Boston College
Department of Physics

Collective Modes of Dipole Oscillations in Dusty Plasmas

A Thesis by

KUMAR SHASHI PRABH

Submitted in partial fulfillment of the requirements for the degree of

Master of Science

December 1998

UMI Number: 1392886

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

BOSTON COLLEGE

GRADUATE SCHOOL OF ARTS & SCIENCES

The thesis of: Kumar Shashi Prabh

entitled: Collective Modes of Dipole Oscillations in Dusty Plasmas

submitted to the Department of: Physics

in partial fulfillment of the requirements for the degree of:

Master of Science

in the Graduate School of Arts and Sciences has been read and approved by the Committee:

Chair: Professor Gabor J. Kalman

Member: Professor Michael J. Graf

Member:

Date: 10 December 1998

गुरूब्रह्मा गुरूविष्णुः गुरूर्देवो महेश्वरः ।
गुरूःसाक्षात परं ब्रह्म तस्मै श्री गुरूवेनमः ।।

A teacher is Brahma (The God who created the Universe), a teacher is Vishnu (The God who looks after the Universe), and a taecher is Maheshwara (The God who is the destroyer). A teacher is the supreme God, and I pay respect to my revered teacher.

Collective Modes of Dipole Oscillations in Dusty Plasmas

By KUMAR SHASHI PRABH

Abstract

Dusty plasma consists of micron size negatively charged particles immersed in plasma. Recent experiments have shown that under appropriate conditions dust grains crystallize. This is significant because it provides an easily observable and manipulable mesoscopic model for studying the dynamics and structure of crystal lattices, solid- liquid phase transitions and to investigate basic plasma interactions. Experiments also suggest that the crystallization is affected by dipole-dipole interaction between dust grains. I have studied the collective modes governed by the dipole-dipole interaction over linear (1D), square (2D) and cubic (3D) lattices. The lattice structure is fixed by the combined Coulomb and dipole interactions and the grains are assumed to carry permanent dipole moments which undergo directional oscillations. The interaction of these dipoles can lead to collective behavior. The dispersion relations for these wobbling modes are discussed. The longitudinal and transverse modes, and instability domains which delineate the different structure boundaries are identified.

Thesis Supervisor: Professor Gabor J. Kalman, Boston College

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prabh@excite.com

Acknowledgements

Prof. Gabor Kalman provided important ideas during my work on this thesis. I thank him for his support and constructive criticisms. I thank Prof. Michael Graf for serving on the thesis committee. I thank Prof. L. Tisza for his interest in my work. I also thank the Department of Physics, Boston College for giving me the opportunity to study here.

Deepak Goel has always been an immensely dedicated friend of mine. I am greatly indebted to him for all that he has done for me over years. I also thank my friends Gilad Kuperman, Jennifer Gupta, Saju Puthussery, Sujata Puthussery, Giulio Gambarota, Hongbo Zhao, and Chun-Lai Yang for their constant help and support especially during difficult times.

I thank US-DOE for supporting this work through DOE Grant

DE-FG02-98ER54501

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Chapter 1

Introduction

1.1 Dusty Plasma

A dusty plasma consists of micron/ sub-micron size particles immersed in plasma. Plasma electrons, ions and neutrals are constantly hit these particles. Electrons, being much lighter than ions, move faster. Therefore, more electrons than ions are collected on the dust grains making them negatively charged. The other charging mechanisms are photoionization (generally by UV radiation), and secondary electron emission due to impact of energetic electrons and ions. The grains acquire positive charge because of these two charging processes. Overall charge on the grain then depends on the relative strength of these competing charging processes. Laboratory created dusty plasma grains are negatively charged because of the dominance of the first mechanism over the other two. The dust particles are screened by surrounding plasma, primarily by electrons. Therefore, the interaction potential is Yukawa type.

In nature, dusty plasmas exist in interstellar clouds, planetary rings (notably Saturn), comets, atmospheric aerosols, noctilucent clouds etc. [Goretz, 1989]. The composition and size of grains of a dusty plasma in space depends on its surroundings.

The charge on a grain is typically 10,000 times the charge of an electron. In general, the strength of interaction in a plasma is defined by Γ which is equal to $\{(Z e)^2 / a k T\}$; Z e is the charge on the grain, a is the interparticle distance, k is the Boltzmann constant, and T is the temperature of the system. Because of the large value of Z, Γ can be quite large. However, because of the screening, it is not a good measure of the ratio of potential energy over kinetic energy. The true measure Γ_{eff} is defined as $\Gamma \exp(-\kappa_D a)$ where κ_D is the inverse of Debye length of the electrons. In the one – component plasma (OCP) limit, when there is no screening, the critical value of Γ when the system crystallizes is about 180, as obtained by molecular dynamics simulation [Hamaguchi, 1998]. In general the value of Γ_{eff} is much higher than this and the system therefore crystallizes. It is known that the structure for three dimensional Coulomb crystals is bcc. Experimentally, some systems are thin in perpendicular direction, i.e. there are only a few layers [Pieper, 1996]. These systems are practically two-dimensional. If the only interaction present is electrostatic, they crystallize in triangular structure. Generally, the lattice points of a layer should lie above the centers of the neighboring layer to minimize interaction energy, but it has been found that they lie on the top of the neighboring layers' lattice sites. This implies attractive interaction between the grains. Furthermore, if there are many layers, they have been found to crystallize in simple cubic [Lee 1997] in some cases. The fact that the dust grain system can crystallize is very significant because it provides a mesoscopic model for studying the dynamics and structure of atoms in solids. However, this same crystal becomes an unwanted object during the manufacturing process of semiconductor devices. These crystals carry momentum large enough to produce defects when they impact the wafer. In addition, the charged grains deflect the ions and make them fall at wrong places.

Since dust particles are negatively charged, the presence of dust particles in plasma allows the formation of electric fields within the plasma due to the excess of local positive charge. Additionally, electric and magnetic fields can affect these particles. The combined dust and plasma system has now new degrees of freedom as compared to the plasma – alone system. Therefore, the collective behavior of the plasma is altered by the appearance of new dust plasma modes, by the modification of the ion-plasma waves etc. The transport properties of a plasma and its potential profile are also altered.

It is understood that dusty plasma is an important agent in structure formation in the universe, in nebulae and possibly in quasars.

1.2 Motivation

The size of a unit cell in a dusty plasma crystal is of the order of 100 microns across and hence they can be observed by eye directly [Thomas,1994]. Typically, laser beams are used for illumination and the arrangement and motion of grains are recorded by a video camera for further analysis [Hayashi, 1996]. These crystals form a mesoscopic system and provide a very convenient model to study phase transitions and dynamics of the lattice configuration. Recent experiments have shown that there is attractive interaction between the grains in a dusty plasma [Mohideen,1998]. One possible explanation of this is dipole-dipole interaction. Grains acquire dipole moment because of uneven impact of streaming ions and polarization due to external electric field used to levitate them in laboratory. However, these two mechanisms produce anti parallel dipole moments. The resulting orientation of dipole moment depends on the net effect of these competing mechanisms. In laboratory experiments, the orientation of the dipoles has not been determined experimentally as yet. Furthermore, it has been verified experimentally that increasing electric field results in shrinking of the lattice which means that the dipole-dipole interaction is operating. It has also been observed that the dust grains always line up parallel to the field irrespective of the shape of the electric field lines. These facts favor the dipole-dipole interaction hypothesis.

It may be possible to drop ferromagnetic grains in plasma and study the magnetic dipole-dipole interaction between the grains. A UC-Irvine group (Rahman, Mohiddeen et.al.) is planning to do this experiment. This experiment has also been suggested independently by Resendes (Instituto Superior Tecnico).

As a consequence of the proposed dipole-dipole interaction, collective modes associated with this interaction can exist. One possible collective excitation corresponds to directional wobbling of the dipoles without any displacement from their equilibrium position. This mode is referred to as wobbling mode. The main motivation behind my work is to study these collective mode oscillations in a dusty plasma crystal. Not much work has been done in this area although this problem of dipole lattices and their collective excitations has been around for many years [Luttinger, 1946], [Sauer, 1940], [Rozenbaum, 1996].

1.3 The Model

The following describes the essential features of the model used in this thesis. (a) The dust grains have permanent dipole moments. In the case of an external electric field, the grains would have both permanent and induced dipole moments. We disregard the latter, the reason being that induced dipole moment will always realign itself according to the direction of the field. In case of magnetic dipoles, the dipole moment would be permanent. In the thesis, I have studied linear, square and cubic lattice structures. Although the system is three dimensional, in order to gain better understanding of the physical process I studied linear structure. The used cubic structure is one of the experimentally observed structures. (b) The lattice sites are fixed by combined coulomb and dipole-dipole interactions. (c) The dipoles in my models are arranged in minimum energy configuration. (d) Screening is not accounted for.

Further details and calculations can be found in chapters 2, 3 and 4.

1.4 The Method

Collective coordinates describing the cooperative wobbling of the grains are introduced. The dynamical matrix $\mathbf{D}\mu\nu(\mathbf{K})$, commonly used in the theory of harmonic phonons in crystal lattices is calculated [Pines, 1962]. In addition to the anisotropy induced by the lattice structure, the dynamical matrix has an inherent anisotropy of the dipole-dipole interaction. Then the terms associated with each site are summed over a large number of dipoles. The summation was checked for convergence. Finally, the eigenvalues of $\mathbf{D}\mu\nu(\mathbf{K})$ are determined to obtain the frequencies of collective modes.

1.5 Instability

If the chosen initial configuration of the dipoles is not the minimum energy configuration, then the dipoles will tend to reorient themselves to lower the energy of the configuration. This shows up as a dynamical instability in collective mode analysis. In subsequent chapters the instability domains are identified.

Chapter 2

One Dimensional Model

In this chapter we consider an infinitely long one dimensional array of dipoles having parallel aligned permanent dipole moment which approximates a dusty plasma crystal in one dimension.

The Model

Fig2.1: The minimum energy configuration for dipoles in a dusty plasma crystal in one dimensional approximation.

Let $p_i{}^{'s}$ be the dipole moments of individual dipoles, $I_i{}^{'s}$ their moments of inertia, $\Theta_i{}^{'s}$ their orientation with respect to z - axis and r_{ij} the radius vector from dipole i to dipole j. c is the lattice constant. In the following treatment, first Lagrangian is introduced (2.1). Then the equation of motion for θ is obtained (2.3). The equation (2.3) is then linearized for small oscillations (2.5). Following this, collective coordinates are introduced (2.6). (2.6) is substituted in (2.5) and the resulting equation is (2.7). (2.8) is sum over lattice sites. (2.11) is the final dispersion relation.

