 = \{\rho_0, \delta_1\} \\
 G_3 = \{\rho_0, \delta_2\} \\
 G_4 = \{\rho_0, \delta_1\} \\
 G_{s_1} = \{\rho_0, \mu_1\}
 \end{array}
 \left| \begin{array}{l}
 G_{s_2} = \{\rho_0, \mu_2\} \\
 G_{s_3} = \{\rho_0, \mu_1\} \\
 G_{s_4} = \{\rho_0, \mu_2\} \\
 G_{m_1} = \{\rho_0, \rho_2, \mu_1, \mu_2\} \\
 G_{m_2} = \{\rho_0, \rho_2, \mu_1, \mu_2\}
 \end{array} \right.
 \left| \begin{array}{l}
 G_{d_1} = \{\rho_0, \rho_2, \delta_1, \delta_2\} \\
 G_{d_2} = \{\rho_0, \rho_2, \delta_1, \delta_2\} \\
 G_C = G = D_4 \\
 G_{P_1} = \{\rho_0, \mu_1\} \\
 G_{P_2} = \{\rho_0, \mu_2\}
 \end{array} \right.
 \left| \begin{array}{l}
 G_{P_3} = \{\rho_0, \mu_1\} \\
 G_{P_4} = \{\rho_0, \mu_2\}
 \end{array} \right.$$

Finally, let's determine the orbits in X under D_4 . The orbits in X under D_4 are defined as the subset of X , $O(x) = \{gx \mid g \in D_4\}$ by Definition 1.13. By this definition, we can see that $O(1)$, for example, is $\{1, 2, 3, 4\}$ because $\rho_0(1) = 1$, $\rho_1(1) = 2$, $\rho_2(1) = 3$, $\rho_3(1) = 4$, $\mu_1(1) = 2$, $\mu_2(1) = 4$, $\delta_1(1) = 3$, and $\delta_2(1) = 1$. Then $O(1) = O(2) = O(3) = O(4) = \{1, 2, 3, 4\}$. Similarly, $O(s_1) = \{s_1, s_2, s_3, s_4\} = O(s_2) = O(s_3) = O(s_4)$, $O(m_1) = \{m_1, m_2\} = O(m_2)$, $O(d_1) = \{d_1, d_2\} = O(d_2)$, $O(C) = \{C\}$, $O(P_1) = \{P_1, P_2, P_3, P_4\} = O(P_2) = O(P_3) = O(P_4)$.

We should also note that if X is a G -set and $H \leq G$, then X can also be regarded as an H set. Since the elements of G all act on X , clearly the elements of H would act on

X also according to the action of G on X . Thus, X would be an H -set too. However, the orbits in X under H are not necessarily the same as the orbits in X under G . If H is missing an element which provides a particular permutation in G , the orbits could be different. For example, let $X = \{1, 2, 3, 4, s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}$ from Example 1.3 and recall that D_4 acts on X . Now let $H \leq D_4$, such that $H = \{\rho_0, \rho_2, \delta_1, \delta_2\}$. Then $d_1 \in X$, and $O_H(d_1) = \{d_1\}$, but $O_{D_4}(d_1) = \{d_1, d_2\}$. Thus, the orbits are not the same.

Notice that when a group G acts on a set X , the orbits partition X .

Proposition 1.18 ([6]). *Let a group G act on a set X . Define a relation \equiv on X by, given $x, y \in X$, $x \equiv y$ in case there exists $g \in G$ with $y = gx$. Then \equiv is an equivalence relation on X whose equivalence classes are the orbits.*

Proof. Let G act on a set X . Define a relation \equiv on X by, given $x, y \in X$, $x \equiv y$ if there exists $g \in G$ such that $y = gx$. For every $x \in X$, $1x = x$ by Definition 1.1(ii). Thus for every $x \in X$, $x \equiv x$, so \equiv is reflexive.

Let $x, y \in X$ with $x \equiv y$. Then there exists $g \in G$ such that $y = gx$. Then since G is a group and is closed under inverses, $g^{-1} \in G$.

$$\begin{aligned} y &= gx \\ g^{-1}y &= g^{-1}(gx) \\ g^{-1}y &= (g^{-1}g)x \text{ by Definition 1.1(i),} \\ g^{-1}y &= 1x \\ g^{-1}y &= x \text{ by Definition 1.1(ii).} \end{aligned}$$

Thus $y \equiv x$ and \equiv is symmetric. Finally, let $x, y, z \in X$, and assume $x \equiv y$ and $y \equiv z$. Then

there exist $g, h \in G$ such that $y = gx$ and $z = hy$. Then

$$\begin{aligned} y &= gx \\ hy &= h(gx) \\ hy &= (hg)x \text{ by Definition 1.1(i)} \\ z &= (hg)x. \end{aligned}$$

Because G must be closed under multiplication, $hg \in G$, and so $x \equiv z$. Thus \equiv is transitive. Therefore, \equiv forms an equivalence relation on X . Since the orbit of $x \in X$ is defined as $O(x) = \{gx : g \in G\}$, it is easy to see that the equivalence classes under \equiv are the orbits. \square

Corollary 1.19. *Suppose that a group G acts on a set X . Suppose $x, y \in X$ and $g \in G$ such that $y = gx$ (i.e., x and y are in the same orbit). Then $O(x) = O(y)$, and hence $|O(x)| = |O(y)|$.*

Since the orbits partition X , then X is the disjoint union of the orbits.

Corollary 1.20 ([6], Proposition 2.97). *If a group G acts on a set X , then X is the disjoint union of the orbits. If X is finite, then*

$$|X| = \sum_i |O(x_i)|,$$

where one x_i is chosen from each orbit.

Proof. ([6], Proposition 2.97) As proved in Proposition 1.18, the relation on X , given by $x \equiv y$ if there exists $g \in G$ with $y = gx$, is an equivalence relation whose equivalence classes are the orbits. By a well-known theorem, every element of X is in exactly one equivalence class under \equiv . That is, the equivalence classes partition X into mutually disjoint nonempty subsets.

Since the orbits are disjoint, no element in X is counted twice. Therefore, the sum of the orders of all orbits will equal the order of X when X is finite. \square

In fact, for any element $x \in X$, the number of elements in the orbit of x multiplied by the number of elements in the stabilizer of x equals the number of elements in G .

Theorem 1.21 (Orbit-Stabilizer Theorem: [6], Theorem 2.98). *If a group G acts on a set X and $x \in X$, then*

$$|O(x)| = [G : G_x],$$

the index of the stabilizer G_x in G .

Proof. ([6], Theorem 2.98) Let G act on X and let $x \in X$. Let G/G_x denote the family of all left cosets of G_x in G . An element of G/G_x has the form gG_x for $g \in G$. Define $\varphi : G/G_x \rightarrow O(x)$ by $\varphi(gG_x) = gx$. We must show that φ is well-defined. The function φ is well-defined if when $gG_x = hG_x$ for $gG_x, hG_x \in G/G_x$, $\varphi(gG_x) = \varphi(hG_x)$. If $gG_x = hG_x$, by [[7], Exercise 9.13(c)], $h^{-1}g \in G_x$ and similarly $g^{-1}h \in G_x$. Then

$$\begin{aligned} h &= (gg^{-1})h \\ &= g(g^{-1}h) \text{ by associativity in } G_x. \end{aligned}$$

Let $f = g^{-1}h$. Then $fx = x$ by the definition of a stabilizer, and thus

$$\begin{aligned} hx &= (gf)x \\ &= g(fx) \text{ by Definition 1.1(i),} \\ &= gx. \end{aligned}$$

Therefore, $\varphi(gG_x) = \varphi(hG_x)$, and so φ is well-defined. Next we show that φ is an injection.

Let $gG_x, hG_x \in G/G_x$. If $\varphi(gG_x) = \varphi(hG_x)$, then $gx = hx$, and hence

$$\begin{aligned} h^{-1}(gx) &= h^{-1}(hx) \\ (h^{-1}g)x &= (h^{-1}h)x \text{ by Definition 1.1(i),} \\ (h^{-1}g)x &= 1x \\ (h^{-1}g)x &= x \text{ by Definition 1.1(ii).} \end{aligned}$$

Hence, $h^{-1}g \in G_x$ by definition. Let \equiv_{G_x} denote the equivalence relation given by $g \equiv_{G_x} h$ iff

$h^{-1}g \in G$. Then $g \equiv_{G_x} h$, which implies that $gG_x = hG_x$ ([7], Exercise 9.13(c)). Finally, we show that φ is a surjection. If $y \in O(x)$, then $y = gx$ for some $g \in G$, and thus $y = \varphi(gG_x)$. Therefore, φ is a bijection. Hence, $|G/G_x| = |O(x)|$, and since $|G/G_x| = [G : G_x]$ by definition, $|O(x)| = [G : G_x]$ as desired. \square

Then given X , a G -set, and $x \in X$, note that if G is finite, then $|G| = |O(x)| \cdot |G_x|$. This is because $|O(x)| = |G/G_x|$ by Theorem 1.21, so $|O(x)| = |G|/|G_x|$ by Lagrange's Theorem since $|G|$ is finite, and thus $|G| = |O(x)| \cdot |G_x|$. Take $s_1 \in D_3$ as an example from Example 1.16. Let $G = D_3$. Earlier we saw that $O(s_1) = \{s_1, s_2, s_3\}$ and $G_{s_1} = \{\rho_0, \mu_1\}$. Note that the size of each of the orbits corresponds with the order of the group and the order of the stabilizer, as explained by the Orbit-Stabilizer Theorem (Theorem 1.21). Because $|G| = 6$ and $|G_{s_1}| = 2$, $|O(s_1)| = |G|/|G_{s_1}| = 6/2 = 3$, as expected.

We conclude this chapter with a few results on stabilizers. If two elements are in the same orbit, the number of elements in their stabilizers is also equal.

Proposition 1.22 ([6], Exercise 2.99(ii)). *Let G be a finite group acting on a set X . If $x, y \in X$ lie in the same orbit, then $|G_x| = |G_y|$.*

Proof. Let G be a finite group acting on a set X . Assume $x, y \in X$ lie in the same orbit. Then since G is a finite group that acts on X , by the Orbit-Stabilizer Theorem (Theorem 1.21), $|O(x)| = [G : G_x]$ and $|O(y)| = [G : G_y]$. Then

$$|O(x)| = [G : G_x]$$

$$|O(x)| = |G/G_x|$$

$$|O(x)| = |G|/|G_x| \text{ by Lagrange's Theorem since } G \text{ is finite.}$$

and similarly,

$$|O(y)| = [G : G_y]$$

$$|O(y)| = |G/G_y|$$

$$|O(y)| = |G|/|G_y| \text{ by Lagrange's Theorem since } G \text{ is finite.}$$

Then $|G| = |O(x)| \cdot |G_x| = |O(y)| \cdot |G_y|$, and since x and y are in the same orbit, as shown in Corollary 1.19, $|O(x)| = |O(y)|$. Then we are left with $|G_x| = |G_y|$ as desired. \square

As an example, let's again consider the group D_3 . In Example 1.16, we saw that 1, 2, and 3 lie in the same orbit. Then $O(1) = O(2) = O(3) = \{1, 2, 3\}$, so $|O(1)| = |O(2)| = |O(3)| = 3$. Since $|D_3| = 6$, by Theorem 1.21, $|G_1| = |G_2| = |G_3| = 2$.

Next we show that any conjugate of a stabilizer is also a stabilizer.

Proposition 1.23 ([6], Exercise 2.88). *Let G be a group and X a G -set. If $x \in X$, and $h \in G$, then $h(G_x)h^{-1} = G_{hx}$.*

Proof. Let $x \in X$ and $h \in G$. We will first show that $h(G_x)h^{-1} \subseteq G_{hx}$. Let $a \in h(G_x)h^{-1}$. $h(G_x)h^{-1} = \{a \in G \mid (\exists g \in G_x)(a = hgh^{-1})\}$. So fix $g \in G_x$ so that $a = hgh^{-1}$. Since $a = hgh^{-1}$, $ah = hg$. Then we want to show that $a(hx) = hx$.

$$\begin{aligned} a(hx) &= (ah)x \text{ by Definition 1.1(i);} \\ &= (hg)x \text{ by substitution;} \\ &= h(gx) \text{ by Definition 1.1(ii);} \\ &= hx \text{ since } g \in G_x, \end{aligned}$$

and thus $a \in G_{hx}$.

Now we will show that $G_{hx} \subseteq h(G_x)h^{-1}$. Assume that $a \in G_{hx}$. Then $a(hx) = hx$. We want to show that there exists $g \in G$ such that $a = hgh^{-1}$ and g has the property that $gx = x$.

Let $g = h^{-1}ah$. Then

$$\begin{aligned}g &= h^{-1}ah \\hg &= ah \\hgh^{-1} &= a,\end{aligned}$$

because a, g , and h are in the group G . From this we can see,

$$\begin{aligned}gx &= (h^{-1}ah)x \\&= ((h^{-1}a)h)x \\&= (h^{-1}a)(hx) \text{ by Definition 1.1(i);} \\&= h^{-1}(a(hx)) \text{ by Definition 1.1(i);} \\&= h^{-1}(hx) \\&= (h^{-1}h)x \text{ by Definition 1.1(i);} \\&= x\end{aligned}$$

Thus, since $g \in G_x$ and $a = hgh^{-1}$, $a \in hG_xh^{-1}$. Therefore, $h(G_x)h^{-1} = G_{hx}$. \square

When G acts on a set X and when x, y are in the same orbit, the stabilizers of x and y are conjugates.