The Lagrangian for this system is:

$$L = \frac{1}{2} \sum_{\substack{i,j=-\infty \\ i\neq j}}^{\infty} \left(I_i (\dot{\theta}_i)^2 - \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}{}^3} + 3\, \frac{(\vec{p}_i \cdot \vec{r}_{ij})(\vec{p}_i \cdot \vec{r}_{ij})}{r_{ij}{}^5} \right) \qquad -(2.1)$$

$$= \frac{1}{2} \sum_{\substack{i,j=-\infty \\ i\neq j}}^{\infty} \left(I_i (\dot{\theta}_i)^2 - \frac{p_i\, p_j \operatorname{Cos}[\theta_i - \theta_j]}{r_{ij}{}^3} + 3\, \frac{p_i\, p_j \operatorname{Cos}[\theta_i] \operatorname{Cos}[\theta_j]}{r_{ij}{}^3} \right) \qquad -(2.2)$$

$$\Rightarrow I_i \ddot{\theta}_i =$$

$$\sum_{\substack{j=-\infty \\ i\neq j}}^{\infty} \left(\frac{p_i\, p_j \operatorname{Sin}[\theta_i - \theta_j]}{r_{ij}{}^3} - 3\, \frac{p_i\, p_j \operatorname{Sin}[\theta_i] \operatorname{Cos}[\theta_j]}{r_{ij}{}^3} \right) \qquad -(2.3)$$

Assuming small θ and $p_i = p_j = p$, $I_i = I_j = I$: –(2.4)

$$\ddot{\theta}_i = \frac{p^2}{2I}\sum_j \frac{-2\theta_i - \theta_j}{r_{ij}^3} \qquad -(2.5)$$

Assuming $\theta_j = \sum_k \theta_0(k)\, e^{i(\omega t - kz)}$ we get – (2.6)

$$-\omega^2 = \frac{p^2}{2I}\sum_j \frac{-2 - e^{-ikr_{ij}}}{r_{ij}^3} \qquad -(2.7)$$

$$= \frac{p^2}{2Ia^3}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{-2 - e^{-ikan}}{n^3}\ ; \quad |\mathbf{r_{ij}}| = na \qquad -(2.8)$$

$$= \frac{p^2}{2Ia^3}\sum_{n=1}^{\infty} \frac{-4 - e^{ikan} - e^{-ikan}}{n^3} \qquad -(2.9)$$

$$= \frac{p^2}{2Ia^3}\sum_{n=1}^{\infty} \frac{-4 - 2\,\mathrm{Cos}[k\,a\,n]}{n^3} \qquad -(2.10)$$

$$\boxed{\omega^2 = \frac{p^2}{2Ia^3}\left(4\,\xi(3) + \sum_{n=1}^{\infty}\frac{2\,\mathrm{Cos}[k\,a\,n]}{n^3}\right) \qquad -(2.11)}$$

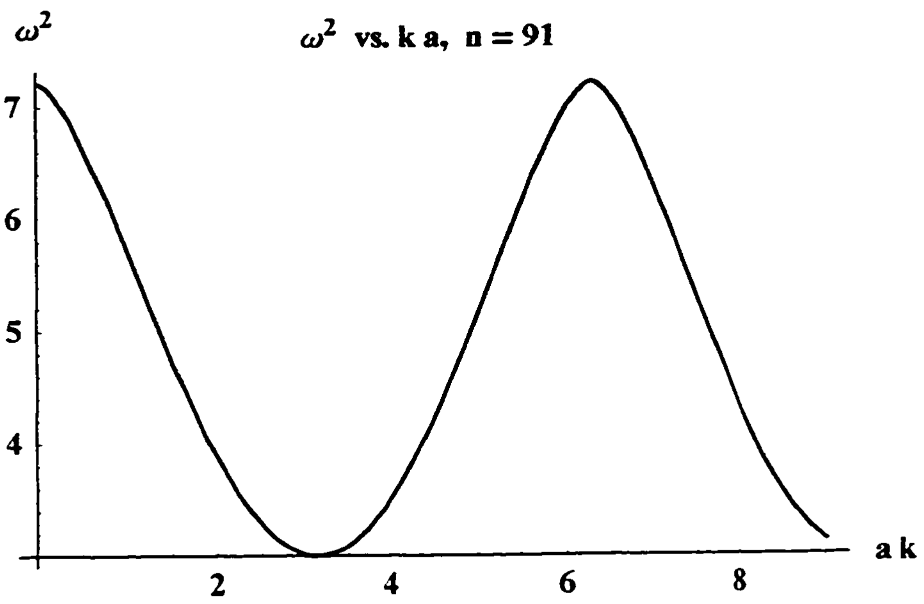


Fig 2.2 : Dispersion Curve (ω^2 in units of $p^2 / 2\,Ia^3$)

From the dispersion curve it can be concluded that in-phase oscillation, $k^*a = 2\,n\,\pi$, corresponds to the maximum frequency and out of phase oscillation, $k^*a = (2n+1)\pi$, corresponds to the minimum frequency. It was observed that the value of ω^2 saturates very fast with the number of dipoles (see the table and graph below).

n	ka = π	ka = 2π
21	2.98789	7.18519
61	3.00302	7.20911
101	3.00436	7.21116
141	3.00474	7.21173
181	3.00489	7.21197

Fig 2.3 : Table : Conversion of the sum for ω^2

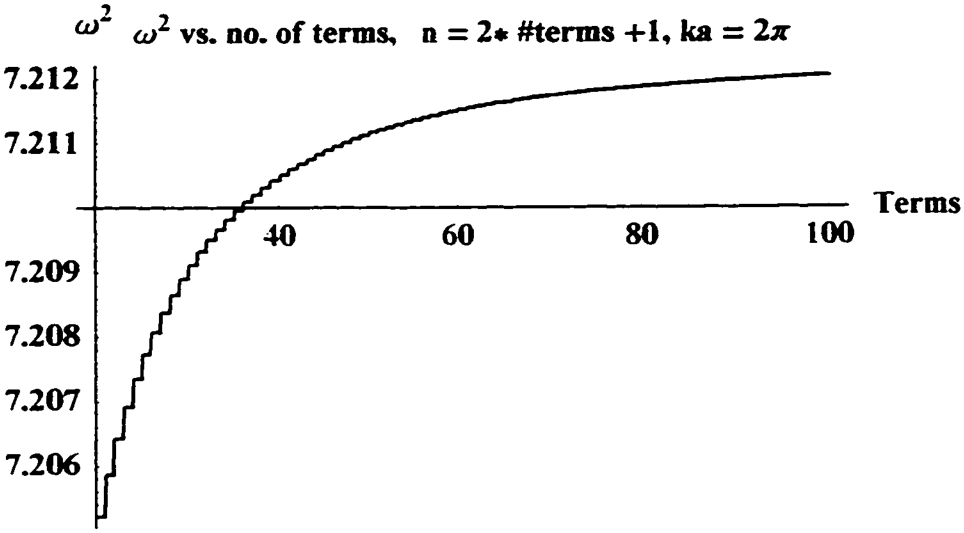


Fig 2.4 : Conversion of the sum for ω^2

Chapter 3

Two Dimensional Model

In this chapter we consider an infinite two dimensional array of dipoles having permanent dipole moment in a configuration which approximates a dusty plasma crystal in two dimensions having minimum energy.

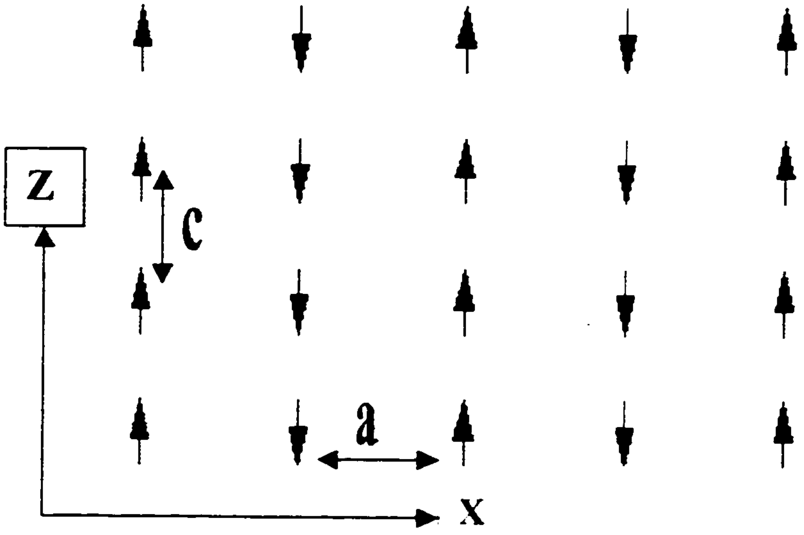


Fig3.1: The minimum energy configuration for dipoles in a dusty plasma crystal in two dimensional approximation.

Let p be the dipole moment of individual dipoles, I their moment of inertia, $\Theta_i{}'^s$ their orientation with respect to z - axis, r_{ij} the radius vector from dipole i to dipole j and $\phi_{ij}{}'^s$ the orientation of the radius vector with respect to z-axis. Let a be the length of unit cell along x-axis and c the length of unit cell along z-axis; m be the index for columns (x-axis) and l the index for rows (z-axis). It should be noted that i and j run over two indices. In the following treatment, first Lagrangian is introduced (3.2). Then the equation of motion for θ is obtained (3.4). The equation (3.5) is then linearized for small oscillations (3.8). Following this, collective coordinates are introduced (3.11). (3.11) is substituted in (3.10) and the resulting equation is (3.12). The sum is over lattice sites. (3.14) is the final dispersion relation.

From the figure above,

$$\cos \phi_{ij} = \frac{lc}{\sqrt{(ma)^2 + (lc)^2}} \qquad - (3.1)$$

The Lagrangian for this system is:

$$L = \frac{1}{2} \sum_{\substack{i,j=-\infty \\ i\neq j}}^{\infty} I_i (\dot{\theta}_i)^2 -$$

$$\frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}^3} + 3 \frac{(\vec{p}_i \cdot \vec{r}_{ij})(\vec{p}_j \cdot \vec{r}_{ij})}{r_{ij}^5} \qquad -(3.2)$$

$$= \frac{1}{2} \sum_{\substack{i,j=-\infty \\ i\neq j}}^{\infty} I (\dot{\theta}_i)^2 -$$

$$\frac{p^2 \operatorname{Cos}[\theta_i - \theta_j]}{r_{ij}^3} + 3 \frac{p^2 \operatorname{Cos}[\theta_i - \phi_{ij}] \operatorname{Cos}[\theta_j - \phi_{ij}]}{r_{ij}^3} \qquad -(3.3)$$

$$\Rightarrow \ddot{\theta}_i = \frac{p^2}{2I} \sum_{\substack{j=-\infty \\ i\neq j}}^{\infty}$$

$$\frac{1}{r_{ij}^3}[\operatorname{Sin}[\theta_i - \theta_j] - 3 \operatorname{Sin}[\theta_i - \phi_{ij}] \operatorname{Cos}[\theta_j - \phi_{ij}]] \qquad -(3.4)$$

$$= \frac{p^2}{2I} \sum_{\substack{j=-\infty \\ i\neq j}}^{\infty} \frac{1}{r_{ij}^3} (\operatorname{Cos}[\theta_j] \operatorname{Sin}[\theta_i] -$$

$$\operatorname{Cos}[\theta_i] \operatorname{Sin}[\theta_j] - 3 \{\operatorname{Cos}[\phi_{ij}] \operatorname{Sin}[\theta_i] - \operatorname{Cos}[\theta_i] \operatorname{Sin}[\phi_{ij}]\}$$

$$\{\operatorname{Cos}[\theta_j] \operatorname{Cos}[\phi_{ij}] + \operatorname{Sin}[\theta_j] \operatorname{Sin}[\phi_{ij}]\}) \qquad -(3.5)$$

Define:

$\theta_j = m\pi + \theta_{sj}$ where θ_{sj} is small. $\qquad -(3.6)$

$$\Rightarrow \operatorname{Cos}[\theta_j] = (-1)^m \operatorname{Cos}[\theta_{sj}] \approx (-1)^m$$

$$\mathrm{Sin}[\theta_j] = (-1)^m \,\mathrm{Sin}[\theta_{sj}] \approx (-1)^m \,\theta_{sj}. \qquad -(3.7)$$

In this approximation :

$$\ddot{\theta}_i =$$

$$\frac{p^2}{2\,I} \sum_{\substack{j,m=-\infty \\ i\neq j}}^{\infty} \frac{1}{r_{ij}{}^3} ((-1)^m \theta_i - (-1)^m \theta_{sj} - 3\ \{\mathrm{Cos}[\phi_{ij}]\ \theta_i - \mathrm{Sin}[\phi_{ij}]\}$$

$$\{(-1)^m \mathrm{Cos}[\phi_{ij}] + (-1)^m \theta_{sj}\ \mathrm{Sin}[\phi_{ij}]\})$$

$$-(3.8)$$

$$= \frac{p^2}{2\,I} \sum_{\substack{j,m=-\infty \\ i\neq j}}^{\infty} \frac{(-1)^m}{r_{ij}{}^3}$$

$$(\ \theta_i - \theta_{sj} - 3\ \{\mathrm{Cos}[\phi_{ij}]\ \theta_i - \mathrm{Sin}[\phi_{ij}]\}\{\ \mathrm{Cos}[\phi_{ij}] + \theta_{sj}\ \mathrm{Sin}[\phi_{ij}]\})$$

$$= \frac{p^2}{2\,I} \sum_{\substack{j,m=-\infty \\ i\neq j}}^{\infty} \frac{(-1)^m}{r_{ij}{}^3} (\ \theta_i - \theta_{sj} -$$

$$3\ \theta_i\ \ \mathrm{Cos}^2[\phi_{ij}] + 3\,\theta_{sj}\ \mathrm{Sin}^2[\phi_{ij}] + (\ \ )\,\mathrm{Sin}[\phi_{ij}]\,\mathrm{Cos}[\phi_{ij}])$$

$$-$$

(3. 9)

The last term is odd fn. of ϕ, therefore is equal to zero.

$$\Rightarrow\ \ddot{\theta}_i = \frac{p^2}{2\,I} \sum_{\substack{j,m=-\infty \\ i\neq j}}^{\infty}$$

$$\frac{(-1)^m}{r_{ij}{}^3} (\theta_i - \theta_{sj} - 3\ \theta_i\ \ \mathrm{Cos}^2[\phi_{ij}] + 3\,\theta_{sj}\ \mathrm{Sin}^2[\phi_{ij}]) \quad -(3.\ 10)$$

Introducing $\theta_{sj} = \sum_{k_x, k_z} \theta_0 (k_x, k_z)\, e^{i(\omega t - k_x m a - k_z l c)}$,

we get -(3.11)

$$-\omega^2 = \frac{p^2}{2I} \sum_{\substack{m,l \\ m\&l\neq 0}} \frac{(-1)^m}{(m^2 a^2 + l^2 c^2)^{\frac{3}{2}}}$$

$$\Bigg[1 - e^{i(-k_x m a - k_z l c)} - \frac{3\, l^2 c^2}{m^2 a^2 + l^2 c^2} + e^{i(-k_x m a - k_z l c)} \frac{3\, m^2 a^2}{m^2 a^2 + l^2 c^2}\Bigg]$$

-(3.12)

$$\Rightarrow\quad -\omega^2 = \frac{p^2}{2I} \sum_{m.l=0}^{\infty}{}' \frac{(-1)^m}{(m^2 a^2 + l^2 c^2)^{-}} [$$

$$m^2 a^2 - 2\, l^2 c^2 + e^{i(-k_x m a - k_z l c)} (2\, m^2 a^2 - l^2 c^2)]$$

-(3.13)

$$\Rightarrow\quad \omega^2 = \frac{p^2}{2I} \sum_{m.l=0}^{\infty}{}' \left(\frac{4 \operatorname{Cos}[\pi\, m]\, (-m^2 a^2 + 2\, l^2 c^2)}{(m^2 a^2 + l^2 c^2)^{5/2}} + \right.$$

$$\left. \frac{4 \operatorname{Cos}[(k_x a - \pi)\, m] \operatorname{Cos}[k_z\, l\, c] (-2\, m^2 a^2 + l^2 c^2)}{(m^2 a^2 + l^2 c^2)^{5/2}} \right)$$

-(3.14)

Setting $c/a = \gamma$:

$$\boxed{\omega^2 = \frac{p^2}{2\,I\,a^3} \sum_{m.l=0}^{\infty}{}' \left(\frac{4\,\mathrm{Cos}[\pi\, m]\,(-m^2 + 2\, l^2 \gamma^2)}{(m^2 + l^2 \gamma^2)^{5/2}} + \frac{4\,\mathrm{Cos}[(k_x a - \pi)\, m]\,\mathrm{Cos}[k_z\, l\, c]\,(-2\, m^2 + l^2 \gamma^2)}{(m^2 + l^2 \gamma^2)^{5/2}} \right)}$$

- (3.15)

Setting $a/c = \alpha$:

$$\boxed{\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m.l=0}^{\infty}{}' \left(\frac{4\,\mathrm{Cos}[\pi\, m]\,(-m^2 \alpha^2 + 2\, l^2)}{(m^2 \alpha^2 + l^2)^{5/2}} + \frac{4\,\mathrm{Cos}[(k_x a - \pi)\, m]\,\mathrm{Cos}[k_z\, l\, c]\,(-2\, m^2 \alpha^2 + l^2)}{(m^2 \alpha^2 + l^2)^{5/2}} \right)}$$

– (3. 16)

Variation in ω^2 with lattice parameters

Two dimensional model

Case 1: $k_x a = 0 = k_z c;\ c = 1$

$$
\begin{aligned}
\omega^2 = {} & \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\ m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4\,\mathrm{Cos}[(\,k_x\,a\ -\ \pi)\,m]}{(m^2\,\alpha^2)^{3/2}} \right) + \\
& \sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\ \mathrm{Cos}[\,k_z\,l\,c\,]}{l^3} \right) + \\
& \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\,\pi\ m]\ (\,-m^2\,\alpha^2 + 2\ l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \right. \\
& \left. \frac{4\,\mathrm{Cos}[(\,k_x\,a\ -\ \pi)\,m]\,\mathrm{Cos}[\,k_z\,l\,c\,](-2\ m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,6\,\mathrm{Cos}[\,\pi\ m]}{(m^2\,\alpha^2)^{3/2}} \right) + \\
& \sum_{l=1}^{p} \frac{6}{l^3} + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{12\,\mathrm{Cos}[\,\pi\ m]\ (\,-m^2\,\alpha^2\ +\ l^2)}{(m^2\,\alpha^2\ + l^2)^{5/2}} \right)
\end{aligned}
$$

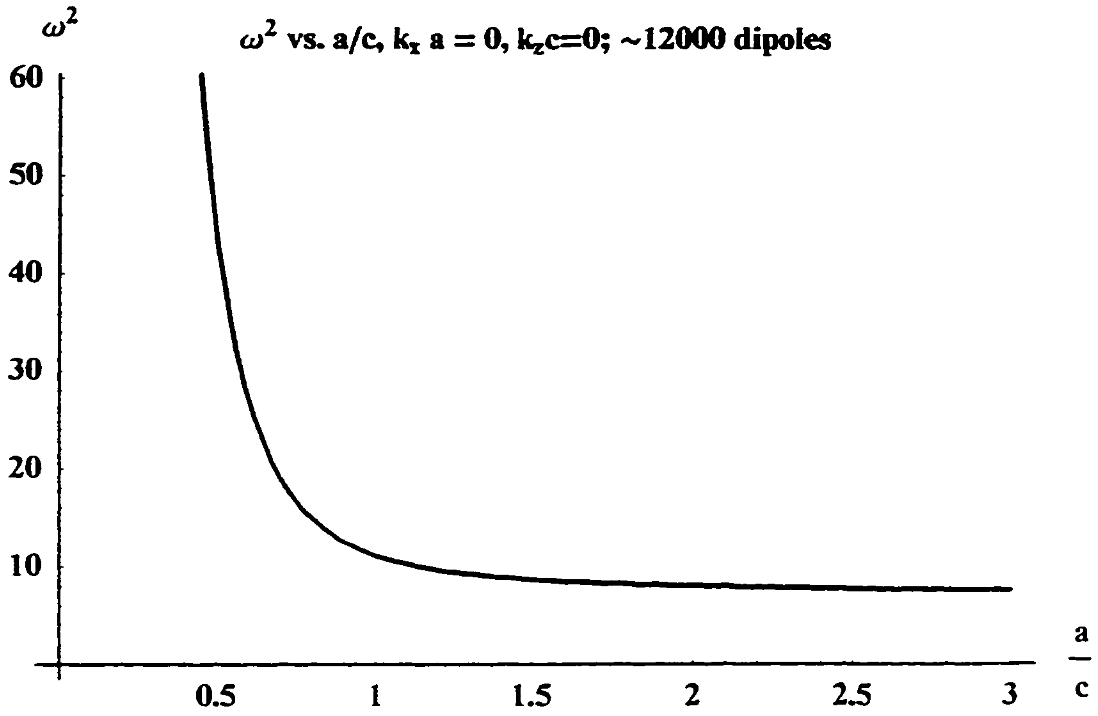


Fig 3.2: ω^2 vs the ratio a/c (ω^2 in units of $p^2/2\ I\ c^3$)