Proposition 1.24 ([6], Exercise 2.99(i)). *Let a group G act on a set X , and suppose that $x, y \in X$ lie in the same orbit: $y = gx$ for some $g \in G$. Then $G_y = gG_xg^{-1}$.*

Proof. Suppose $x, y \in X$ lie in the same orbit so that $y = gx$ for some $g \in G$. We will first show that $G_y \subseteq gG_xg^{-1}$. Let's assume that $a \in G_y$. Then $ay = y$. We want to show that there

exists $h \in G$ such that $a = ghg^{-1}$, and h has the property that $hx = x$. Let $h = g^{-1}ag$. Then

$$\begin{aligned} h &= g^{-1}ag \\ gh &= ag \\ ghg^{-1} &= a, \end{aligned}$$

because a, g , and h are in the group G . From this we can see,

$$\begin{aligned} hx &= (g^{-1}ag)x \\ &= ((g^{-1}a)g)x \\ &= (g^{-1}a)(gx) \text{ by Definition 1.1(i);} \\ &= (g^{-1}a)y \\ &= g^{-1}(ay) \text{ by Definition 1.1(i);} \\ &= g^{-1}y \text{ since } a \in G_y; \\ &= g^{-1}(gx) \\ &= (g^{-1}g)x \text{ by Definition 1.1(i);} \\ &= x, \end{aligned}$$

as desired. Thus, since $h \in G_x$ and $a = ghg^{-1}$, $a \in gG_xg^{-1}$. Therefore, $G_y \subseteq gG_xg^{-1}$.

Next we will show that $gG_xg^{-1} \subseteq G_y$. Assume that $a \in gG_xg^{-1}$. We know $g(G_x)g^{-1} = \{a \in G \mid (\exists h \in G_x)(a = ghg^{-1})\}$, so fix $h \in G_x$ so that $a = ghg^{-1}$. Since $a = ghg^{-1}$, $ag = gh$.

We want to show that $ay = y$.

$$\begin{aligned} ay &= a(gx) \\ &= (ag)x \text{ by Definition 1.1(i);} \\ &= (gh)x \\ &= g(hx) \text{ by Definition 1.1(ii);} \\ &= gx \text{ since } h \in G_x; \\ &= y \end{aligned}$$

Thus, $a \in G_y$.

Therefore, $G_y = gG_xg^{-1}$.

□

2 Burnside's Theorem

William Burnside was born on July 2, 1852 in London. He attended college and was later appointed lecturer at Cambridge University. Later he took a position at the Royal Naval College at Greenwich. During his life, Burnside wrote more than 150 research papers in many fields, but he is best remembered for his work in group theory. Because of his emphasis on abstract approaches, Burnside is considered by many as the first pure group theorist [3].

Although the theorem that is the focus of this paper is called “Burnside’s Theorem,” William Burnside was not the originator of the proof. The approach for counting unique permutations was proved by Georg Frobenius in 1887. However, the approach did not become widely known until it appeared in a book on group theory by William Burnside in 1911. Thus, it became known as “Burnside’s Theorem”, although it is also called the “Cauchy-Frobenius Lemma” or “not-Burnside’s Lemma” [3]. Many counting problems can be solved using observation, but more complicated ones require a more sophisticated approach that Burnside’s Theorem provides.

Theorem 2.1 ([6], **Theorem 2.113**). *Let G be a finite group that acts on a finite set X . If N is the number of orbits, then*

$$N = \frac{1}{|G|} \sum_{\tau \in G} |\text{Fix}(\tau)|,$$

where $|\text{Fix}(\tau)|$ is the number of $x \in X$ fixed by τ .

Proof. ([6], Theorem 2.113) Create a table and list the elements of X as follows: Choose $x_1 \in X$, and then list all the elements x_1, x_2, \dots, x_r in the orbit $O(x_1)$. Then choose $x_{r+1} \notin O(x_1)$. List all elements in $O(x_{r+1})$: x_{r+1}, x_{r+2}, \dots . Each element lies in the first row of the table, acting as a label. Continue this procedure until all of the elements of X are listed. In the leftmost column of the table, list the elements $\tau_1, \tau_2, \dots, \tau_n$ of G . To fill the table, form the following array, where

$$f_{i,j} = \begin{cases} 1 & \text{if } \tau_i \text{ fixes } x_j, \\ 0 & \text{if } \tau_i \text{ moves } x_j. \end{cases}$$

	x_1	x_2	\cdots	x_{r+1}	x_{r+2}	\cdots
τ_1	$f_{1,1}$	$f_{1,2}$	\cdots	$f_{1,r+1}$	$f_{1,r+2}$	\cdots
τ_2	$f_{2,1}$	$f_{2,2}$	\cdots	$f_{2,r+1}$	$f_{2,r+2}$	\cdots
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
τ_i	$f_{i,1}$	$f_{i,2}$	\cdots	$f_{i,r+1}$	$f_{i,r+2}$	\cdots
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
τ_n	$f_{n,1}$	$f_{n,2}$	\cdots	$f_{n,r+1}$	$f_{n,r+2}$	\cdots

The number $|\text{Fix}(\tau_i)|$ is the number of x fixed by τ_i . Then $|\text{Fix}(\tau_i)|$ equals the number of 1's in the i th row of the array. Thus, $\sum_{\tau \in G} |\text{Fix}(\tau)|$ is the total number of 1's in the array.

Now let's look at the columns. The number of τ_i that fix x_1 equals the number of 1's in the first column. Thus, $|G_{x_1}|$, the size of the stabilizer of x_1 , equals the number of 1's in column one. Similarly, $|G_{x_2}|$ equals the number of 1's in column two. Since x_1 lies in the same orbit as x_2 , $|G_{x_1}| = |G_{x_2}|$ by Proposition 1.24. Since there are r columns labeled by the $x_i \in O(x_1)$, $r|G_{x_1}| = |O(x_1)| \cdot |G_{x_1}| = ([G : G_{x_1}]|G_{x_1}|)$, by the Orbit-Stabilizer Theorem (Theorem 1.21). Then $([G : G_{x_1}]|G_{x_1}| = (|G/G_{x_1}|)|G_{x_1}| = (|G|/|G_{x_1}|)|G_{x_1}| = |G|$.

Analogously, every orbit of $x \in X$ corresponds to exactly $|G|$ 1's in the table. Thus, if there are N orbits, there are $N|G|$ 1's in the array. Then $N|G| = \sum_{\tau \in G} |\text{Fix}(\tau)|$. Therefore,

$$N = \frac{1}{|G|} \sum_{\tau \in G} |\text{Fix}(\tau)|.$$

□

In Proposition 1.8, we showed that if a group G acts on a set of n elements and C is any set, then G also acts on the set of all n -tuples formed from the elements of C . This proposition can be reworded for use in coloring problems.

Proposition 2.2 ([6]). *Note that this statement corrects an error in the source.*

If a group G acts on a set $X = \{1, \dots, n\}$, and if C is a set of q colors, then G acts on the set C^n of all n -tuples of colors by

$$\tau(c_1, \dots, c_n) = (c_{\tau^{-1}1}, c_{\tau^{-1}2}, \dots, c_{\tau^{-1}n}) \text{ for all } \tau \in G.$$

Definition 2.3 ([6]). An orbit of $(c_1, \dots, c_n) \in C^n$ is called a (\mathbf{q}, G) -coloring of X .

Using this notation along with Burnside's Theorem, let's consider a simple example of counting the number of ways to distinctly color the corners of a hexagon with two colors.

Example 2.4 ([3], based on Exercise 29.1). Determine the number of ways in which the six corners of a hexagon can be colored with two colors. It is permissible to use a single color on all six corners.

Let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the six corners of the hexagon. Let C consist of the two colors, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where c_1, c_2, c_3, c_4, c_5 and c_6 are each one of two colors.

Imagine that the hexagon is sitting on a plane. Since the hexagon is in a 2-dimensional environment, the only way to move the hexagon involves rotating it. Let τ be the permutation in S_6 associated with rotating the hexagon 60° clockwise.

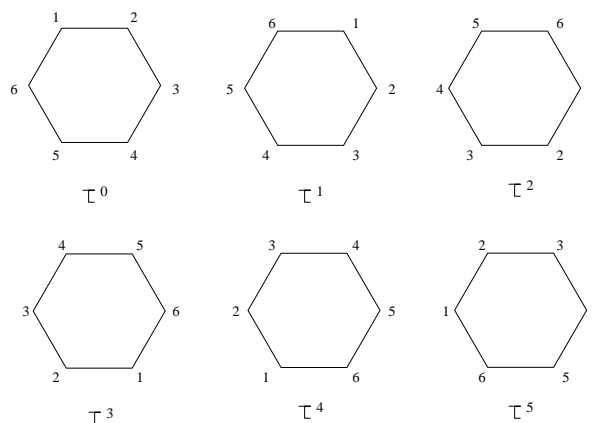


Figure 3: The rotation of a hexagon with corners 1, 2, 3, 4, 5, 6

The permutations in S_6 can be written in cycle notation as:

$$\tau = (1, 2, 3, 4, 5, 6)$$

$$\tau^2 = (1, 3, 5)(2, 4, 6)$$

$$\tau^3 = (1, 4)(2, 5)(3, 6)$$

$$\tau^4 = (1, 5, 3)(2, 6, 4)$$

$$\tau^5 = (1, 6, 5, 4, 3, 2)$$

Figure 3 demonstrates how the five rotations act on the hexagon.

Let $G = \langle \tau \rangle$, which acts on X . Then $|G| = |\langle \tau \rangle| = 6$. We can view the coloring of the corners of the hexagon as a $(2, G)$ -coloring of X . Since G acts on X , by Proposition 2.2, G acts on C^6 , the set of all 6-tuples of colors. The orbit of $(c_1, c_2, c_3, c_4, c_5, c_6) \in C^6$ corresponds to the six ways of viewing the hexagon. By Burnside's Theorem, the number N of ways that the corners can be colored is

$$N = \frac{1}{6} [|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)|].$$

The identity permutation (1) fixes every $x \in C^6$, so $|\text{Fix}((1))| = 2^6$ since each of the six corners can be one of two colors. If $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ is fixed by τ , then $c_1 = c_2 = c_3 = c_4 = c_5 = c_6$; there are two choices of color for c_1 , but c_2, c_3, c_4, c_5 and c_6 are bound to c_1 , so c_2, c_3, c_4, c_5 and c_6 have only one choice of color each. Thus, $|\text{Fix}(\tau)| = 2^1$. The same is the case for $|\text{Fix}(\tau^5)|$ because $\tau^5 = \tau^{-1}$, so $|\text{Fix}(\tau^5)| = |\text{Fix}(\tau)| = 2^1$. Next, if $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ is fixed by τ^2 , $c_1 = c_3 = c_5$ and $c_2 = c_4 = c_6$. Then c_1 and c_2 each have two choices of color. However, c_3 and c_5 are bound to c_1 , and c_4 and c_6 are bound to c_2 , so these elements have only one color choice each. It then follows that $|\text{Fix}(\tau^2)| = 2^2$. The same is the case for $|\text{Fix}(\tau^4)|$ since $\tau^4 = \tau^{-2}$, so $|\text{Fix}(\tau^4)| = |\text{Fix}(\tau^2)| = 2^2$. Finally, if $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ is fixed by τ^3 , $c_1 = c_4$, $c_2 = c_5$, and $c_3 = c_6$. Then c_1, c_2 , and c_3 each have two choices of color. However, c_4 is bound to c_1 , c_5 is bound to c_2 , and c_6 is bound to c_3 , so these elements have only one color choice each. It then follows that $|\text{Fix}(\tau^3)| = 2^3$.

Therefore the number of ways in which the six corners of a hexagon can be colored with two colors is

$$N = \frac{1}{6} [2^6 + 2^1 + 2^2 + 2^3 + 2^2 + 2^1] = \frac{84}{6} = 14.$$

As another example, how many striped flags are there having five stripes of equal width, each of which can be colored one of q colors?

Example 2.5 ([6], based on example following Theorem 2.113). Let C^5 be the set of all 5-tuples of colors; if $x \in C^5$, then

$$x = (c_1, c_2, c_3, c_4, c_5)$$

where $c_1, c_2, c_3, c_4,$ and c_5 are each one of q colors.

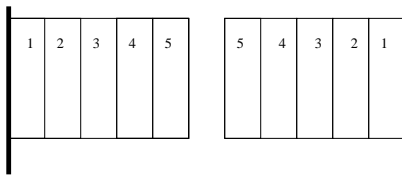


Figure 4: Flipping over of a flag with 5 stripes.

Imagine that the flag is on a flagpole. A flag can only move by turning over to the other side of the flag pole, which reveals a new coloring. This is represented in Figure 4.

Let τ be the permutation that reverses all the indices, representing flipping the flag over.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1, 5)(2, 4)(3), \quad \text{a product of disjoint cycles.}$$

The cyclic group $G = \langle \tau \rangle$ defines the action of turning over each 5-tuple x of colored stripes. This is a group action by Proposition 1.8.

Since $\tau = (1, 5)(2, 4)(3)$, $\tau^2 = (1, 5)(2, 4)(3)(1, 5)(2, 4)(3) = (1)(2)(3)(4)(5) = \tau^0$. Then $|G| = |\langle \tau \rangle| = 2$, and so the orbit of any 5-tuple x consists of either 1 or 2 elements; either τ fixes x or it does not. The number of distinct flags is thus the number N of orbits.

By Burnside's theorem,

$$N = \frac{1}{2} [|\text{Fix}((1))| + |\text{Fix}(\tau)|].$$

The identity permutation (1) fixes every $x \in C^n$, so $|\text{Fix}((1))| = q^5$ since there are q colors, where each of the five stripes can be one of q colors. If $x = (c_1, c_2, c_3, c_4, c_5)$ is fixed by $\tau = (1, 5)(2, 4)(3)$, then $c_1 = c_5$ and $c_2 = c_4$. It follows that $|\text{Fix}(\tau)| = q^3$, for there are q choices for each of c_1, c_2, c_3 . Since the color c_5 is bound to c_1 and c_4 is bound to c_2 , c_4 and c_5

only have one choice for a color. Thus the number of flags is

$$N = \frac{1}{2}(q^5 + q^3).$$

Now let's consider a more general case of the flag problem.