Case 2: $k_x a = 0$, $k_z c = \pi$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\ m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4\,\mathrm{Cos}[(k_x\,a\,-\,\pi)\,m]}{(m^2\,\alpha^2)^{3/2}} \right) +$$

$$\sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\ \mathrm{Cos}[\,k_z\,l\,c\,]}{l^3} \right) + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\ m]\ (-m^2\,\alpha^2 + 2\ l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \right.$$

$$\left. \frac{4\,\mathrm{Cos}[(k_x\,a\,-\,\pi)\,m]\,\mathrm{Cos}[\,k_z\,l\,c\,]\,(-2\ m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,6\,\mathrm{Cos}[\pi\ m]}{(m^2\,\alpha^2)^{3/2}} \right) +$$

$$\sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\ \mathrm{Cos}[\,\pi\,l\ ]}{l^3} \right) + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\ m]\ (-m^2\,\alpha^2 + 2\ l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \right.$$

$$\left. \frac{4\ \mathrm{Cos}[\pi\ m]\,\mathrm{Cos}[\pi\,l\ ]\,(-2\ m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

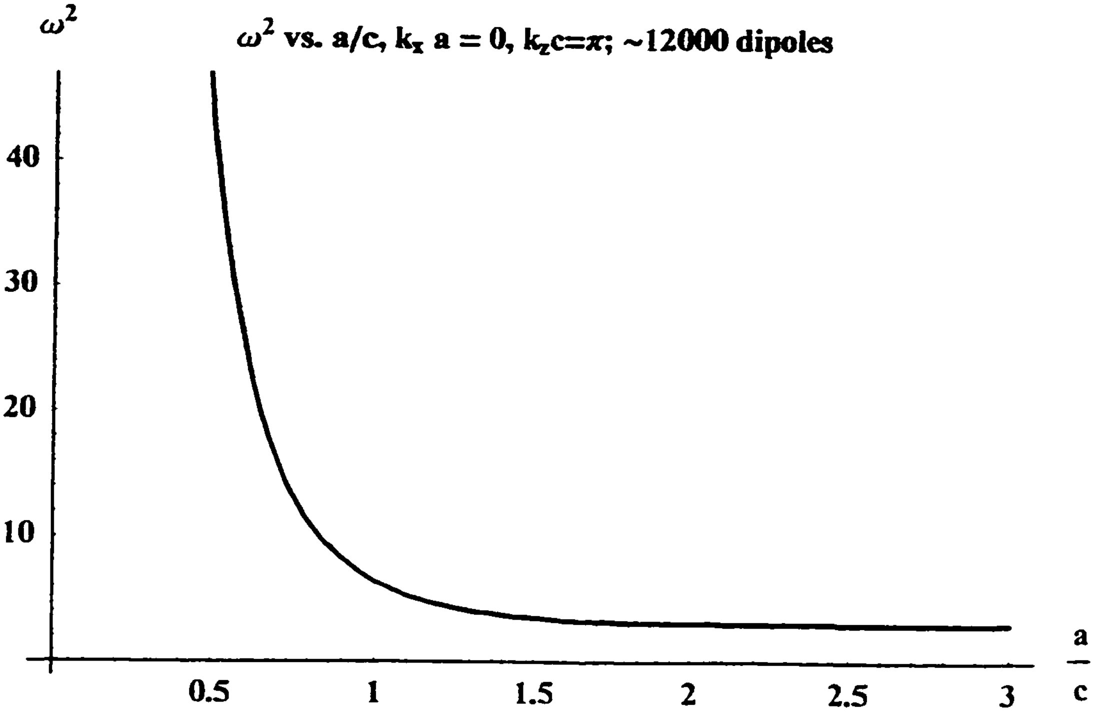


Fig 3.3 : ω^2 vs the ratio a / c (ω^2 in units of $p^2/2\ I\ c^3$)

Case 3 : $\boldsymbol{k_x \alpha = \pi,\ k_z = 0}$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\ m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4\,\mathrm{Cos}[(k_x\,a\ -\ \pi)\,m]}{(m^2\,\alpha^2)^{3/2}} \right) +$$

$$\sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\ \mathrm{Cos}[\,k_z\,l\,c\,]}{l^3} \right) + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\ m]\ (-m^2\,\alpha^2 + 2\ l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \right.$$

$$\left. \frac{4\,\mathrm{Cos}[(\,k_x\,a\ -\ \pi)\,m]\,\mathrm{Cos}[\,k_z\,l\,c\,](-2\ m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\ m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4}{(m^2\,\alpha^2)^{3/2}} \right) + \sum_{l=1}^{p} \frac{6}{l^3} +$$

$$\sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\ m]\ (-m^2\,\alpha^2 + 2\ l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \frac{4\ (-2\ m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

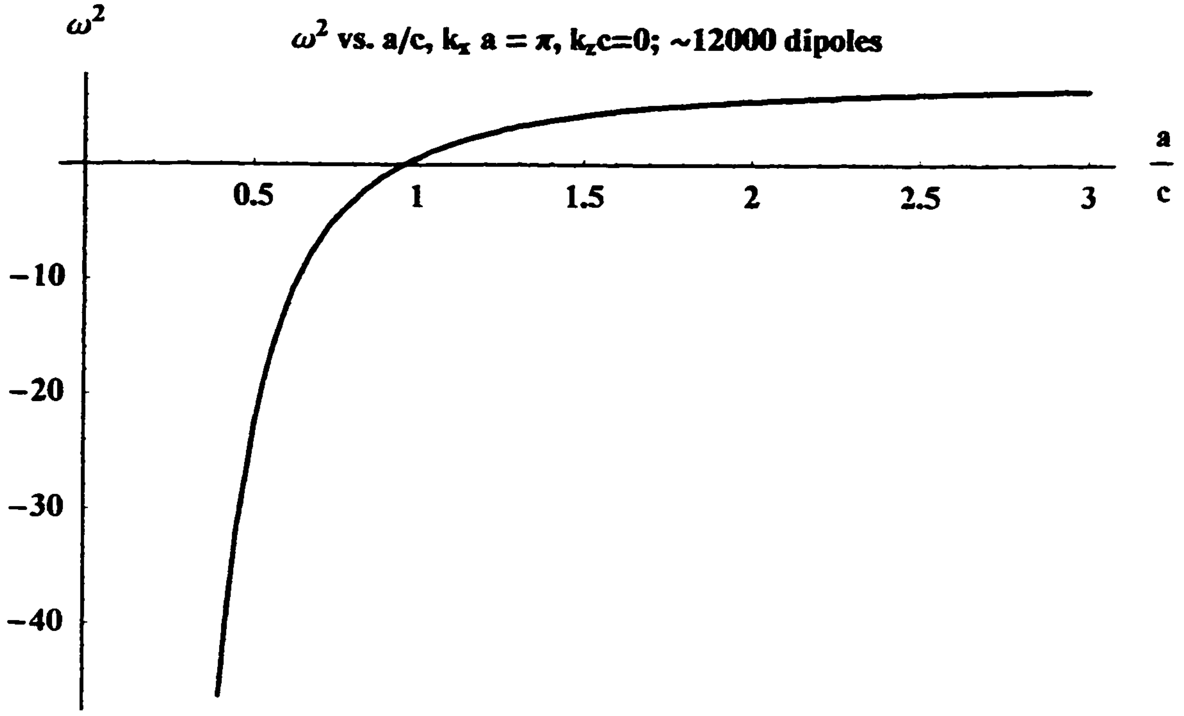


Fig 3.4 : ω^2 vs the ratio a / c (ω^2 in units of $p^2/2\ I\ c^3$)

Case 4: $k_x a = \pi,\ k_z = \pi$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\,m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4\,\mathrm{Cos}[(k_x\,a\,-\,\pi)\,m]}{(m^2\,\alpha^2)^{3/2}} \right) + \sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\,\mathrm{Cos}[k_z\,l\,c]}{l^3} \right) + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\,m]\,(-m^2\,\alpha^2 + 2\,l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \frac{4\,\mathrm{Cos}[(k_x\,a\,-\,\pi)\,m]\,\mathrm{Cos}[k_z\,l\,c]\,(-2\,m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

$$\omega^2 = \frac{p^2}{2\,I\,c^3} \sum_{m=1}^{p} \left(\frac{-\,2\,\mathrm{Cos}[\pi\,m]}{(m^2\,\alpha^2)^{3/2}} - \frac{4}{(m^2\,\alpha^2)^{3/2}} \right) + \sum_{l=1}^{p} \left(\frac{4}{l^3} + \frac{2\,\mathrm{Cos}[\pi\,l\,]}{l^3} \right) + \sum_{l=1}^{p} \sum_{m=1}^{p} \left(\frac{4\,\mathrm{Cos}[\pi\,m]\,(-m^2\,\alpha^2 + 2\,l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} + \frac{4\,\mathrm{Cos}[\pi\,l\,]\,(-2\,m^2\,\alpha^2 + l^2)}{(m^2\,\alpha^2 + l^2)^{5/2}} \right)$$

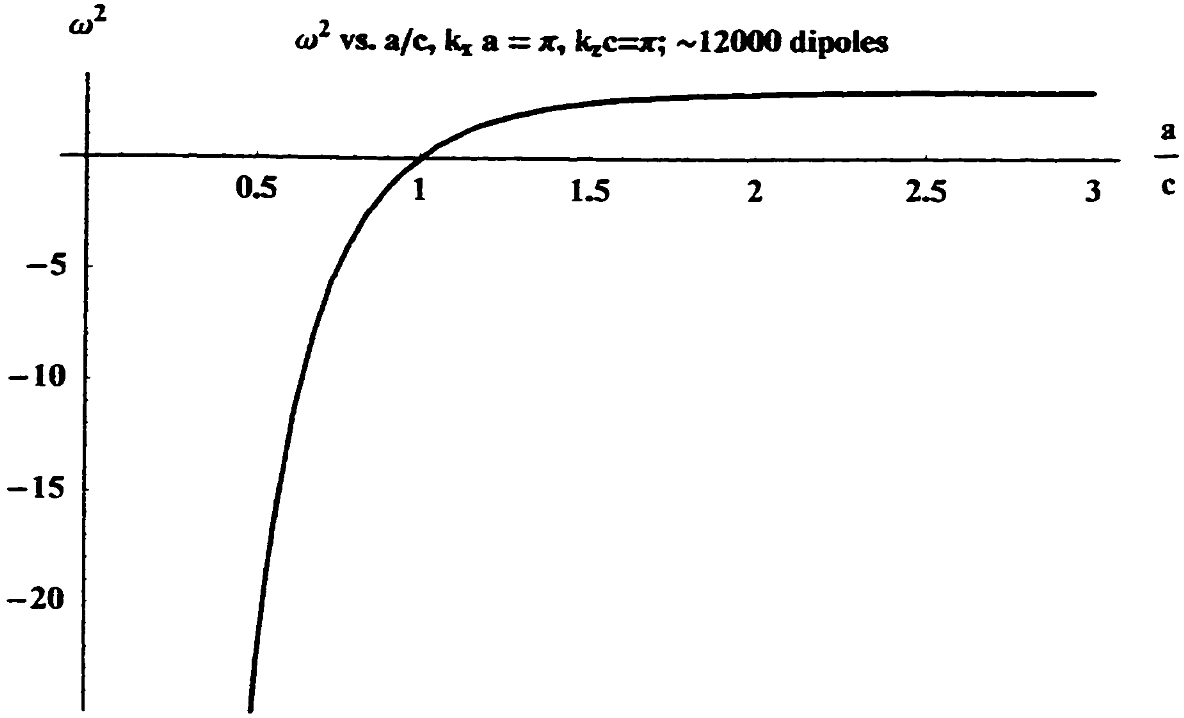


Fig 3.5 : ω^2 vs the ratio a / b (ω^2 in units of $p^2 / 2\, Ia^3$)