Example 2.6 ([6], Exercise 2.100). We show that there are

$$N = \frac{1}{2} \left(q^n + q^{\lceil \frac{n}{2} \rceil} \right)$$

flags with n stripes, each of which can be colored any one of q given colors.

There are two cases.

1. n is even
2. n is odd

First consider when n is even.

Let $X = \{1, 2, \dots, n\}$ represent the stripes of the flag. Let C be the set of q colors, and let C^n be the set of all n -tuples of colors; if $x \in C^n$, then $x = (c_1, c_2, \dots, c_n)$ where c_1, c_2, \dots, c_n are each one of q colors. Let τ be the permutation that reverses all the indices as in Example 2.5, representing turning the flag over on the flag pole:

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix} = (1, n) (2, n-1) \dots \left(\frac{n}{2}, \frac{n}{2} + 1 \right).$$

The cyclic group $G = \langle \tau \rangle$ acts on C^n as shown in Example 2.5.

$$\begin{aligned} \tau &= (1, n) (2, n-1) \dots \left(\frac{n}{2}, \frac{n}{2} + 1 \right), \quad \text{so} \\ \tau^2 &= (1, n) (2, n-1) \dots \left(\frac{n}{2}, \frac{n}{2} + 1 \right) (1, n) (2, n-1) \dots \left(\frac{n}{2}, \frac{n}{2} + 1 \right) \\ &= \tau^0. \end{aligned}$$

By multiplication of disjoint cycles, $\frac{n}{2} + 1$ gets sent to $\frac{n}{2}$, then back to itself. Similarly, $\frac{n}{2}$

gets sent to $\frac{n}{2} + 1$ and then back to itself. The same happens for each element $i \in X$, where $1 \leq i \leq n$. Thus, $\tau^2 = \tau^0$, so $|G| = |\langle \tau \rangle| = 2$. The orbit of any n -tuple x consists of either 1 or 2 elements; either τ fixes x or it does not. The number of distinct flags is thus the number N of orbits.

By Burnside's Theorem, $N = \frac{1}{2} [|\text{Fix}((1))| + |\text{Fix}(\tau)|]$. The identity permutation (1) fixes every $x \in C^n$, so $|\text{Fix}((1))| = q^n$ since each of the n stripes can be colored one of q colors. If $x = (c_1, c_2, \dots, c_n)$ is fixed by $\tau = (1, n)(2, n-1) \dots (\frac{n}{2}, \frac{n}{2} + 1)$, then $c_1 = c_n$, $c_2 = c_{n-1}$, $c_3 = c_{n-2}$, \dots , and $c_{\frac{n}{2}} = c_{\frac{n}{2}+1}$. It follows that $|\text{Fix}(\tau)| = q^{\frac{n}{2}}$, for there are q choices for each of $c_1, c_2, \dots, c_{\frac{n}{2}}$. The number of flags is thus:

$$N = \frac{1}{2} (q^n + q^{\frac{n}{2}}).$$

Next we consider the case when n is odd.

Let $X = \{1, 2, \dots, n\}$ represent the stripes of the flag. Let C be the set of q colors, and again let C^n be the set of all n -tuples of colors; if $x \in C^n$, then $x = (c_1, c_2, \dots, c_n)$ where c_1, c_2, \dots, c_n are each one of q colors. Let τ be the permutation that reverses all the indices:

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix} = (1, n)(2, n-1) \dots \left(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil \right).$$

The cyclic group $G = \langle \tau \rangle$ acts on C^n .

$$\begin{aligned} \tau &= (1, n)(2, n-1) \dots \left(\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil \right), \text{ so} \\ \tau^2 &= \tau^0. \end{aligned}$$

By multiplication of disjoint cycles, $\left\lceil \frac{n}{2} \right\rceil$ gets sent only to itself. As an example of the rest of the elements, $\left\lceil \frac{n}{2} \right\rceil + 1$ gets sent to $\left\lceil \frac{n}{2} \right\rceil - 1$, then back to itself. Each element $i \in X$, where $1 \leq i \leq n$, gets sent to itself in the product. Thus, $\tau^2 = \tau^0$, so $|G| = |\langle \tau \rangle| = 2$. The orbit of any n -tuple x consists of either 1 or 2 elements; either τ fixes x or it does not. The number of distinct flags is thus the number N of orbits.

By Burnside's Theorem, $N = \frac{1}{2} [|\text{Fix}((1))| + |\text{Fix}(\tau)|]$. The identity permutation (1) fixes every $x \in C^n$, so $|\text{Fix}((1))| = q^n$ since each of the n stripes can be colored one of q colors. If $x = (c_1, c_2, \dots, c_n)$ is fixed by $\tau = (1, n)(2, n-1) \dots (\lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil + 1)(\lceil \frac{n}{2} \rceil)$, then $c_1 = c_n$, $c_2 = c_{n-1}$, $c_3 = c_{n-2}, \dots$, and $c_{\lceil \frac{n}{2} \rceil + 1} = c_{\lceil \frac{n}{2} \rceil - 1}$. Because $c_{\lceil \frac{n}{2} \rceil}$ is not bound to any other color in x , $c_{\lceil \frac{n}{2} \rceil}$ can be any of q colors.

It follows that $|\text{Fix}(\tau)| = q^{\lceil \frac{n}{2} \rceil}$, for there are q choices for each of $c_1, c_2, \dots, c_{\lceil \frac{n}{2} \rceil}$. The number of flags is thus

$$N = \frac{1}{2} \left(q^n + q^{\lceil \frac{n}{2} \rceil} \right).$$

We will now recast Burnside's Theorem to make it easier to apply. Recall that the set $X = \{1, \dots, n\}$ contains the elements under the action of a group G , and C^n is the set of all n -tuples of possible colorings. In order to apply Burnside's Theorem, we wrote each element $\tau \in G$ as a product of disjoint cycles, and we needed to know the number of disjoint cycles.

Definition 2.7 ([7]). Two cycles $\alpha = (x_1, x_2, \dots, x_r), \beta = (y_1, y_2, \dots, y_s) \in S_n$ are **disjoint** if no element of $\{1, 2, \dots, n\}$ is moved by both α and β . In other words, two cycles are disjoint if for every $i \in \{1, 2, \dots, n\}$, one of the following is true: $\alpha(i) = i$ and $\beta(i) = i$; if $\alpha(i) \neq i$, then $\beta(i) = i$; or if $\beta(i) \neq i$, then $\alpha(i) = i$.

By Theorem 5.1 [3], every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. When writing τ as a product of disjoint cycles, it is important to write the complete factorization of τ .

Definition 2.8 ([5]). A **complete factorization** of a permutation α into a product of disjoint cycles is a factorization of τ that contains the 1-cycle (i) for every i fixed by τ .

The importance of including the 1-cycles can be observed in Example 2.5. In this example, when calculating $|\text{Fix}(\tau)|$, one must remember that c_3 is not bound to any other element of the n -tuple. Thus, c_3 can be any one of q colors, just as c_1 and c_2 . By not including 1-cycles, we would have forgotten to include the possible colorings of c_3 in the calculation of $|\text{Fix}(\tau)|$.

Therefore, all permutations should be completely factored as a product of disjoint cycles.

As an example, consider

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}.$$

Written as a product of disjoint cycles, but not as a complete factorization, $\tau = (1, 2, 4)$.

However, we would want to write τ as $\tau = (1, 2, 4)(3)(5)$.

Let q equal the number of colors in a set. We will now show that $|\text{Fix}(\tau)|$ equals $q^{t(\tau)}$, where $t(\tau)$ is the number of cycles in the complete factorization of τ as a product of disjoint cycles. This will then allow us to recast Burnside's Theorem.

Theorem 2.9 ([6], Lemma 2.115). *Let C be a set of q colors, and let G be a subgroup of S_n . If $\tau \in G$, then*

$$|\text{Fix}(\tau)| = q^{t(\tau)},$$

where $t(\tau)$ is the number of cycles in the complete factorization of τ as a product of disjoint cycles.

Proof. ([6], Lemma 2.115) Let $x = (c_1, c_2, \dots, c_n)$ be an n -tuple fixed by τ . By Proposition 2.2, $\tau(c_1, \dots, c_n) = (c_{\tau^{-1}1}, \dots, c_{\tau^{-1}n}) = (c_1, \dots, c_n)$. Then $c_{\tau^{-1}i} = c_i$ for all $1 \leq i \leq n$. The group $G = \langle \tau \rangle$ acts on $\{1, 2, \dots, n\}$. Given $i \in \{1, \dots, n\}$, $O(i) = \{\tau^k i : k \in \mathbb{Z}\}$. Then all associated symbols involved in the orbit of i under the action of $\langle \tau \rangle$ have the same color. In other words, if $O(2) = \{2, 3, 4\}$, the corresponding symbols in x , c_2, c_3, c_4 , will be bound to each other (i.e., $c_2 = c_3 = c_4$). Thus, for all k , $c_{\tau^k i}$ is the same color as c_i .

Let the complete factorization of τ as a product of disjoint cycles be $\tau = \beta_1 \cdots \beta_{t(\tau)}$ where $t(\tau)$ is the number of cycles. Since τ is a product of disjoint cycles, i occurs in exactly one of the cycles, say β_j . Let $\beta_j = (i_1, i_2, \dots, i_r)$, where $i = i_1$. Then $i_{k+1(\text{mod } r)} = \tau^k(i_1)$ for all k . Therefore $O(i) = \{i_1, i_2, \dots, i_r\}$ and $c_{i_1}, c_{i_2}, \dots, c_{i_r}$ all are the same color. Thus, for an n -tuple fixed by τ , all symbols involved in each of the $t(\tau)$ cycles must have the same color. Therefore, as there are q colors from which to choose, there are then $q^{t(\tau)}$ n -tuples fixed by τ . \square

Thus, the cycle structure of a permutation τ allows one to calculate $|\text{Fix}(\tau)|$. Now we can restate Burnside's Theorem in the following way.

Corollary 2.10 ([6], **Corollary 2.116**). *Let G be a subgroup of S_n that acts on a finite set X . If N is the number of (q, G) -colorings of X , then*

$$N = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)},$$

where $t(\tau)$ is the number of cycles in the complete factorization of τ as a product of disjoint cycles.

Proof. By Theorem 2.9, $|\text{Fix}(\tau)| = q^{t(\tau)}$. A (q, G) -coloring of X is an orbit of (c_1, \dots, c_n) in the set of all n -tuples of colors. If N is the number of all such colorings, then N equals the number of orbits. Thus, $q^{t(\tau)}$ can be substituted into Burnside's Theorem (Theorem 2.1), and we can rewrite the number of orbits N as

$$N = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}.$$

□

Using this new notation, consider the example of a 5×5 grid with 25 squares, each of which can be colored with one of 2 colors.

Example 2.11 ([6], based on **Example 2.114**). Color each square of a 5×5 grid red or black where any square may be either color. Let X be the set containing the 25 squares and let C contain the two colors, red and black.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

21	16	11	6	1
22	17	12	7	2
23	18	13	8	3
24	19	14	9	4
25	20	15	10	5

Figure 5: Rotation of a 5×5 grid 90° clockwise.

Rotating the grid clockwise 90° possibly yields a new coloring, as can be seen in Figure 5. We

are going to use the corollary of Burnside's Theorem (Corollary 2.10) to compute the number of distinct colorings of 5×5 grids.

Let τ be the clockwise rotation of the grid by 90° . Then τ can be written in cycle notation as:

$$\tau = (1, 5, 25, 21)(2, 10, 24, 16)(3, 15, 23, 11)(4, 20, 22, 6)(7, 9, 19, 17)(8, 14, 18, 12)(13)$$

Multiplying the cycles, we get:

$$\begin{aligned} \tau &= (1, 5, 25, 21)(2, 10, 24, 16)(3, 15, 23, 11)(4, 20, 22, 6)(7, 9, 19, 17)(8, 14, 18, 12)(13) \\ \tau^2 &= (1, 25)(5, 21)(2, 24)(10, 16)(3, 23)(15, 11)(4, 22)(20, 6)(7, 19)(9, 17)(8, 18)(14, 12)(13) \\ \tau^3 &= (1, 21, 25, 5)(2, 16, 24, 10)(3, 11, 23, 15)(4, 6, 22, 20)(7, 17, 19, 9)(8, 12, 18, 14)(13) \\ \tau^4 &= (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)(17)(18)(19)(20)(21)(22)(23)(24)(25) \\ &= \tau^0 \end{aligned}$$

The grid is 1-sided, as opposed to the flags in Exercises 2.5 and 2.6, which are 2-sided. We are visualizing the grid as living in a two dimensional plane. Thus, the only way to change the position of the squares is to rotate the grid. The rotation of the grid 90° clockwise is τ^1 , 180° is τ^2 , 270° is τ^3 , and 360° is τ^4 . Thus $\tau^0 = \tau^4$, which is the original position.

Because X is the set of squares, we can view the grid as a 2-coloring of X . The orbit of $(c_1, \dots, c_{25}) \in C^{25}$ corresponds to the four ways of viewing the board. The group $G = \langle \tau \rangle$ acts on X , so by Proposition 2.2, G also acts on C^{25} , the set of all 25-tuples of colors.

By Corollary 2.10, the number of grids is

$$\frac{1}{4} \left[q^{t(\tau^0)} + q^{t(\tau)} + q^{t(\tau^2)} + q^{t(\tau^3)} \right].$$

Using this formula makes it much easier to calculate the distinct number of grids than using the original statement of Burnside's Theorem. There are 25 cycles in the complete factorization of τ^0 since there are 25 elements in the initial grid. Thus, $q^{t(\tau^0)} = 2^{25}$. Next, $q^{t(\tau)} = 2^7$ since there are 7 cycles in the complete factorization of τ . Similarly, $q^{t(\tau^2)} = 2^{13}$ and $q^{t(\tau^3)} = 2^7$.