Dispersion curves for two dimensional model

(ω^2 in units of $p^2 / 2 I a^3$)

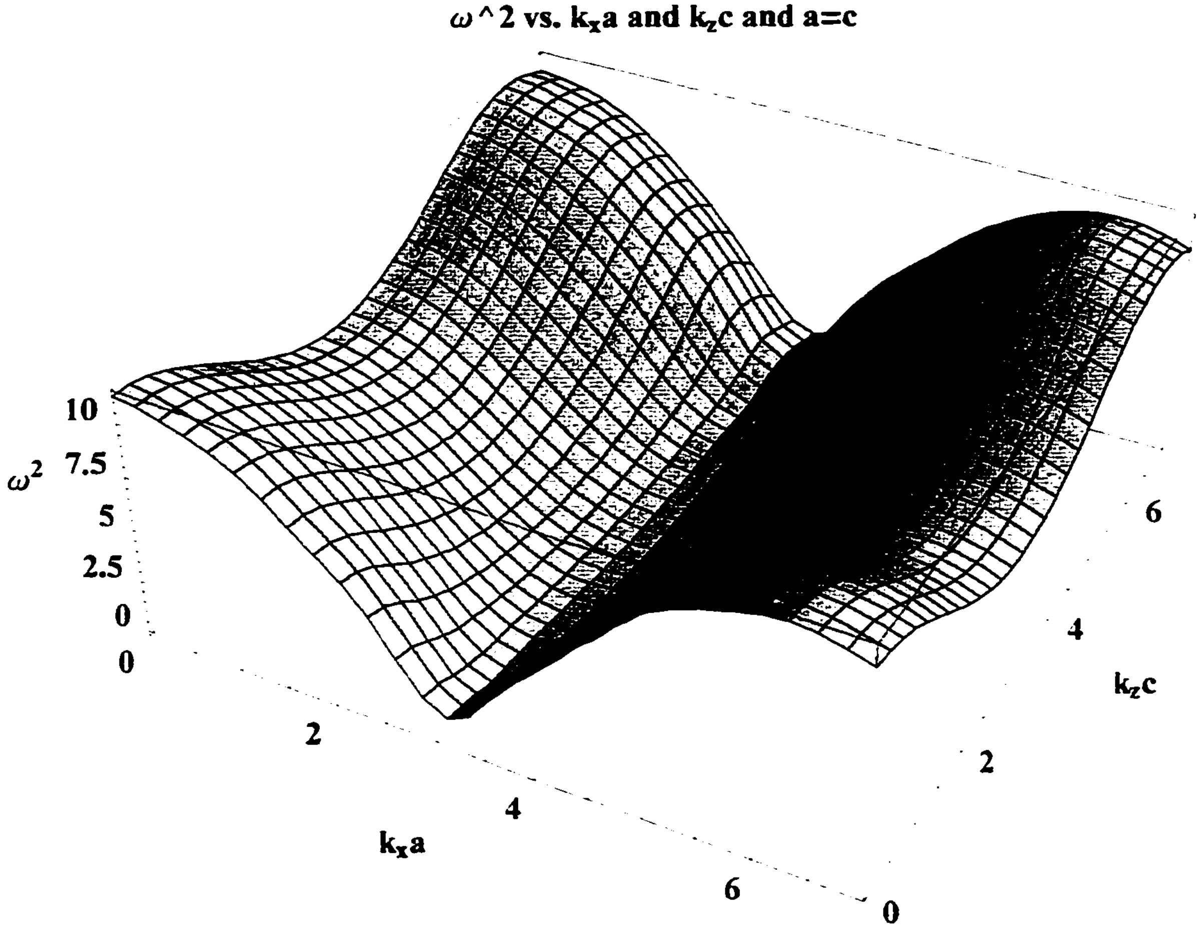


Fig 3.6 : ω^2 vs k_x a and k_z c

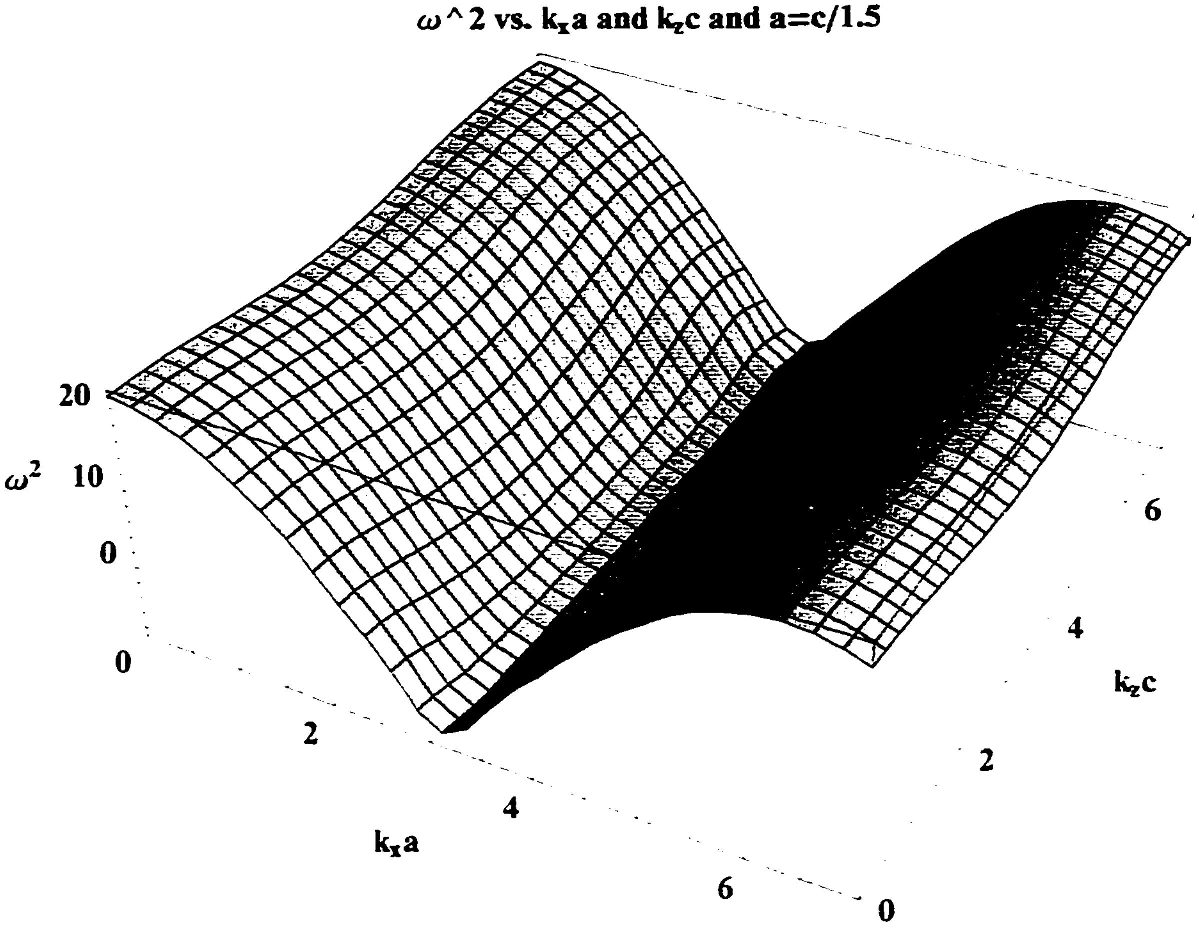


Fig 3.7: ω^2 vs $k_x a$ and $k_z c$ (unstable near $k_x a = \pi$);
(ω^2 in units of $p^2/2\,l\,c^3$)

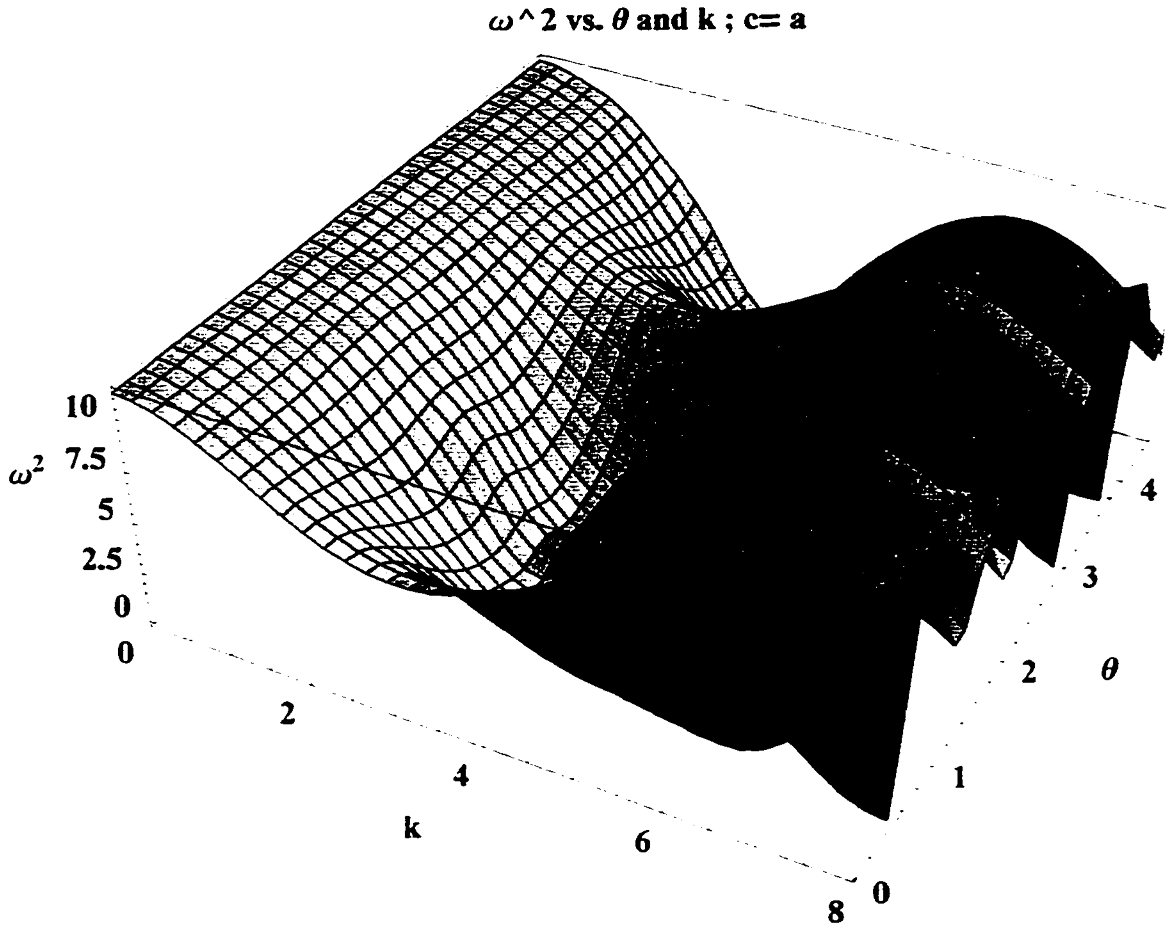


Fig 3.7 : Dispersion curve in polar coordinates :
ω^2 vs k and θ ; (ω^2 in units of $p^2/2$ I a^3)

Chapter 4

Three Dimensional Model

In this chapter we consider an infinite three dimensional array of dipoles having permanent dipole moment in a configuration which approximates a dusty plasma crystal in three dimension having minimum energy.