Therefore, the number N of distinct grids is

$$N = \frac{1}{4} [2^{25} + 2^7 + 2^{13} + 2^7] = 8,390,720.$$

Finally, consider a more general example, an $n \times n$ grid in which each square can be either red or black.

Example 2.12 ([6], Exercise 2.101). Let X be the set containing the n^2 squares and let C contain the two colors, red and black. As in Example 2.11, rotating the grid clockwise 90° possibly yields a new view of the board. We are going to use the corollary of Burnside's Theorem (Corollary 2.10) to compute the number of distinct $n \times n$ grids.

The grid is in two-dimensional space so, the only way to change the position of the squares is to rotate the grid. Let τ be the 90° clockwise rotation of the grid. Because X is the set of squares, we can view the grid as a 2-coloring of X . The group $G = \langle \tau \rangle$ acts on X , and the orbit of $(c_1, \dots, c_{n^2}) \in C^{n^2}$ corresponds to the four ways of viewing the board. Since G acts on X , by Proposition 1.8, G also acts on C^{n^2} , the set of all n^2 -tuples of colors.

By Corollary 2.10, the number of distinct grids is

$$N = \frac{1}{4} [q^{t(\tau^0)} + q^{t(\tau)} + q^{t(\tau^2)} + q^{t(\tau^3)}].$$

We show that

$$N = \frac{1}{4} \left[2^{n^2} + 2^{\lfloor \frac{n^2+1}{2} \rfloor} + 2 \cdot 2^{\lfloor \frac{n^2+3}{4} \rfloor} \right].$$

In particular, we prove by induction that

$$t(\tau) = \left\lfloor \frac{n^2 + 3}{4} \right\rfloor$$

and

$$t(\tau^2) = \left\lfloor \frac{n^2 + 1}{2} \right\rfloor.$$

The formula for N follows immediately.

Let's first consider the case where n is even. Then $n = 2l$ for some $l \in \mathbb{Z}^+$. We do induction on l . First, test the base case when $l = 1$. Then we have a 2×2 grid. We claim

$$\begin{aligned} t(\tau) &= \left\lfloor \frac{(2 \cdot 1)^2 + 3}{4} \right\rfloor \\ &= \left\lfloor \frac{2^2 + 3}{4} \right\rfloor \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} t(\tau^2) &= \left\lfloor \frac{(2 \cdot 1)^2 + 1}{2} \right\rfloor \\ &= \left\lfloor \frac{2^2 + 1}{2} \right\rfloor \\ &= 2. \end{aligned}$$

Since $\tau \in G$ is the permutation that rotates the grid 90° clockwise, $\tau = (1, 2, 4, 3)$, as can be seen in Figure 6. Then $\tau^2 = (1, 4)(2, 3)$. Because $t(\tau)$ is the number of cycles in the complete factorization of τ and $t(\tau^2)$ is the number of cycles in the complete factorization of τ^2 , $t(\tau) = 1$ and $t(\tau^2) = 2$, which satisfies the claim. Thus, the base case holds.

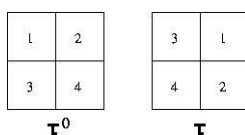


Figure 6: Rotation of a 2×2 grid 90°

Next, let $k \geq 1$, and assume the result for the $2k \times 2k$ grid. That is, assume that

$$t(\tau) = \left\lfloor \frac{(2k)^2 + 3}{4} \right\rfloor$$

and

$$t(\tau^2) = \left\lfloor \frac{(2k)^2 + 1}{2} \right\rfloor.$$

Show that the claim holds for $k + 1$. In other words, show that the number of cycles in the

complete factorization of τ for the $2(k + 1) \times 2(k + 1)$ grid is

$$t(\tau) = \left\lfloor \frac{(2(k + 1))^2 + 3}{4} \right\rfloor$$

and that the number of cycles in the complete factorization of τ^2 is

$$t(\tau^2) = \left\lfloor \frac{(2(k + 1))^2 + 1}{2} \right\rfloor.$$

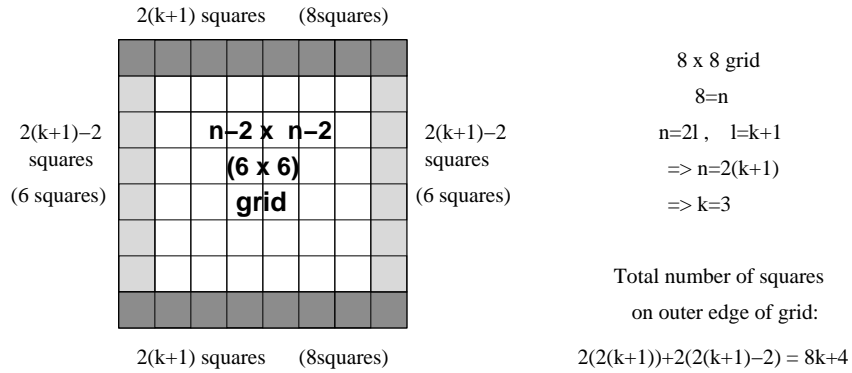


Figure 7: An $2(k + 1) \times 2(k + 1)$ grid contains an outer edge of $8k + 4$ squares and a $2k \times 2k$ grid inside.

As can be observed in Figure 7, a grid with $l = k + 1$ has $(2(k + 1))^2$ squares. The outer edge of the grid is comprised of $2(k + 1) + 2(k + 1) + (2(k + 1) - 2) + (2(k + 1) - 2)$ or $8k + 4$ squares. Inside this outer edge lies the $2k \times 2k$ grid. Thus, to compute the number of cycles in the complete factorization of τ for the $2(k + 1) \times 2(k + 1)$ grid, we need only consider the number of cycles added by G 's action on the outer edge of squares.

When τ acts on the outer squares, the square in the top-left moves to the top-right, the square originally in the top-right moves to the bottom-right, the square originally in the bottom-right corner moves to the bottom-left corner, and the square originally in the bottom-left corner moves to the top-left corner. Thus, the top-left square is in a cycle with the other squares at the corners. The same sort of movement happens to the rest of the outer squares when the grid is rotated. Thus, the permutation caused by τ on the outer edge of the grid can be described

by $n - 1 = 2l - 1 = 2(k + 1) - 1 = 2k + 1$ disjoint 4-cycles. Then τ 's action on the entire $2(k + 1) \times 2(k + 1)$ grid can be described by $2k + 1$ disjoint 4-cycles in addition to the cycles describing τ 's action on the interior $2k \times 2k$ grid. Thus,

$$\begin{aligned}
t(\tau) &= \left\lfloor \frac{(2k)^2 + 3}{4} \right\rfloor + 2k + 1 \quad \text{by the induction hypothesis,} \\
&= \left\lfloor \frac{(2k)^2 + 3}{4} \right\rfloor + \frac{8k + 4}{4} \\
&= \left\lfloor \frac{4k^2 + 8k + 4 + 3}{4} \right\rfloor \\
&= \left\lfloor \frac{(2(k + 1))^2 + 3}{4} \right\rfloor.
\end{aligned}$$

We observed in Example 2.11 that the square of a 4-cycle is a product of two disjoint 2-cycles. Since τ 's action on the outer edge can be described in $2k + 1$ disjoint 4-cycles, τ^2 's permutation of the outer edge can be described by $2(2k + 1) = 4k + 2$ disjoint 2-cycles. Then τ^2 's action on the entire $2(k + 1) \times 2(k + 1)$ grid can be described by $4k + 2$ disjoint 4-cycles in addition to the cycles describing τ^2 's action on the interior $2k \times 2k$ grid. Then

$$\begin{aligned}
t(\tau^2) &= \left\lfloor \frac{(2k)^2 + 1}{2} \right\rfloor + 4k + 2 \quad \text{by the induction hypothesis,} \\
&= \left\lfloor \frac{(2k)^2 + 1}{2} \right\rfloor + \frac{8k + 4}{2} \\
&= \left\lfloor \frac{4k^2 + 8k + 4 + 1}{2} \right\rfloor \\
&= \left\lfloor \frac{(2(k + 1))^2 + 1}{2} \right\rfloor.
\end{aligned}$$

This completes the induction. Hence, when n is even, $t(\tau) = \left\lfloor \frac{n^2+3}{4} \right\rfloor$, and $t(\tau^2) = \left\lfloor \frac{n^2+1}{2} \right\rfloor$. Clearly, τ^0 fixes all n^2 squares in the grid. Thus, $t(\tau^0) = n^2$. Next, $t(\tau) = t(\tau^{-1})$, and since $\tau^{-1} = \tau^3$, $t(\tau) = t(\tau^3)$. Therefore,

$$N = \frac{1}{4} \left[2^{n^2} + 2 \left\lfloor \frac{n^2+1}{2} \right\rfloor + 2 \cdot 2 \left\lfloor \frac{n^2+3}{4} \right\rfloor \right],$$

as desired.

Next, we consider the case where n is odd. Then $n = 2l + 1$ for some $l \in \mathbb{Z}$. We again do induction on l . First, test the base case when $l = 0$. Then we have a 1×1 grid. We claim

$$\begin{aligned} t(\tau) &= \left\lfloor \frac{(2 \cdot 0 + 1)^2 + 3}{4} \right\rfloor \\ &= \left\lfloor \frac{1^2 + 3}{4} \right\rfloor \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} t(\tau^2) &= \left\lfloor \frac{(2 \cdot 0 + 1)^2 + 1}{2} \right\rfloor \\ &= \left\lfloor \frac{1^2 + 1}{2} \right\rfloor \\ &= 1. \end{aligned}$$

Clearly, $\tau = (1)$ and $\tau^2 = (1)$, so $t(\tau) = t(\tau^2) = 1$. Thus, the base case holds.

Next, let $k \geq 0$, and assume the result for the $(2k + 1) \times (2k + 1)$ grid. That is, assume that

$$t(\tau) = \left\lfloor \frac{(2k + 1)^2 + 3}{4} \right\rfloor$$

and

$$t(\tau^2) = \left\lfloor \frac{(2k + 1)^2 + 1}{2} \right\rfloor.$$

Show that the claim holds for $k + 1$. In other words, show that the number of cycles in the complete factorization of τ for the $(2(k + 1) + 1) \times (2(k + 1) + 1)$ grid is

$$t(\tau) = \left\lfloor \frac{(2(k + 1) + 1)^2 + 3}{4} \right\rfloor$$

and that the number of cycles in the complete factorization of τ^2 is

$$t(\tau) = \left\lfloor \frac{(2(k + 1) + 1)^2 + 1}{2} \right\rfloor.$$

permutation caused by τ on the outer edge of the $(2(k+1)+1) \times (2(k+1)+1)$ grid can be described in $2k+2$ 4-cycles. Since we assumed that $k \geq 0$, $k+1 \geq 1$, so for $k+1$, τ 's action on the outer edge of the $(2(k+1)+1) \times (2(k+1)+1)$ grid can always be described in disjoint 4-cycles. Thus, τ^2 's action on the outer edge can be described by $2(2k+2) = 4k+4$ disjoint 2-cycles. Then τ^2 's action on the entire grid can be described by $4k+4$ disjoint 2-cycles in addition to the cycles describing τ^2 's action on the interior $(2k+1) \times (2k+1)$ grid. Then

$$\begin{aligned}
t(\tau^2) &= \left\lfloor \frac{(2k+1)^2 + 1}{2} + 4k + 4 \right\rfloor \quad \text{by the induction hypothesis,} \\
&= \left\lfloor \frac{(2k+1)^2 + 1}{2} + \frac{8k+8}{2} \right\rfloor \\
&= \left\lfloor \frac{(2k+1)^2 + 1 + 8k + 8}{2} \right\rfloor \\
&= \left\lfloor \frac{4k^2 + 4k + 1 + 1 + 8k + 8}{2} \right\rfloor \\
&= \left\lfloor \frac{4k^2 + 12k + 9 + 1}{2} \right\rfloor \\
&= \left\lfloor \frac{(2(k+1)+1)^2 + 1}{2} \right\rfloor.
\end{aligned}$$

This completes the induction. Hence, when n is odd, $t(\tau) = \left\lfloor \frac{n^2+3}{4} \right\rfloor$, and $t(\tau^2) = \left\lfloor \frac{n^2+1}{2} \right\rfloor$. Clearly, τ^0 fixes all n^2 squares in the grid. Thus, $t(\tau^0) = n^2$. Next, $t(\tau) = t(\tau^{-1})$, and since $\tau^{-1} = \tau^3$, $t(\tau) = t(\tau^3)$.

Thus,

$$N = \frac{1}{4} \left[2n^2 + 2 \left\lfloor \frac{n^2+1}{2} \right\rfloor + 2 \cdot 2 \left\lfloor \frac{n^2+3}{4} \right\rfloor \right],$$

as desired.

Therefore, for any $n \times n$ grid, there are

$$N = \frac{1}{4} \left[2n^2 + 2 \left\lfloor \frac{n^2+1}{2} \right\rfloor + 2 \cdot 2 \left\lfloor \frac{n^2+3}{4} \right\rfloor \right]$$

distinct ways to color the $n \times n$ grid with 2 colors.

3 Less Routine Examples

So far, we have been looking at problems of the sort where any color from a set of q colors can be used to color an object without restriction. Take, for example, coloring the edges of a square.

Example 3.1 ([3], **Exercise 29.8**). Determine the number of ways in which the edges of a square can be colored with six colors with no restriction placed on the number of times a color can be used. Let $X = \{1, 2, 3, 4\}$ be the set containing the 4 edges of the square, let C be the set of six colors, and let C^4 be the set of all 4-tuples of colors. If $x \in C^4$, then $x = (c_1, c_2, c_3, c_4)$ where c_1, c_2, c_3 , and c_4 are each one of six colors. Visualize the square as living in a two-dimensional plane. Then the square can only be permuted by rotation. Let τ represent the clockwise rotation of the square 90° . Then

$$\begin{aligned}\tau &= (1, 2, 3, 4) \\ \tau^2 &= (1, 3)(2, 4) \\ \tau^3 &= (1, 4, 3, 2).\end{aligned}$$

Note that $\tau^4 = \tau^0 = (1)(2)(3)(4)$. Let $G = \langle \tau \rangle$. Then G acts on X and hence, G acts on C^4 by Proposition 1.8. By Corollary 2.10, the number of distinct colorings of the edges of the square is

$$N = \frac{1}{|G|} \left(q^{t(\tau^0)} + q^{t(\tau)} + q^{t(\tau^2)} + q^{t(\tau^3)} \right),$$

where q is the number of colors. Since the identity permutation consists of 4 cycles when completely factored, $q^{t(\tau^0)} = 6^4$. Similarly, $q^{t(\tau)} = 6^1$, $q^{t(\tau^2)} = 6^2$, and $q^{t(\tau^3)} = 6^1$. Because $G = \langle \tau \rangle$, $|G| = 4$. Therefore, the number N of ways to color the edges of a square with six

colors is

$$\begin{aligned} N &= \frac{1}{4} (6^4 + 6^1 + 6^2 + 6^1) \\ &= \frac{1}{4} (1296 + 6 + 36 + 6) \\ &= \frac{1}{4} (1344) \\ &= 336. \end{aligned}$$

How does counting change when there are constraints placed on the number of times a color is used? Let's consider Example 3.1 again with the added constraint that no color can be used on more than one edge.

Example 3.2 ([3], Exercise 29.7). Let $X = \{1, 2, 3, 4\}$ be the set containing the four edges of the square, let C be the set of six colors, and let C^4 be the set of all 4-tuples of colors. If $x \in C^4$, $x = (c_1, c_2, c_3, c_4)$ where c_1, c_2, c_3 , and c_4 are each one of 6 colors. We want to specifically consider the subset $Y \subseteq C^4$ where $y = (c_1, c_2, c_3, c_4) \in Y$ and $c_1 \neq c_2 \neq c_3 \neq c_4$. Since the square is being visualized in two-dimensional space, the square can only be moved by rotation. So let τ again represent the clockwise rotation of the square 90° . Then

$$\begin{aligned} \tau &= (1, 2, 3, 4) \\ \tau^2 &= (1, 3)(2, 4) \\ \tau^3 &= (1, 4, 3, 2). \end{aligned}$$

Note that $\tau^4 = \tau^0 = (1)(2)(3)(4)$. Let $G = \langle \tau \rangle$, so G acts on X . By Proposition 1.8, G then acts on C^4 . The subset $Y \subseteq C^4$ is closed under the action of G on C^4 since for every $\tau \in G$, $\tau y \in Y$ for all $y \in Y$, so by Remark 1.5, G acts on Y . We now count the number of ways in which the edges of the square can be colored with six colors so that no color is used on more than one edge. Because of the added restriction, we need to determine the number of such colorings fixed by a permutation. Thus, we must revert to the original form of Burnside's

Theorem. The number of distinct colorations of the square is

$$N = \frac{1}{|G|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)|).$$

Once the color for the first edge is selected, there are 5 colors left for the second edge, 4 for the third edge, and 3 for the fourth edge. Thus, $|\text{Fix}((1))| = 6 * 5 * 4 * 3 = 360$. Next, consider $|\text{Fix}(\tau)|$. By the complete factorization of τ , we see that τ binds the colors of all four edges to each other so that $c_1 = c_2 = c_3 = c_4$. Thus, τ fixes none of the 360 possible designs. Similarly, τ^3 forces $c_1 = c_2 = c_3 = c_4$ and τ^2 causes $c_1 = c_3$ and $c_2 = c_4$. So τ^2 and τ^3 also fix none of the possible designs. Therefore, the number of distinct squares is

$$\begin{aligned} N &= \frac{1}{4} (360 + 0 + 0 + 0) \\ &= \frac{360}{4} \\ &= 90. \end{aligned}$$

In this chapter, we will consider how more difficult counting problems like Example 3.2 can be solved. To get started, consider the colorings of the beads of a bracelet with all beads evenly spaced throughout the bracelet. Because it exists in three-dimensional space, a bracelet poses a new challenge. First, determine the number of ways in which the six beads of a bracelet can be colored with 6 colors with no restriction on the number of times a color can be used.

Example 3.3. Let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the 6 beads of the bracelet, let C be the set consisting of the 6 colors, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where each element of x is one of 6 colors. A bracelet can be moved by rotation, reflection across a diagonal, or reflection across a side bisector, as seen in Figure 9. For the purposes of analysis, we may arrange the beads in the shape of a regular hexagon.

The permutations on X can be represented by the dihedral group, D_6 . Clearly, D_6 acts on X , so by Proposition 1.8, D_6 acts on C^6 also.

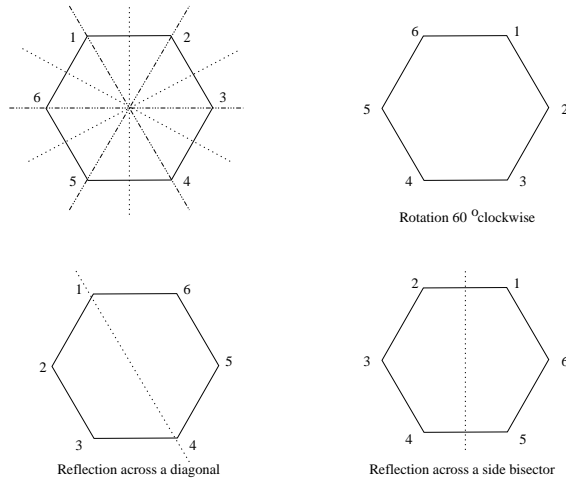


Figure 9: Permutations of a bracelet.

Let τ be an element of D_6 representing clockwise rotation 60° . Then

$$\tau = (1, 2, 3, 4, 5, 6)$$

$$\tau^2 = (1, 3, 5)(2, 4, 6)$$

$$\tau^3 = (1, 4)(2, 5)(3, 6)$$

$$\tau^4 = (1, 5, 3)(2, 6, 4)$$

$$\tau^5 = (1, 6, 5, 4, 3, 2).$$

Next, let δ represent reflecting the bracelet across the diagonal from the first to the fourth bead. Then $\delta = (2, 6)(3, 5)(1)(4)$. Note that δ^2 corresponds to flipping the bracelet back to its original position.

In order to simulate all permutations caused by reflecting the bracelet across a diagonal or across a side bisector, first flip the bracelet, then rotate it as demonstrated in Figure 10.

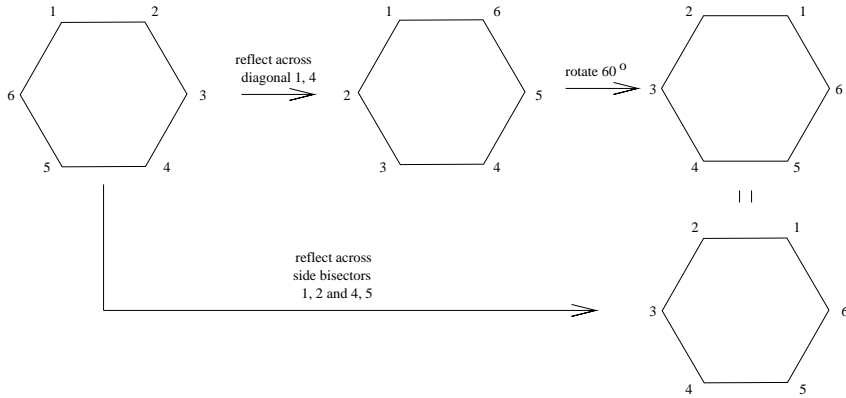


Figure 10: Reflection of a bracelet by $\tau\delta$.

Then

$$\begin{aligned}
 \delta &= (2, 6)(3, 5)(1)(4) \\
 \tau\delta &= (1, 2)(3, 6)(4, 5) \\
 \tau^2\delta &= (1, 3)(4, 6)(2)(5) \\
 \tau^3\delta &= (1, 4)(2, 3)(5, 6) \\
 \tau^4\delta &= (1, 5)(2, 4)(3)(6) \\
 \tau^5\delta &= (1, 6)(2, 5)(3, 4).
 \end{aligned}$$

As can be seen in Figure 11, these permutations represent all permutations caused by reflecting the bracelet across a diagonal or across a side bisector.

Then $D_6 = \{1, \tau, \tau^2, \tau^3, \tau^4, \tau^5, \delta, \delta\tau, \delta\tau^2, \delta\tau^3, \delta\tau^4, \delta\tau^5\}$.

By the corollary to Burnside's Theorem (Corollary 2.10), the number of distinct bracelets is

$$\begin{aligned}
 N &= \frac{1}{|D_6|} (6^{t((1))} + 6^{t(\tau)} + 6^{t(\tau^2)} + 6^{t(\tau^3)} + 6^{t(\tau^4)} + 6^{t(\tau^5)} + \\
 &\quad 6^{t(\delta)} + 6^{t(\delta\tau)} + 6^{t(\delta\tau^2)} + 6^{t(\delta\tau^3)} + 6^{t(\delta\tau^4)} + 6^{t(\delta\tau^5)}).
 \end{aligned}$$

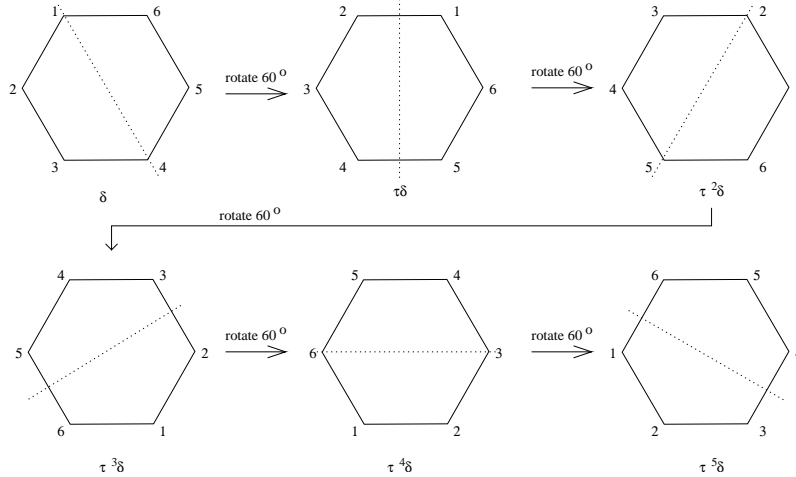


Figure 11: Reflections of a bracelet.

Thus, we can easily calculate the number of distinct bracelets with 6 colors as

$$\begin{aligned}
 N &= \frac{1}{12}(6^6 + 6^1 + 6^2 + 6^3 + 6^2 + 6^1 + 6^4 + 6^3 + 6^4 + 6^3 + 6^4 + 6^3) \\
 &= \frac{51492}{12} \\
 &= 4291.
 \end{aligned}$$

A much different number is calculated if the number of colors that can be used is restricted. As an example, how many bracelets are there having 6 beads, 3 of which are white and 3 black?

Example 3.4 ([3], based on example on page 493). Let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the 6 beads of the bracelet, let C consist of the colors white and black, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where each one of c_1, c_2, c_3, c_4, c_5 , and c_6 are one of two colors. Specifically, we want $y \in Y \subseteq C^6$ where 3 elements of the tuple are one color, and the other three are another color.

As in Example 3.3, a bracelet can be moved by rotation, reflection across a diagonal, or reflection across a side bisector. Thus, the permutations on X can be represented by the dihedral group, D_6 . As already shown, D_6 acts on X , and thus also acts on C^6 . The subset $Y \subseteq C^6$ is closed under the action of D_6 on C^6 since clearly for every $g \in D_6$, $gy \in Y$ for all $y \in Y$. So by Remark 1.5, D_6 acts on Y as well.

Let τ be an element of D_6 representing clockwise rotation 60° . Then

$$\begin{aligned}\tau &= (1, 2, 3, 4, 5, 6) \\ \tau^2 &= (1, 3, 5)(2, 4, 6) \\ \tau^3 &= (1, 4)(2, 5)(3, 6) \\ \tau^4 &= (1, 5, 3)(2, 6, 4) \\ \tau^5 &= (1, 6, 5, 4, 3, 2).\end{aligned}$$

The permutations caused by reflecting the bracelet across a diagonal or across a side bisector are

$$\begin{aligned}\delta &= (2, 6)(3, 5)(1)(4) \\ \tau\delta &= (1, 2)(3, 6)(4, 5) \\ \tau^2\delta &= (1, 3)(4, 6)(2)(5) \\ \tau^3\delta &= (1, 4)(2, 3)(5, 6) \\ \tau^4\delta &= (1, 5)(2, 4)(3)(6) \\ \tau^5\delta &= (1, 6)(2, 5)(3, 4).\end{aligned}$$

As in Example 3.3, by Burnside's Theorem, the number of distinct bracelets with three white beads and three black beads is

$$N = \frac{1}{|D_6|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)| + |\text{Fix}(\delta)| + |\text{Fix}(\tau\delta)| + |\text{Fix}(\tau^2\delta)| + |\text{Fix}(\tau^3\delta)| + |\text{Fix}(\tau^4\delta)| + |\text{Fix}(\tau^5\delta)|).$$

There are $\binom{6}{3} = 20$ possible ways to organize 3 white beads and 3 black beads. Obviously, the identity fixes all 20 designs, so $|\text{Fix}((1))| = 20$. Next, τ and τ^5 cause $c_1 = c_2 = c_3 = c_4 = c_5 = c_6$, so all 6 beads must be the same color. Thus, τ and τ^5 fix none of the 20 possible

designs. Similarly, τ^3 binds c_1 to c_4 , c_2 to c_5 , and c_3 to c_6 . Thus, τ^3 fixes no designs because it causes an even number of beads to be the same color.