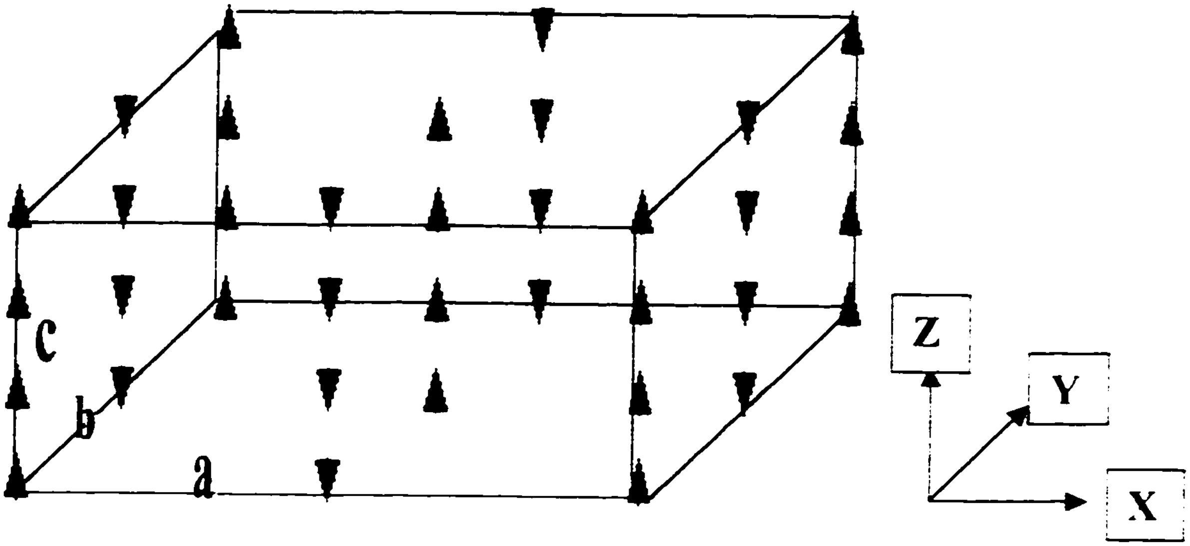


Fig 4.1: The minimum energy configuration for dipoles in a cubic dusty plasma crystal.

Let p be the dipole moment of individual dipoles, I their moment of inertia, θ_i's their orientation with respect to z - axis, φ_i's their orientation with respect to x - axis, $\mathbf{r}_{ij}$ the radius vector from dipole i to dipole j. Let a be the length of unit cell along x-axis, b the length of unit cell along y-axis and c the length of unit cell along z-axis. Let m be the index for columns along x-axis. n the index for columns along y-axis and n the index for rows i.e. z-axis.

In the following treatment, first Lagrangian is introduced (4.1). Then the Lagrangian is linearized for small oscillations in θ (4.5 - 4.7). φ could be anything between 0 and 2π . Therefore the oscillation is broken into two perpendicular parts of small amplitude (4.8 - 4.9). Then eqn. of motion for these coordinates are obtained (4.12 - 4.13). Following this. collective coordinates are introduced (4.14 - 4.15). Dynamical matrix is obtained (4.28 - 4.29). The eigenvalues of the dynamical matrix are (4.30) and (4.31).

The Lagrangian for this system is:

$$L = \frac{1}{2} \sum_{\substack{i, j = -\infty \\ i \neq j}}^{\infty} \left(I_i \left(\dot{\theta}_i^{\,2} + \dot{\varphi}_i^{\,2} \operatorname{Sin}^2 \theta_i \right) - \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}^{\,3}} + 3 \, \frac{\left(\vec{p}_i \cdot \vec{r}_{ij} \right) \left(\vec{p}_j \cdot \vec{r}_{ij} \right)}{r_{ij}^{\,5}} \right) \quad -(4.1)$$

Defining:

$$\Theta_j = (m + n)\,\pi + \theta_{sj} \text{ where } \theta_{sj} \text{ is small}$$

$$\Rightarrow \operatorname{Cos}[\theta_j] = (-1)^{m+n} \operatorname{Cos}[\theta_{sj}] \approx (-1)^{m+n} \left(1 - \frac{\theta_{sj}^{\,2}}{2} \right)$$

$$\text{and } \operatorname{Sin}[\theta_j] = (-1)^{m+n} \operatorname{Sin}[\theta_{sj}] \approx (-1)^{m+n}\, \theta_{sj} \quad -(4.2)$$

Assuming:

$$p_i = p_j = p, \; I_i = I_j = I \qquad -(4.3)$$

Then:

$$L = \frac{1}{2} \sum_{\substack{i, j, m, n, l = -\infty \\ i \neq j}}^{\infty} \Big(I \left(\dot{\theta}_i^{\,2} + \dot{\varphi}_i^{\,2} \operatorname{Sin}^2[\theta_i] \right) -$$

$$\frac{p^2}{r_{ij}^{\,3}} \left(\operatorname{Sin}[\theta_i] \operatorname{Sin}[\theta_j] \operatorname{Cos}[\varphi_i - \varphi_j] + \operatorname{Cos}[\theta_i] \operatorname{Cos}[\theta_j] \right) +$$

$$3 \, \frac{p^2}{r_{ij}^{\,5}} \left(m\, a \operatorname{Sin}[\theta_i] \operatorname{Cos}[\varphi_i] + n\, b \operatorname{Sin}[\theta_i] \operatorname{Sin}[\varphi_i] + l\, c \operatorname{Cos}[\theta_i] \right)$$

$$\left(m\, a \operatorname{Sin}[\theta_j] \operatorname{Cos}[\varphi_j] + \right.$$

$$\left. n\, b \operatorname{Sin}[\theta_j] \operatorname{Sin}[\varphi_j] + l\, c \operatorname{Cos}[\theta_j] \right) \qquad \_(4.4)$$

$$L = \frac{1}{2} \sum_{\substack{i,\, j,\, m,\, n,\, l = -\infty \\ i \neq j}}^{\infty}$$

$$\left(I \left(\dot{\theta}_i^{\,2} + \dot{\varphi}_i^{\,2} \, \theta_{si}^{\,2} \right) - \frac{(-1)^{m+n} \, p^2}{r_{ij}^{\,3}} \left(\theta_{si} \, \theta_{sj} \, \mathrm{Cos}[\varphi_i - \varphi_j] + \left(1 - \frac{\theta_{si}^{\,2}}{2} - \frac{\theta_{sj}^{\,2}}{2} \right) \right) + \right.$$

$$3 \, \frac{(-1)^{m+n} \, p^2}{r_{ij}^{\,5}} \left(m \, a \, \theta_{si} \, \mathrm{Cos}[\varphi_i] + n \, b \, \theta_{si} \, \mathrm{Sin}[\varphi_i] + l \, c \left(1 - \frac{\theta_{si}^{\,2}}{2} \right) \right)$$

$$\left. \left(m \, a \, \theta_{sj} \, \mathrm{Cos}[\varphi_j] + n \, b \, \theta_{sj} \, \mathrm{Sin}[\varphi_j] + l \, c \left(1 - \frac{\theta_{sj}^{\,2}}{2} \right) \right) \right)$$

– (4.5)

$$= \frac{1}{2} \sum{}'$$

$$\left(I \left(\dot{\theta}_i^{\,2} + \dot{\varphi}_i^{\,2} \, \theta_{si}^{\,2} \right) - \frac{(-1)^{m+n} \, p^2}{r_{ij}^{\,3}} \left(\theta_{si} \, \theta_{sj} \, \mathrm{Cos}[\varphi_i - \varphi_j] + \left(1 - \frac{\theta_{si}^{\,2}}{2} - \frac{\theta_{sj}^{\,2}}{2} \right) \right) + \right.$$

$$3 \, \frac{(-1)^{m+n} \, p^2}{r_{ij}^{\,5}} \Big(a^2 \, m^2 \, \mathrm{Cos}[\varphi_i] \, \mathrm{Cos}[\varphi_j] \, \theta_{si} \, \theta_{sj} +$$

$$b^2 \, n^2 \, \mathrm{Sin}[\varphi_i] \, \mathrm{Sin}[\varphi_j] \, \theta_{si} \, \theta_{sj} + c^2 \, l^2 - \frac{1}{2} \, c^2 \, l^2 \, \theta_{si}^2 -$$

$$\frac{1}{2} \, c^2 \, l^2 \, \theta_{sj}^2 + \frac{1}{4} \, c^2 \, l^2 \, \theta_{si}^2 \, \theta_{sj}^2 + a \, c \, l \, m \, \mathrm{Cos}[\varphi_i] \, \theta_{si} +$$

$$b \, c \, l \, n \, \mathrm{Sin}[\varphi_i] \, \theta_{si} + a \, c \, l \, m \, \mathrm{Cos}[\varphi_j] \, \theta_{sj} + b \, c \, l \, n \, \mathrm{Sin}[\varphi_j] \, \theta_{sj} +$$

$$a \, b \, m \, n \, \mathrm{Cos}[\varphi_j] \, \mathrm{Sin}[\varphi_i] \, \theta_{si} \, \theta_{sj} + a \, b \, m \, n \, \mathrm{Cos}[\varphi_i] \, \mathrm{Sin}[\varphi_j] \, \theta_{si} \, \theta_{sj} -$$

$$\frac{1}{2} \, a \, c \, l \, m \, \mathrm{Cos}[\varphi_j] \, \theta_{si}^2 \, \theta_{sj} - \frac{1}{2} \, b \, c \, l \, n \, \mathrm{Sin}[\varphi_j] \, \theta_{si}^2 \, \theta_{sj} -$$

$$\left. \frac{1}{2} \, a \, c \, l \, m \, \mathrm{Cos}[\varphi_i] \, \theta_{si} \, \theta_{sj}^2 - \frac{1}{2} \, b \, c \, l \, n \, \mathrm{Sin}[\varphi_i] \, \theta_{si} \, \theta_{sj}^2 \Big) \right)$$