However, τ^2 and τ^4 cause $c_1 = c_3 = c_5$ and $c_2 = c_4 = c_6$. Thus, there are two color choices for either c_1 or c_2 . Without loss of generality, assume c_1 is black. Then c_2 has only one choice of color, white. The rest of the elements of the 6-tuple, c_3, c_4, c_5 , and c_6 , are bound to either c_1 or c_2 , so each has only one choice for a color. Thus, $|\text{Fix}(\tau^2)| = |\text{Fix}(\tau^4)| = 2^1$.

Now consider the permutations caused by reflection. The three permutations $\tau\delta$, $\tau^3\delta$, and $\tau^5\delta$ corresponding to reflection across the side bisectors are each factored into three disjoint 2-cycles. Thus, $\tau\delta$, $\tau^3\delta$, and $\tau^5\delta$ fix none of the 20 designs since each one causes an even number of beads to be the same color.

On the other hand, the three permutations δ , $\tau^2\delta$, and $\tau^4\delta$ corresponding to reflection across the diagonals are factored into two disjoint 2-cycles and two 1-cycles. Let's focus on δ . Factoring δ into disjoint cycles reveals when δ acts on x , $c_2 = c_6$ and $c_3 = c_5$. There are 2 color choices for either c_2 or c_3 . If both beads were the same color, since c_6 is bound to c_2 and c_5 is bound to c_3 , then 4 beads would be the same color.

Without loss of generality, assume c_2 is white. Then there is only one choice of color for c_3, c_5 , and c_6 . Setting c_2 as white, we then have two white beads and two black beads. Because c_1 and c_2 are bound to nothing, there are two color choices for c_1 and c_2 , but if both are the same color, there will not be three black beads and three white beads as desired. In other words, in order to fix three black beads and three white beads, one of the four beads that is bound to another bead may be white (i.e., one of the four beads has the choice of being white). Because they are bound, we then have two white beads, and one more is needed to make the desired bead arrangement. Two beads are bound to nothing, so one of the two has the choice of being white. Therefore, 2^2 designs are fixed by each of δ , $\tau^2\delta$, and $\tau^4\delta$. Thus, $|\text{Fix}(\delta)| = |\text{Fix}(\tau^2\delta)| = |\text{Fix}(\tau^4\delta)| = 2^2$.

Therefore, the number N of ways to color a bracelet with three white beads and three black

beads is

$$\begin{aligned}
 N &= \frac{1}{12} (20 + 0 + 2 + 0 + 2 + 0 + 4 + 0 + 4 + 0 + 4 + 0) \\
 &= \frac{36}{12} \\
 &= 3.
 \end{aligned}$$

Indeed, the patterns are as pictured in Figure 12.

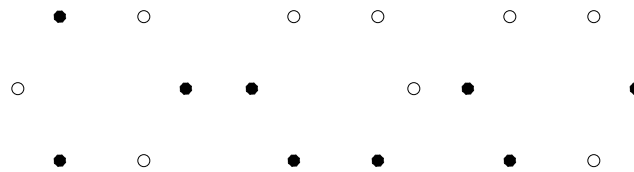


Figure 12: The distinct patterns with 3 white beads and 3 black beads.

As slight variation, let's calculate the number of distinct ways to color a bracelet with 4 white beads and 2 black beads.

Example 3.5 ([3], variation of example on page 493). Once again let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the 6 beads of the bracelet, let C consist of the colors white and black, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where each one of c_1, c_2, c_3, c_4, c_5 , and c_6 are one of two colors. Specifically, we want $y \in Y \subseteq C^6$ where 4 elements of the tuple are white, and the other two are black. As explained in Example 3.3, the permutations on X can be represented by the dihedral group, D_6 . Let the elements of D_6 be defined as in Example 3.4. The subset $Y \subseteq C^6$ is closed under the action of D_6 on C^6 since clearly for every $g \in D_6$, $gy \in Y$ for all $y \in Y$. So by Remark 1.5, D_6 acts on Y .

As in Example 3.4, by Burnside's Theorem, the number of distinct bracelets with four white beads and two black beads is

$$\begin{aligned}
 N &= \frac{1}{|D_6|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)| + \\
 &\quad |\text{Fix}(\delta)| + |\text{Fix}(\tau\delta)| + |\text{Fix}(\tau^2\delta)| + |\text{Fix}(\tau^3\delta)| + |\text{Fix}(\tau^4\delta)| + |\text{Fix}(\tau^5\delta)|).
 \end{aligned}$$

There are $\binom{6}{4} = 15$ possible ways to organize 4 white beads and 2 black beads. Obviously, the identity fixes all 15 designs, so $|\text{Fix}((1))| = 15$. Next, τ and τ^5 cause $c_1 = c_2 = c_3 = c_4 = c_5 = c_6$, so all 6 beads must be the same color. Thus, τ and τ^5 fix none of the 15 possible designs. Similarly, τ^2 and τ^4 cause $c_1 = c_3 = c_5$ and $c_2 = c_4 = c_6$, so c_1, c_3, c_5 are the same color and c_2, c_4, c_6 are the same color. Thus, τ^2 and τ^4 cannot cause exactly 4 colors to be the same.

However, τ^3 binds c_1 to c_4 , c_2 to c_5 , and c_3 to c_6 , so c_1, c_2 , and c_3 each have 2 color choices. Since $c_1 = c_4$, $c_2 = c_5$, and $c_3 = c_6$, two out of the three pairs can be white in order to have one of the 15 possible patterns. Then there are $\binom{3}{2} = 3$ ways to create an equivalent design. Thus, $|\text{Fix}(\tau^3)| = 3$.

Now consider the permutations caused by reflection. The three permutations $\tau\delta$, $\tau^3\delta$, and $\tau^5\delta$ corresponding to reflection across the side bisectors are each factored into three disjoint 2-cycles, as was τ^3 . In the same way, $|\text{Fix}(\tau\delta)| = |\text{Fix}(\tau^3\delta)| = |\text{Fix}(\tau^5\delta)| = 3$

On the other hand, the three permutations δ , $\tau^2\delta$, and $\tau^4\delta$ corresponding to reflection across the diagonals are factored into two disjoint 2-cycles and two 1-cycles. Let's focus on δ . Factoring δ into disjoint cycles reveals that when δ acts on x , $c_2 = c_6$ and $c_3 = c_5$. Then either c_2 and c_3 can be white, leaving c_1 and c_4 to be black, or one of c_2 and c_3 is black, forcing c_1 and c_4 to be white. Thus, 3 designs are fixed by δ . Since $\tau^2\delta$ and $\tau^4\delta$ follow the same pattern, $|\text{Fix}(\delta)| = |\text{Fix}(\tau^2\delta)| = |\text{Fix}(\tau^4\delta)| = 3$.

Therefore, the number N of ways to color a bracelet with four white beads and two black beads is

$$\begin{aligned} N &= \frac{1}{12} (15 + 0 + 0 + 3 + 0 + 0 + 3 + 3 + 3 + 3 + 3 + 3) \\ &= \frac{36}{12} \\ &= 3. \end{aligned}$$

Indeed, the patterns are as pictured in Figure 13.

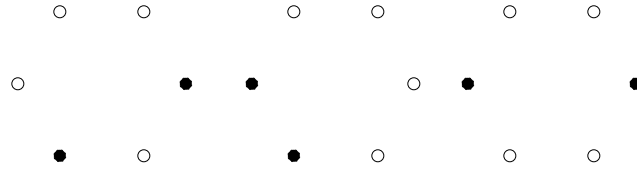


Figure 13: The distinct patterns with 4 white beads and 2 black beads.

Next, determine the number of distinct ways to color a bracelet with 5 white beads and 1 black bead.

Example 3.6 ([3], variation of example on page 493). Once again let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the 6 beads of the bracelet, let C consist of the colors white and black, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where each one of c_1, c_2, c_3, c_4, c_5 , and c_6 are one of two colors. Specifically, we want $y \in Y \subseteq C^6$ where five elements of the tuple are white, and one is black. As explained in Example 3.3, the permutations on X can be represented by the dihedral group, D_6 . Let the elements of D_6 be defined as in Example 3.4. The subset $Y \subseteq C^6$ is closed under the action of D_6 on C^6 since clearly for every $g \in D_6$, $gy \in Y$ for all $y \in Y$. So by Remark 1.5, D_6 acts on Y .

By Burnside's Theorem, the number of distinct bracelets with five white beads and one black bead is

$$N = \frac{1}{|D_6|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)| + |\text{Fix}(\delta)| + |\text{Fix}(\tau\delta)| + |\text{Fix}(\tau^2\delta)| + |\text{Fix}(\tau^3\delta)| + |\text{Fix}(\tau^4\delta)| + |\text{Fix}(\tau^5\delta)|).$$

There are $\binom{6}{5} = 6$ possible ways to organize 5 white beads and 1 black bead. Obviously, the identity fixes all 6 designs, so $|\text{Fix}((1))| = 6$. Next, τ and τ^5 cause $c_1 = c_2 = c_3 = c_4 = c_5 = c_6$, so all 6 beads must be the same color. Then τ and τ^5 fix none of the 6 possible designs. Similarly, τ^2 and τ^4 cause $c_1 = c_3 = c_5$ and $c_2 = c_4 = c_6$, so 3 of the beads are one color the other 3 can either be the same color or the second color. However, we want 5 beads to be all one color. Thus, τ^2 and τ^4 fix none of the 6 designs. Next, τ^3 binds c_1 to c_4 , c_2 to c_5 , and c_3 to c_6 , so c_1, c_2 , and c_6 each have 2 color choices. This provides no way of obtaining a bracelet

with 5 beads of one color, so again, no designs are fixed.

Next consider the permutations caused by reflection. The three permutations $\tau\delta$, $\tau^3\delta$, and $\tau^5\delta$ corresponding to reflection across the side bisectors are each factored into three disjoint 2-cycles, as was τ^3 . Thus, $\tau\delta$, $\tau^3\delta$, and $\tau^5\delta$ fix no designs.

On the other hand, the three permutations δ , $\tau^2\delta$, and $\tau^4\delta$ corresponding to reflection across the diagonals are factored into two disjoint 2-cycles and two 1-cycles. As before, let's focus on δ . Factoring δ into disjoint cycles reveals that when δ acts on x , $c_2 = c_6$ and $c_3 = c_5$. Then c_2 and c_3 can be white, leaving either c_1 or c_4 to be black. If either c_2 or c_3 were black, there would be too many black beads, so there are only 2 designs that can be fixed by δ . Since $\tau^2\delta$ and $\tau^4\delta$ follow the same pattern, $|\text{Fix}(\delta)| = |\text{Fix}(\tau^2\delta)| = |\text{Fix}(\tau^4\delta)| = 2$.

Therefore, the number N of distinct ways to color a bracelet with five white beads and one black bead is

$$\begin{aligned} N &= \frac{1}{12} (6 + 0 + 0 + 0 + 0 + 0 + 2 + 0 + 2 + 0 + 2 + 0) \\ &= \frac{12}{12} \\ &= 1. \end{aligned}$$

In the introduction, we mentioned that if we were working with a cube, all six colorings in which one face is white and all others are red would be equivalent. As demonstrated in this example, this is indeed that case. All 6 colorings in which one bead is black and the other five are white are equivalent.

Finally, consider how one would calculate the number of distinct ways to color a bracelet with 6 white beads and 0 black beads.

Example 3.7 ([3], variation of example on page 493). Again let $X = \{1, 2, 3, 4, 5, 6\}$ be the set containing the 6 beads of the bracelet, let C consist of the colors white and black, and let C^6 be the set of all 6-tuples of colors. If $x \in C^6$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6)$ where each one of c_1, c_2, c_3, c_4, c_5 , and c_6 are one of two colors. Specifically, we want $y \in Y \subseteq C^6$ where six elements of the tuple are white, and none are black. As explained in Example 3.3, the

permutations on X can be represented by the dihedral group, D_6 . Let the elements of D_6 be defined as in Example 3.4. The subset $Y \subseteq C^6$ is closed under the action of D_6 on C^6 since clearly for every $g \in D_6$, $gy \in Y$ for all $y \in Y$. So by Remark 1.5, D_6 acts on Y .

There is only one possible design that can be made with 6 white beads in a six bead bracelet. We can stop here because obviously, since there is only one possible ordering, it is unique.

Burnside's Theorem supports this answer. Each permutation fixes exactly one design with six white beads. Thus, by Burnside's Theorem, the number N of distinct ways to color a bracelet with six white beads and no black beads is

$$\begin{aligned}
 N &= \frac{1}{|D_6|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)| + \\
 &\quad |\text{Fix}(\delta)| + |\text{Fix}(\tau\delta)| + |\text{Fix}(\tau^2\delta)| + |\text{Fix}(\tau^3\delta)| + |\text{Fix}(\tau^4\delta)| + |\text{Fix}(\tau^5\delta)|) \\
 &= \frac{1}{12} (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) \\
 &= \frac{12}{12} \\
 &= 1.
 \end{aligned}$$

Next we will look at the case where the bracelet has an odd number of beads.

Example 3.8 ([3], variation of example on page 493). Determine the number of distinct ways to color a bracelet with 5 white beads and 4 black beads. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set containing the 9 beads of the bracelet, let C consist of the colors white and black, and let C^9 be the set of all 9-tuples of colors. If $x \in C^9$, then $x = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9)$ where each one of $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and c_9 are one of two colors. Specifically, we want $y \in Y \subseteq C^9$ where five elements of the tuple are white, and four are black.

A bracelet with an odd number of beads can be moved by rotation or by reflection across a side bisector, as seen in Figure 14.

The permutations on X can be represented by the dihedral group, D_9 . Clearly, D_9 acts on X , so by Proposition 1.8, D_9 acts on C^9 also. The subset $Y \subseteq C^9$ is closed under the action of D_9 on C^9 since clearly for every $g \in D_9$, $gy \in Y$ for all $y \in Y$. So by Remark 1.5, D_9 acts on Y too.

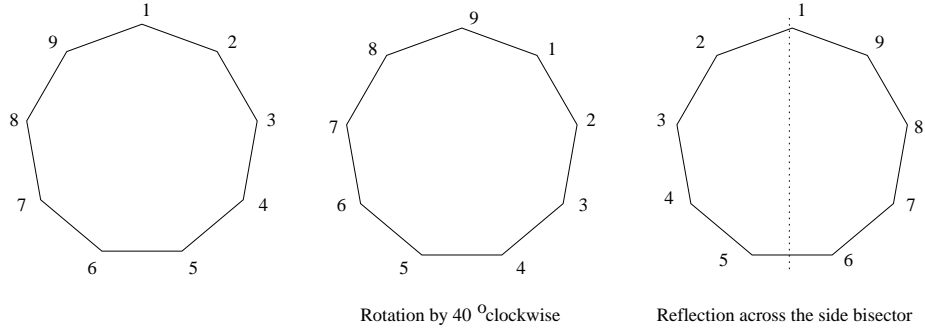


Figure 14: Bead arrangements of a bracelet with 9 beads.

Let τ be an element of D_9 representing clockwise rotation 40° . Then

$$\begin{aligned} \tau &= (1, 2, 3, 4, 5, 6, 7, 8, 9) \\ \tau^2 &= (1, 3, 5, 7, 9, 2, 4, 6, 8) \\ \tau^3 &= (1, 4, 7)(2, 5, 8)(3, 6, 9) \\ \tau^4 &= (1, 5, 9, 4, 8, 3, 7, 2, 6) \\ \tau^5 &= (1, 6, 2, 7, 3, 8, 4, 9, 5) \\ \tau^6 &= (1, 7, 4)(2, 8, 5)(3, 9, 6) \\ \tau^7 &= (1, 8, 6, 4, 2, 9, 7, 5, 3) \\ \tau^8 &= (1, 9, 8, 7, 6, 5, 4, 3, 2). \end{aligned}$$

Next, let δ represent reflecting the bracelet across the side bisector through vertex 1. Then $\delta = (2, 9)(3, 8)(4, 7)(5, 6)(1)$. Note that δ^2 corresponds to flipping the bracelet back to its original position.

In order to simulate all permutations caused by reflecting the bracelet across a side bisector, first flip the bracelet, then rotate it as explained in Example 3.3.

Then

$$\begin{aligned}
\delta &= (2, 9)(3, 8)(4, 7)(5, 6)(1) \\
\tau\delta &= (1, 2)(3, 9)(4, 8)(5, 7)(6) \\
\tau^2\delta &= (1, 3)(4, 9)(5, 8)(6, 7)(2) \\
\tau^3\delta &= (1, 4)(2, 3)(5, 9)(6, 8)(7) \\
\tau^4\delta &= (1, 5)(2, 4)(6, 9)(7, 8)(3) \\
\tau^5\delta &= (1, 6)(2, 5)(3, 4)(7, 9)(8) \\
\tau^6\delta &= (1, 7)(2, 6)(3, 5)(8, 9)(4) \\
\tau^7\delta &= (1, 8)(2, 7)(3, 6)(4, 5)(9) \\
\tau^8\delta &= (1, 9)(2, 8)(3, 7)(4, 6)(5).
\end{aligned}$$

Note that $D_9 = \{1, \tau, \tau^2, \tau^3, \tau^4, \tau^5, \tau^6, \tau^7, \tau^8\delta, \tau\delta, \tau^2\delta, \tau^3\delta, \tau^4\delta, \tau^5\delta, \tau^6\delta, \tau^7\delta, \tau^8\delta\}$. By Burnside's Theorem, the number of distinct bracelets with five white beads and four black beads is

$$\begin{aligned}
N &= \frac{1}{|D_9|} (|\text{Fix}((1))| + |\text{Fix}(\tau)| + |\text{Fix}(\tau^2)| + |\text{Fix}(\tau^3)| + |\text{Fix}(\tau^4)| + |\text{Fix}(\tau^5)| + \\
&\quad |\text{Fix}(\tau^6)| + |\text{Fix}(\tau^7)| + |\text{Fix}(\tau^8)| + |\text{Fix}(\delta)| + |\text{Fix}(\tau\delta)| + |\text{Fix}(\tau^2\delta)| + \\
&\quad |\text{Fix}(\tau^3\delta)| + |\text{Fix}(\tau^4\delta)| + |\text{Fix}(\tau^5\delta)| + |\text{Fix}(\tau^6\delta)| + |\text{Fix}(\tau^7\delta)| + |\text{Fix}(\tau^8\delta)|).
\end{aligned}$$

There are $\binom{9}{5} = 126$ possible ways to organize 5 white beads and four black beads. Clearly, the identity fixes all 126 designs, so $|\text{Fix}((1))| = 126$. Next, $\tau, \tau^2, \tau^4, \tau^5, \tau^7$, and τ^8 cause every bead to be the same color. Thus, $|\text{Fix}(\tau)| = |\text{Fix}(\tau^2)| = |\text{Fix}(\tau^4)| = |\text{Fix}(\tau^5)| = |\text{Fix}(\tau^7)| = |\text{Fix}(\tau^8)| = 0$. Out of the bead arrangements caused by rotation, this leaves, τ^3 and τ^6 , which can each be factored into three disjoint 3-cycles. Beads 1, 4, and 7 must share one color, as do 2, 8, and 5 and 3, 9, and 6. Thus, there is no way to get a design with 5 white beads and 4 black beads. So $|\text{Fix}(\tau^3)| = |\text{Fix}(\tau^6)| = 0$.

Next consider the permutations caused by reflection. Each of these permutations can be factored into four 2-cycles and one 1-cycle. The bead associated with the 1-cycle must always be white because, for the design to be equivalent to one of the 126 possible designs, it must contain 5 white beads. This means that two out of the four pairs must be white in order to have one of the 126 possible patterns. Thus, there are $\binom{4}{2} = 6$ ways to create an equivalent design. Since $\delta, \tau\delta, \tau^2\delta, \tau^3\delta, \tau^4\delta, \tau^5\delta, \tau^6\delta, \tau^7\delta, \tau^8\delta$ all have this cycle structure, $|\text{Fix}(\delta)| = |\text{Fix}(\tau\delta)| = |\text{Fix}(\tau^2\delta)| = |\text{Fix}(\tau^3\delta)| = |\text{Fix}(\tau^4\delta)| = |\text{Fix}(\tau^5\delta)| = |\text{Fix}(\tau^6\delta)| = |\text{Fix}(\tau^7\delta)| = |\text{Fix}(\tau^8\delta)| = 6$.

Therefore, the number N of distinct ways to color a bracelet with five white beads and four black beads is

$$\begin{aligned} N &= \frac{1}{18}(126 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6) \\ &= \frac{180}{18} \\ &= 10. \end{aligned}$$

Notice that in the examples involving bracelets, for each $\tau \in S_n$, the number of r -cycles, $1 \leq r \leq n$, are counted in order to figure out if the desired bead arrangement can be fixed by τ or not.

Definition 3.9 ([5]). If the complete factorization $\tau \in S_n$ has $e_r(\tau) \geq 0$ r -cycles, then the **index** of τ is the monomial

$$\text{ind}(\tau) = x_1^{e_1(\tau)} x_2^{e_2(\tau)} \dots x_n^{e_n(\tau)}.$$

If G is a subgroup of S_n , then the **cycle index** of G is the polynomial in n variables with coefficients in \mathbb{Q} :

$$P_G(x_1, \dots, x_n) = \frac{1}{|G|} \sum_{\tau \in G} \text{ind}(\tau).$$

Example 3.10. For instance, in the discussion of the 6 bead bracelet in Example 3.3, $G = D_6$

and

$$\begin{aligned}
\tau &= (1, 2, 3, 4, 5, 6) \\
\tau^2 &= (1, 3, 5)(2, 4, 6) \\
\tau^3 &= (1, 4)(2, 5)(3, 6) \\
\tau^4 &= (1, 5, 3)(2, 6, 4) \\
\tau^5 &= (1, 6, 5, 4, 3, 2) \\
\delta &= (2, 6)(3, 5)(1)(4) \\
\tau\delta &= (1, 2)(3, 6)(4, 5) \\
\tau^2\delta &= (1, 3)(4, 6)(2)(5) \\
\tau^3\delta &= (1, 4)(2, 3)(5, 6) \\
\tau^4\delta &= (1, 5)(2, 4)(3)(6) \\
\tau^5\delta &= (1, 6)(2, 5)(3, 4).
\end{aligned}$$

Thus, $\text{ind}(1) = x_1^6$ since the identity is made up of six 1-cycles. Then $\text{ind}(\tau) = \text{ind}(\tau^5) = x_6$, $\text{ind}(\tau^2) = \text{ind}(\tau^4) = x_3^2$, $\text{ind}(\tau^3) = x_2^3$, $\text{ind}(\delta) = \text{ind}(\delta\tau^2) = \text{ind}(\delta\tau^4) = x_1^2x_2^2$, and $\text{ind}(\delta\tau) = \text{ind}(\delta\tau^3) = \text{ind}(\delta\tau^5) = x_2^3$. The cycle index of D_6 is

$$P_{D_6}(x_1, \dots, x_6) = \frac{1}{12} (x_1^6 + 2x_6 + 2x_3^2 + 3x_1^2x_2^2 + 4x_2^3).$$

Example 3.11. Next, in Example 2.11, we saw that the cyclic group $G = \langle \tau \rangle$ of order 4 acts on a grid with 25 squares where

$$\tau = (1, 5, 25, 21)(2, 10, 24, 16)(3, 15, 23, 11)(4, 20, 22, 6)(7, 9, 19, 17)(8, 14, 18, 12)(13).$$

It follows that

$$\tau^2 = (1, 25)(5, 21)(2, 24)(10, 16)(3, 23)(15, 11)(4, 22)(20, 6)(7, 19)(9, 17)(8, 18)(14, 12)(13)$$

$$\tau^3 = (1, 21, 25, 5)(2, 16, 24, 10)(3, 11, 23, 15)(4, 6, 22, 20)(7, 17, 19, 9)(8, 12, 18, 14)(13)$$

$$\tau^4 = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)(17)(18)(19)(20)(21)(22)(23)(24)(25)$$

The cycle index of G is

$$P_G(x_1, \dots, x_{25}) = \frac{1}{4} (x_1^{25} + x_1 x_2^{12} + 2x_1 x_4^6).$$

We can now rewrite the number of (q, G) -colorings in terms of the cycle index, which means that Burnside's Theorem can also be recast.

Theorem 3.12 ([5], **Theorem 2.73**). *Let X be a finite set with $|X| = n$ and let G be a subgroup of S_n that acts on X . Then the number of (q, G) -colorings of X is $P_G(q, \dots, q)$.*

Proof. ([5], Theorem 2.73) Recall that a (q, G) -coloring of X is an orbit of (c_1, \dots, c_n) in the set C^n of n -tuples. By Proposition 2.2, since G acts on X , G acts on C^n . Then by Burnside's Theorem, since G acts on C^n , the number of (q, G) -colorings of X is

$$N = \frac{1}{|G|} \sum_{\tau \in G} |\text{Fix}(\tau)|.$$

By Lemma 2.9, this number is

$$N = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)},$$

where $t(\tau)$ is the number of cycles in the complete factorization of τ . As just defined,

$$\begin{aligned} P_G(x_1, \dots, x_n) &= \frac{1}{|G|} \sum_{\tau \in G} \text{ind}(\tau) \\ &= \frac{1}{|G|} \sum_{\tau \in G} x_1^{e_1(\tau)} x_2^{e_2(\tau)} \dots x_n^{e_n(\tau)}, \end{aligned}$$

and so

$$\begin{aligned} P_G(q, \dots, q) &= \frac{1}{|G|} \sum_{\tau \in G} q^{e_1(\tau) + e_2(\tau) + \dots + e_n(\tau)} \\ &= \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}. \end{aligned}$$

Thus, the number of (q, G) -colorings of X is $P_G(q, \dots, q)$. □

Example 3.13. Returning to Example 3.11, let's count the number of distinct colorings of a 5×5 grid with 25 squares, each of which can be colored with one of 2 colors using the cycle index. Since $G = \langle \tau \rangle$,

$$P_G(x_1, \dots, x_{25}) = \frac{1}{4} (x_1^{25} + x_1 x_2^{12} + 2x_1 x_4^6).$$

Then the number of distinct grids is

$$\begin{aligned} P_G(2, \dots, 2) &= \frac{1}{4} (2^{25} + 2 * 2^{12} + 2(2 * 2^6)) \\ &= \frac{1}{4} (33554432 + 8192 + 256) \\ &= \frac{33562880}{4} \\ &= 8,390,720. \end{aligned}$$

Note that this is the same result as calculated in Example 2.11.

Thus, we now have another method to calculate the number of distinct colorings. The cycle index makes the calculation a little bit easier, but it also has a key role in a generalization of Burnside's Theorem due to Pólya. Pólya's Theorem is especially useful when solving problems where the use of the colors is restricted. For example, it could be used to calculate the number of colorings for the 6-bead bracelet with 4 white beads and 2 black beads.