– (4.6)

$$
\begin{aligned}
L = \frac{1}{2} \sum{}' \Bigg(& I\left(\dot{\theta}_i^{\,2} + \dot{\varphi}_i^{\,2}\, \theta_{si}^{\,2}\right) - \\
& \frac{(-1)^{m+n}\, p^2}{r_{ij}^{\,3}} \left(\theta_{si}\, \theta_{sj}\ \mathrm{Cos}[\varphi_i - \varphi_j] + \left(1 - \frac{\theta_{si}^{\,2}}{2} - \frac{\theta_{sj}^{\,2}}{2}\right)\right) + \\
& 3\, \frac{(-1)^{m+n}\, p^2}{r_{ij}^{\,5}} \Big(a^2\, m^2\, \mathrm{Cos}[\varphi_i]\, \mathrm{Cos}[\varphi_j]\, \theta_{si}\, \theta_{sj} + \\
& \qquad b^2\, n^2\, \mathrm{Sin}[\varphi_i]\, \mathrm{Sin}[\varphi_j]\, \theta_{si}\, \theta_{sj} + \frac{1}{2}\, c^2\, l^2\, (2 - \theta_{si}^2 - \theta_{sj}^2) + \\
& \qquad a\, b\, m\, n\, (\mathrm{Cos}[\varphi_j]\, \mathrm{Sin}[\varphi_i] + \mathrm{Cos}[\varphi_i]\, \mathrm{Sin}[\varphi_j])\, \theta_{si}\, \theta_{sj}\Big)\Bigg)
\end{aligned}
$$

(Discarding odd and higher order (> 2) terms)

– (4.7)

Substituting:

$\psi_i = \theta_{si}\, \mathrm{Sin}[\varphi_i]$ – (4.8)

$\eta_i = \theta_{si}\, \mathrm{Cos}[\varphi_i]$ – (4.9)

Upon this linearization, we get

$$L = \frac{1}{2}\sum{}' \Bigg(I\left((\dot{\psi}_i)^2 + (\dot{\eta}_i)^2\right) -$$

$$\frac{(-1)^{m+n}\,p^2}{r_{ij}^3}\left(\eta_i\,\eta_j + \psi_i\,\psi_j + 1 - \frac{\psi_i^2 + \eta_i^2}{2} - \frac{\psi_j^2 + \eta_j^2}{2}\right) +$$

$$3\,\frac{(-1)^{m+n}\,p^2}{r_{ij}^5}\Bigg(a^2\,m^2\,\eta_i\,\eta_j + b^2\,n^2\,\psi_i\,\psi_j +$$

$$c^2\,l^2\left(1 - \frac{\psi_i^2 + \eta_i^2}{2} - \frac{\psi_j^2 + \eta_j^2}{2}\right) + a\,b\,m\,n\,(\psi_i\,\eta_j + \psi_j\,\eta_i)\Bigg)\Bigg)$$

$$-\ (4.10)$$

$$= \frac{1}{2}\sum{}' \Bigg(I\left((\dot{\psi}_i)^2 + (\dot{\eta}_i)^2\right) -$$

$$\frac{(-1)^{m+n}\,p^2}{r_{ij}^3}\left(\eta_i\,\eta_j + \psi_i\,\psi_j + 1 - \frac{\psi_i^2 + \eta_i^2}{2} - \frac{\psi_j^2 + \eta_j^2}{2}\right) +$$

$$3\,\frac{(-1)^{m+n}\,p^2}{r_{ij}^5}\Bigg(a^2\,m^2\,\eta_i\,\eta_j + b^2\,n^2\,\psi_i\,\psi_j +$$

$$c^2\,l^2\left(1 - \frac{\psi_i^2 + \eta_i^2}{2} - \frac{\psi_j^2 + \eta_j^2}{2}\right) + a\,b\,m\,n\,(\psi_i\,\eta_j + \psi_j\,\eta_i)\Bigg)\Bigg)$$

$$-\ (4.11)$$

Then the equations of motion for ψ and η are :

$$2\,I\,\ddot{\psi}_i = -\sum{}' \left(\frac{(-1)^{m+n}\,p^2}{r_{ij}^3} (\psi_j - \psi_i) - 3\,\frac{(-1)^{m+n}\,p^2}{r_{ij}^5} \left(b^2 n^2 \psi_j + c^2 l^2 (-\psi_i) + a\,b\,m\,n\,\eta_j\right)\right)$$

i.e.,

$$\boxed{\ddot{\psi}_i = -\frac{p^2}{2I} \sum{}' \frac{(-1)^{m+n}}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \left(\psi_i(-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) + \psi_j (a^2 m^2 - 2 b^2 n^2 + c^2 l^2) + a\,b\,m\,n\,\eta_j\right)}$$

– (4.12)

$$\boxed{\ddot{\eta}_i = -\frac{p^2}{2I} \sum{}' \frac{(-1)^{m+n}}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \left(\eta_i(-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) + \eta_j (-2 a^2 m^2 + b^2 n^2 + c^2 l^2) + a\,b\,m\,n\,\psi_j\right)}$$

– (4.13)

Introducing:

$$\psi_j = \sum_{\vec{k}} \psi_0(\vec{k})\, e^{-i(\omega t - k_x m a - k_y n b - k_z l c)} \qquad -\ (4.14)$$

$$\eta_j = \sum_{\vec{k}} \eta_0(\vec{k})\, e^{-i(\omega t - k_x m a - k_y n b - k_z l c)} \qquad -\ (4.15)$$

$$-\omega^2 \psi_0 =$$

$$-\frac{p^2}{2I} \sum{}' \frac{(-1)^{m+n}}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \Big(\psi_0 (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\psi_0\, e^{-i(-k_x m a - k_y n b - k_z l c)} (a^2 m^2 - 2 b^2 n^2 + c^2 l^2) +$$

$$a\, b\, m\, n\, \eta_0\, e^{-i(-k_x m a - k_y n b - k_z l c)}\Big) \qquad \_(4.16)$$

$$-\omega^2 \eta_0 =$$

$$-\frac{p^2}{2I} \sum{}' \frac{(-1)^{m+n}}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \Big(\eta_0 (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\eta_0\, e^{-i(-k_x m a - k_y n b - k_z l c)} (-2 a^2 m^2 + b^2 n^2 + c^2 l^2) +$$

$$a\, b\, m\, n\, \psi_0\, e^{-i(-k_x m a - k_y n b - k_z l c)}\Big) \qquad \_(4.17)$$

$$\Rightarrow -\omega^2 \psi_0 =$$

$$-\frac{p^2}{2I} \sum{}' \frac{\mathrm{Exp}[i(m+n)\pi]}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} (\psi_0 (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\psi_0 e^{-i(-k_x m a - k_y nb - k_z l c)} (a^2 m^2 - 2 b^2 n^2 + c^2 l^2) +$$

$$a b m n\ \eta_0 e^{-i(-k_x m a - k_y nb - k_z l c)}),$$

-(4.18)

$$-\omega^2 \eta_0 =$$

$$-\frac{p^2}{2I} \sum{}' \frac{\mathrm{Exp}[i(m+n)\pi]}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} (\eta_0 (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\eta_0 e^{-i(-k_x m a - k_y nb - k_z l c)} (-2 a^2 m^2 + b^2 n^2 + c^2 l^2) +$$

$$a b m n \psi_0 e^{-i(-k_x m a - k_y nb - k_z l c)})$$

-(4.19)

$$\Rightarrow -\omega^2 \psi_0 = -\frac{p^2}{2I} \sum{}' \frac{1}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}}$$

$$(\psi_0 \mathrm{Exp}[i(m+n)\pi] (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\psi_0 e^{i((k_x - \pi) m a + (k_y - \pi) nb + k_z l c)} (a^2 m^2 - 2 b^2 n^2 + c^2 l^2) +$$

$$a b m n\ \eta_0 e^{i((k_x - \pi) m a + (k_y - \pi) nb + k_z l c)}),$$

-(4.20)

$$-\omega^2 \eta_0 = -\frac{p^2}{2I} \sum{}' \frac{1}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}}$$

$$(\eta_0 \mathrm{Exp}[i(m+n)\pi] (-a^2 m^2 - b^2 n^2 + 2 c^2 l^2) +$$

$$\eta_0 e^{i((k_x a - \pi) m + (k_y b - \pi) n + k_z l c)} (-2 a^2 m^2 + b^2 n^2 + c^2 l^2) +$$

$$a b m n \psi_0 e^{i((k_x a - \pi) m + (k_y b - \pi) n + k_z l c)})$$

-(4.21)

$$\begin{aligned}\Rightarrow \omega^2 \psi_o = \frac{p^2}{2\,I} \sum_{m,n,l=0}^{\infty}{}' &\frac{1}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \\ &(\psi_o\, 8\, \mathrm{Cos}[m\,\pi]\, \mathrm{Cos}[n\,\pi]\, (-a^2 m^2 - b^2 n^2 + 2\,c^2 l^2) + \\ &\quad \psi_o\, 8\, \mathrm{Cos}[(k_x\, a - \pi)\, m]\, \mathrm{Cos}[(k_y\, b - \pi)\, n] \\ &\qquad \mathrm{Cos}[k_z\, l\, c]\, (a^2 m^2 - 2\, b^2 n^2 + c^2 l^2) + \\ &\quad a\, b\, m\, n\, \eta_o\, 8\, \mathrm{Sin}[(k_x\, a - \pi)\, m]\, \mathrm{Sin}[(k_y\, b - \pi)\, n]\, \mathrm{Cos}[k_z\, l\, c]\,),\end{aligned}$$

−(4.22)

$$\begin{aligned}\omega^2 \eta_o = \frac{p^2}{2\,I} \sum_{m,n,l=0}^{\infty}{}' &\frac{1}{(a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}} \\ &(\eta_o\, 8\, \mathrm{Cos}[m\,\pi]\, \mathrm{Cos}[n\,\pi]\, (-a^2 m^2 - b^2 n^2 + 2\,c^2 l^2) + \\ &\quad \eta_o\, 8\, \mathrm{Cos}[(k_x\, a - \pi)\, m]\, \mathrm{Cos}[(k_y\, b - \pi)\, n] \\ &\qquad \mathrm{Cos}[k_z\, l\, c]\, (-2\, a^2 m^2 + b^2 n^2 + c^2 l^2) + \\ &\quad a\, b\, m\, n\, \psi_o\, 8\, \mathrm{Sin}[(k_x\, a - \pi)\, m]\, \mathrm{Sin}[(k_y\, b - \pi)\, n]\, \mathrm{Cos}[k_z\, l\, c])\end{aligned}$$