Theorem 3.14 ([5], Theorem (Pólya, 1937)). *Let $G \subseteq S_X$, where $|X| = n$, let $|C| = q$, and, for each $i \geq 1$, define $\sigma_i = c_1^i + \dots + c_q^i$. Then the number of (q, G) -colorings of X having f_r elements of color c_r , for every r , is the coefficient of $c_1^{f_1} c_2^{f_2} \dots c_q^{f_q}$ in $P_G(\sigma_1, \dots, \sigma_n)$.*

To solve the bracelet problem posed above, first recall from Example 3.10 that the cycle index for 6-bead bracelets is

$$P_{D_6}(x_1, \dots, x_6) = \frac{1}{12} (x_1^6 + 2x_6 + 2x_3^2 + 3x_1^2x_2^2 + 4x_2^3).$$

Using Pólya's Theorem, the number of bracelets with 4 white beads and 2 black beads is the coefficient of b^2w^4 in

$$P_\sigma(\sigma_1, \dots, \sigma_6) = \frac{1}{12} [(b+w)^6 + 2(b^6+w^6) + 2(b^3+w^3)^2 + 3((b+w)^2(b^2+w^2)^2) + 4(b^2+w^2)^3].$$

Using the binomial theorem, we find the coefficient of b^2w^4 . The binomial theorem states that for positive integers n ,

$$(x+a)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k},$$

where $\binom{n}{k}$ are binomial coefficients. Begin with $(b+w)^6$. Since $n=6$, the coefficient of

b^2w^4 is found when $k=2$. Thus, for this part, the coefficient of b^2w^4 is $\binom{6}{2}$. Next, consider b^6+w^6 . Here $n=1$, and we want to find an integer k such that $b^{6k}w^{6(1-k)} = b^2w^4$. This has no solution, so the coefficient of b^2w^4 is zero. For $(b^3+w^3)^2$, $n=2$, so we want to find an integer k such that $b^{3k}w^{3(2-k)} = b^2w^4$. Again, this has no solution. For $(b+w)^2(b^2+w^2)^2$,

$$\begin{aligned} b^2b^4 &= b^l w^{2-l} b^{2k} w^{2(2-k)} \\ &= b^{2k+l} w^{6-2k-l} \end{aligned}$$

when $k=1$ and $l=0$ or $k=0$ and $l=2$. Then the coefficient of b^2w^4 is $\binom{2}{1} \binom{2}{0} + \binom{2}{0} \binom{2}{2}$. Finally, consider $(b^2+w^2)^3$. Since $n=3$, $b^{2k}w^{2(3-k)} = b^2w^4$ when $k=1$, the coefficient of b^2w^4

is $\binom{3}{1}$.

Thus, the coefficient of b^2w^4 is

$$\begin{aligned} & \frac{1}{12} \left(\binom{6}{2} + 2(0) + 2(0) + 3 \left(\binom{2}{1} \binom{2}{0} + \binom{2}{0} \binom{2}{2} \right) + 4 \left(\binom{3}{1} \right) \right) \\ &= \frac{1}{12} (15 + 2(0) + 2(0) + 3(2 + 1) + 4(3)) \\ &= \frac{36}{12} \\ &= 3. \end{aligned}$$

Note that this is the same result calculated in Example 3.5.

In conclusion, let's consider several more examples of how Pólya's Theorem can be used. A cube could be colored on its 8 vertices, its 12 edges, or its 6 faces. In each case, two colorings are equivalent if one can be obtained from the other by rotation. By Theorem 7.3 of [3], the group G of rotations of a cube is isomorphic to S_4 . Thus, G contains 24 rotations.

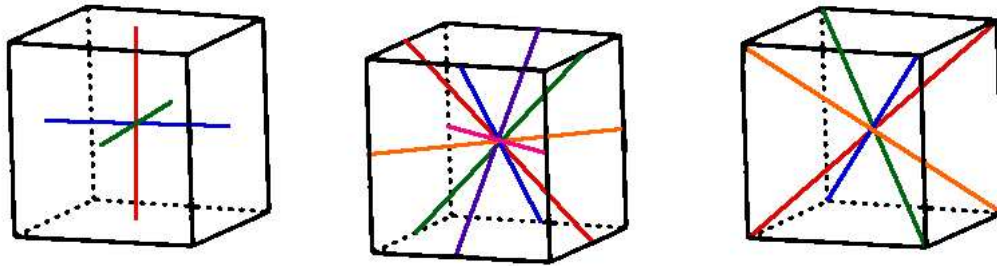


Figure 15: Possible axes of rotation of a cube.

The 24 rotations of the cube whose axes can be seen in Figure 15 include:

1. The identity.
2. Three 180° rotations around lines connecting the centers of opposite faces.
3. Six 90° rotations around lines connecting the centers of the opposite faces.

4. Six 180° rotations around lines connecting the midpoints of opposite edges.
5. Eight 120° rotations around lines connecting opposite vertices.

Example 3.15 ([1], based on example on page 231). Let's first color the vertices of a cube so that there are 6 red and 2 black.

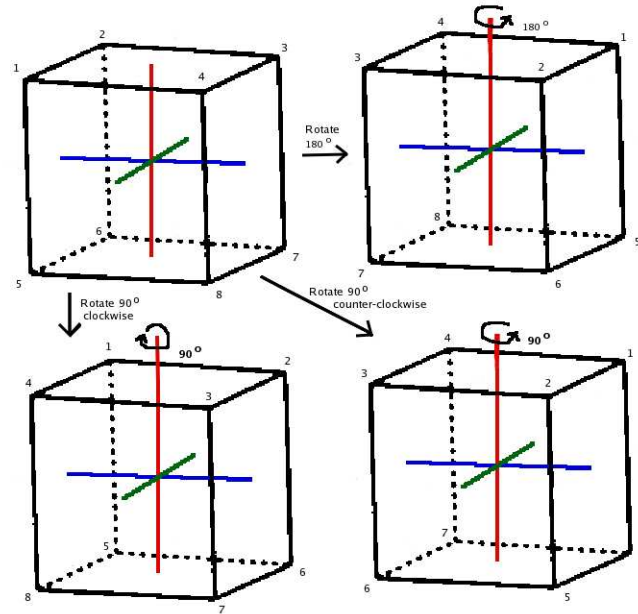


Figure 16: Rotations around lines connecting the centers of opposite faces.

The identity can be described as a product of eight 1-cycles. In Figure 16, we can see that the 180° rotations around lines connecting the centers of opposite faces can be described as the product of four disjoint 2-cycles, and the 90° rotations around lines connecting the centers of the opposite faces can be written as the product of two disjoint 4-cycles.

In Figure 17, we can see that the 180° rotations around lines connecting the midpoints of opposite edges can be described as the product of four disjoint 2-cycles.

Finally, Figure 18 shows that the 120° rotations around lines connecting opposite vertices can each be written as the product of two 1-cycles and two disjoint 3-cycles.

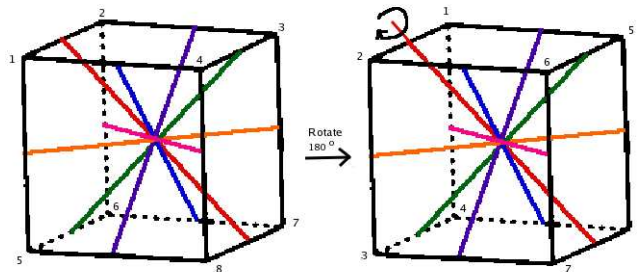


Figure 17: Rotations around lines connecting the midpoints of opposite edges.

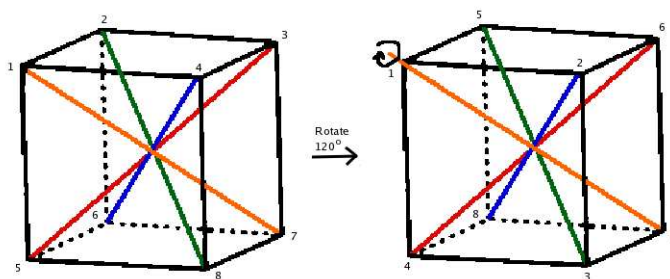


Figure 18: Rotations around lines connecting opposite vertices.

Thus, the cycle index is

$$P_{S_4} = \frac{1}{24} (x_1^8 + 6x_4^2 + 9x_2^4 + 8x_1^2x_3^2).$$

If the cube is colored using 2 colors with no restriction placed on the number of times a color can be used, then by Theorem 3.12 there are $P_{S_4}(2, \dots, 2)$ distinct colorings of the vertices of the cube:

$$\begin{aligned} P_{S_4}(2, \dots, 2) &= \frac{1}{24} (2^8 + 6(2^2) + 9(2^4) + 8(2^2)(2^2)) \\ &= \frac{552}{24} \\ &= 23. \end{aligned}$$

However, we are calculating how many distinct colorings there are with 6 red vertices and 2 black vertices. Thus, we look at the coefficient of b^2r^6 in

$$P_{\sigma}(\sigma_1, \dots, \sigma_{24}) = \frac{1}{24} [(b+r)^8 + 9(b^2+r^2)^4 + 6(b^4+r^4)^2 + 8((b+r)^2(b^3+r^3)^2)].$$

Using the binomial theorem as before, the coefficient of b^2r^6 is

$$\begin{aligned} \frac{1}{24} \left(\binom{8}{2} + 9 \binom{4}{1} + 8 \binom{2}{0} \binom{2}{2} \right) &= \frac{72}{24} \\ &= 3. \end{aligned}$$

Next, we will consider coloring the twelve edges of the cube.

Example 3.16 ([1], based on example on page 231). Color the cube's edges so that 4 edges are green and 8 edges are blue. The identity can be described as a product of twelve 1-cycles. By examining Figures 16, 17, and 18, we can see that the 180° rotations around lines connecting the centers of opposite faces can be described as the product of six disjoint 2-cycles, and the 90° rotations around lines connecting the centers of the opposite faces can be written as the product of three disjoint 4-cycles. The 180° rotations around lines connecting the midpoints of opposite edges can be described as the product of two 1-cycles and five disjoint 2-cycles, and the 120° rotations around lines connecting opposite vertices can each be written as the product of four disjoint 3-cycles.

Thus, the cycle index is

$$P_{S_4} = \frac{1}{24} [x_1^{12} + 3x_2^6 + 6x_4^3 + 6x_1^2x_2^5 + 8x_3^4].$$

If the cube were being colored with 2 colors with no restrictions placed on the colors, by Theorem 3.12 the total number of patterns is

$$\begin{aligned} P_{S_4}(2, \dots, 2) &= \frac{1}{24} [2^{12} + 3(2^6) + 6(2^3) + 6(2^2 * 2^5) + 8(2^4)] \\ &= \frac{5232}{24} \\ &= 218. \end{aligned}$$

However, we are calculating how many distinct colorings there are with 8 blue edges and 4

green edges. Thus, we look at the coefficient of b^8g^4 in

$$P_\sigma(\sigma_1, \dots, \sigma_{24}) = \frac{1}{24} [(b+g)^{12} + 3(b^2+g^2)^6 + 6(b^4+g^4)^3 + 6((b+g)^2(b^2+g^2)^5) + 8(b^3+g^3)^4].$$

Using the binomial theorem as before, the coefficient of b^8g^4 is

$$\begin{aligned} & \frac{1}{24} \left(\binom{12}{8} + 3 \binom{6}{4} + 6 \binom{3}{2} + 6 \left(\binom{2}{0} \binom{5}{4} + \binom{2}{2} \binom{5}{3} \right) \right) \\ &= \frac{1}{24} (495 + 45 + 18 + 6(5 + 10)) \\ &= \frac{648}{24} \\ &= 27. \end{aligned}$$

The same method is used to calculate the number of unique colorings of the faces of the cube.

Example 3.17 ([1], based on example on page 231). Color the cube's 6 faces so that 2 faces are yellow and 4 faces are green. The identity can be described as a product of six 1-cycles. By examining Figures 16, 17, and 18, we can see that the 180° rotations around lines connecting the centers of opposite faces can be described as the product of two 1-cycles and two disjoint 2-cycles, and the 90° rotations around lines connecting the centers of the opposite faces can be written as the product of two 1-cycles and one 4-cycle. The 180° rotations around lines connecting the midpoints of opposite edges can be described as the product of three disjoint 2-cycles, and the 120° rotations around lines connecting opposite vertices can each be written as the product of two disjoint 3-cycles.

Thus, the cycle index of the rotation group is

$$P_{S_4} = \frac{1}{24} (x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2).$$

If the cube were being colored with 2 colors with no restrictions placed on the colors, by

Theorem 3.12 the total number of patterns is

$$\begin{aligned}
 P_{S_4}(2, \dots, 2) &= \frac{1}{24} [2^6 + 3(2^2 2^2) + 6(2^2 2) + 6(2^3) + 8(2^2)] \\
 &= \frac{240}{24} \\
 &= 10.
 \end{aligned}$$

However, we are calculating how many distinct colorings there are with 2 yellow faces and 4 green faces. Thus, we look at the coefficient of $y^2 g^4$ in

$$\begin{aligned}
 P_\sigma(\sigma_1, \dots, \sigma_{24}) &= \\
 \frac{1}{24} [(y + g)^6 + 3((y + g)^2(y^2 + g^2)^2) + 6((y + g)^2(y^4 + g^4)) + 6(y^2 + g^2)^3 + 8(y^3 + g^3)^2].
 \end{aligned}$$

Using the binomial theorem as before, the coefficient of $y^2 g^4$ is

$$\begin{aligned}
 \frac{1}{24} \left(\binom{6}{2} + 3 \left(\binom{2}{0} \binom{2}{1} + \binom{2}{2} \binom{2}{0} \right) + 6 \binom{2}{2} \binom{1}{0} + 6 \binom{3}{1} \right) &= \frac{1}{24} (15 + 9 + 6 + 18) \\
 &= \frac{48}{24} \\
 &= 2.
 \end{aligned}$$

It is easy to see that Pólya's Theorem is quite powerful. Burnside's Theorem is successful for counting the number of non-equivalent colorings caused by a permutation group G acting on a set X of objects to be colored, but its success is dependent on being able to compute the number of colorings of X fixed by a permutation $\tau \in G$. Thus, further consideration of the generalization of Burnside's Theorem, Pólya's Theorem, is beneficial for more difficult counting problems.

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