− (4.23)

For the following substitutions :

$$\begin{aligned}A = \sum_{m,n,l=0}^{\infty}{}' \Big(&8\, \mathrm{Cos}[m\,\pi]\, \mathrm{Cos}[n\,\pi] \\ &(-a^2 m^2 - b^2 n^2 + 2\,c^2 l^2) \Big/ (a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}\Big)\end{aligned}$$

− (4.24)

$$\begin{aligned}B = \sum_{m,n,l=0}^{\infty}{}' \Big(&8\, \mathrm{Cos}[(k_x\, a - \pi)\, m]\, \mathrm{Cos}[(k_y\, b - \pi)\, n]\, \mathrm{Cos}[k_z\, l\, c] \\ &(a^2 m^2 - 2\, b^2 n^2 + c^2 l^2) \Big/ (a^2 m^2 + b^2 n^2 + c^2 l^2)^{\frac{5}{2}}\Big)\end{aligned}$$

– (4.25)

$$C = \sum_{m,n,l=0}^{\infty}{}' \left(8\,\mathrm{Cos}[(k_x\,a-\pi)\,m]\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]\,(-2\,a^2\,m^2+b^2\,n^2+c^2\,l^2)\Big/(a^2\,m^2+b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\right)$$

– (4.26)

$$D = \sum_{m,n,l=0}^{\infty}{}' \left(-\,8\,\mathrm{Sin}[(k_x\,a-\pi)\,m]\,\mathrm{Sin}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]\;a\,b\,m\,n\Big/(a^2\,m^2+b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\right)$$

– (4.27)

$\Rightarrow$

$$\boldsymbol{\omega^2\,\psi_o(\vec{k}) = \{A(\vec{k})+B(\vec{k})\}\,\psi_o(\vec{k}) + D(\vec{k})\,\eta_o(\vec{k})} \quad -(4.28)$$

$$\boldsymbol{\omega^2\,\eta_o(\vec{k}) = D(\vec{k})\,\psi_o(\vec{k}) + \{A(\vec{k})+C(\vec{k})\}\,\eta_o(\vec{k})} \quad -(4.29)$$

The eigenvalues of the matrix
$\begin{pmatrix} A+B & D \\ D & A+C \end{pmatrix}$ give $\omega_{+/-}{}^2$.

Therefore,

$$\omega_-{}^2 = \frac{1}{2}\left(2\,A+B+C-\sqrt{(B-C)^2+4\,D^2}\right) \qquad -(4.30)$$

$$\omega_+{}^2 = \frac{1}{2}\left(2\,A+B+C+\sqrt{(B-C)^2+4\,D^2}\right) \qquad -(4.31)$$

The corresponding eigenvectors are :

$$\left\{-\frac{-B+C+\sqrt{(B-C)^2+4\,D^2}}{2\,D},\ 1\right\},$$

$$\left\{-\frac{-B+C-\sqrt{(B-C)^2+4\,D^2}}{2\,D},\ 1\right\} \qquad -(4.32)$$

Expanding the coefficients, we get :

$$\mathbf{A} = \sum_{m=1}^{p} (-2\,\mathrm{Cos}[m\,\pi]/(a^3\,m^3)) + \sum_{n=1}^{p} (-2\,\mathrm{Cos}[n\,\pi]/(b^3\,n^3)) +$$

$$\sum_{l=1}^{p} (4/(c^3\,l^3)) + \sum_{m=1}^{p}\sum_{n=1}^{p}\left(-\,4\,\mathrm{Cos}[m\,\pi]\,\mathrm{Cos}[n\,\pi]\Big/(a^2\,m^2+\,b^2\,n^2)^{\frac{3}{2}}\right) +$$

$$\sum_{m=1}^{p}\sum_{l=1}^{p}\left(4\,\mathrm{Cos}[m\,\pi]\,(-a^2\,m^2+2\,c^2\,l^2)\Big/(a^2\,m^2+c^2\,l^2)^{\frac{5}{2}}\right) +$$

$$\sum_{n=1}^{p}\sum_{l=1}^{p}\left(4\,\mathrm{Cos}[n\,\pi]\,(-b^2\,n^2+2\,c^2\,l^2)\Big/(\,b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\right) +$$

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{l=1}^{p}\Big(8\,\mathrm{Cos}[m\,\pi]\,\mathrm{Cos}[n\,\pi]$$

$$(-a^2\,m^2-b^2\,n^2+2\,c^2\,l^2)\Big/(a^2\,m^2+\,b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\Big)$$

$$-(4.34)$$

$$\mathbf{B} = \sum_{m=1}^{p} (2\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]/(a^3\,m^3)) + \sum_{n=1}^{p} (-4\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\;/(b^3\,n^3)) +$$

$$\sum_{l=1}^{p}(2\,\mathrm{Cos}[k_z\,l\,c]/(c^3\,l^3)) + \sum_{m=1}^{p}\sum_{n=1}^{p}\Big(4\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]$$

$$\mathrm{Cos}[(k_y\,b-\pi)\,n]\,(a^2\,m^2-2\,b^2\,n^2)\Big/(a^2\,m^2+\,b^2\,n^2)^{\frac{5}{2}}\Big) +$$

$$\sum_{m=1}^{p}\sum_{l=1}^{p}\Big(4\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]\,\mathrm{Cos}[k_z\,l\,c]\Big/(a^2\,m^2+c^2\,l^2)^{\frac{3}{2}}\Big) + \sum_{n=1}^{p}\sum_{l=1}^{p}\Big(4$$

$$\mathrm{Cos}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]\,(-2\,b^2\,n^2+c^2\,l^2)\Big/(\,b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\Big) +$$

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{l=1}^{p}\Big(8\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]$$

$$(a^2\,m^2-2\,b^2\,n^2+c^2\,l^2)\Big/(a^2\,m^2+\,b^2\,n^2+c^2\,l^2)^{\frac{5}{2}}\Big)$$

$$-\,(4.35)$$

$$C =$$

$$\sum_{m=1}^{p}(-4\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]/(a^3\,m^3)) + \sum_{n=1}^{p}(2\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\,/(b^3\,n^3)) +$$

$$\sum_{l=1}^{p}(2\,\mathrm{Cos}[k_z\,l\,c]/(c^3\,l^3)) + \sum_{m=1}^{p}\sum_{n=1}^{p}\Big(4\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]$$

$$\mathrm{Cos}[(k_y\,b-\pi)\,n]\,(-2\,a^2\,m^2+b^2\,n^2)\Big/(a^2\,m^2+\,b^2\,n^2)^{\frac{5}{2}}\Big) +$$

$$\sum_{m=1}^{p}\sum_{l=1}^{p}\Big(4\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]$$

$$\mathrm{Cos}[k_z\,l\,c]\,(-2\,a^2\,m^2+c^2\,l^2)\Big/(a^2\,m^2+c^2\,l^2)^{\frac{5}{2}}\Big) +$$

$$\sum_{n=1}^{p}\sum_{l=1}^{p}\Big(4\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]\Big/(\,b^2\,n^2+c^2\,l^2)^{\frac{3}{2}}\Big) +$$

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{l=1}^{p}\Big(8\,\mathrm{Cos}[(\,k_x\,a-\pi)\,m]\,\mathrm{Cos}[(k_y\,b-\pi)\,n]\,\mathrm{Cos}[k_z\,l\,c]$$

$$(-2\,a^2\,m^2 + b^2\,n^2 + c^2\,l^2)\Big/(a^2\,m^2 + b^2\,n^2 + c^2\,l^2)^{\frac{5}{2}}\Big)$$

$$-\,(4.36)$$

$$\mathbf{D} = \sum_{m=1}^{p}\sum_{n=1}^{p}\Big(-\,4\,\mathrm{Sin}[(\,k_x\,a - \pi)\,m]$$

$$\mathrm{Sin}[(k_y\,b - \pi)\,n]\;\;a\,b\,m\,n\Big/(a^2\,m^2 + b^2\,n^2)^{\frac{5}{2}}\Big) +$$

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{l=1}^{p}\Big(-\,8\,\mathrm{Sin}[(\,k_x\,a - \pi)\,m]\,\mathrm{Sin}[(k_y\,b - \pi)\,n]$$

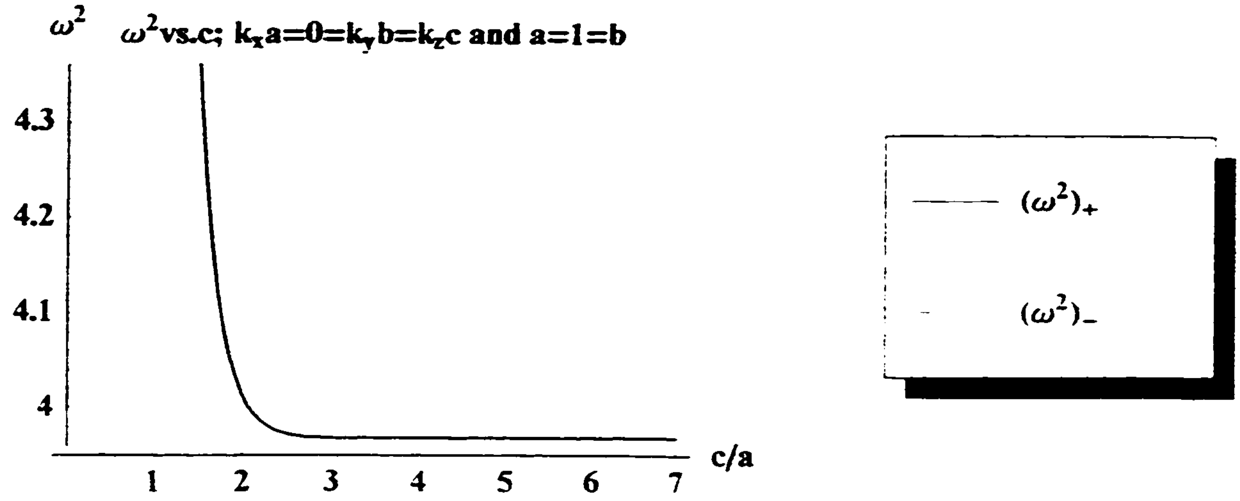
$$\mathrm{Cos}[k_z\,l\,c]\;\;a\,b\,m\,n\Big/(a^2\,m^2 + b^2\,n^2 + c^2\,l^2)^{\frac{5}{2}}\Big)$$

$$-\;(4.37)$$

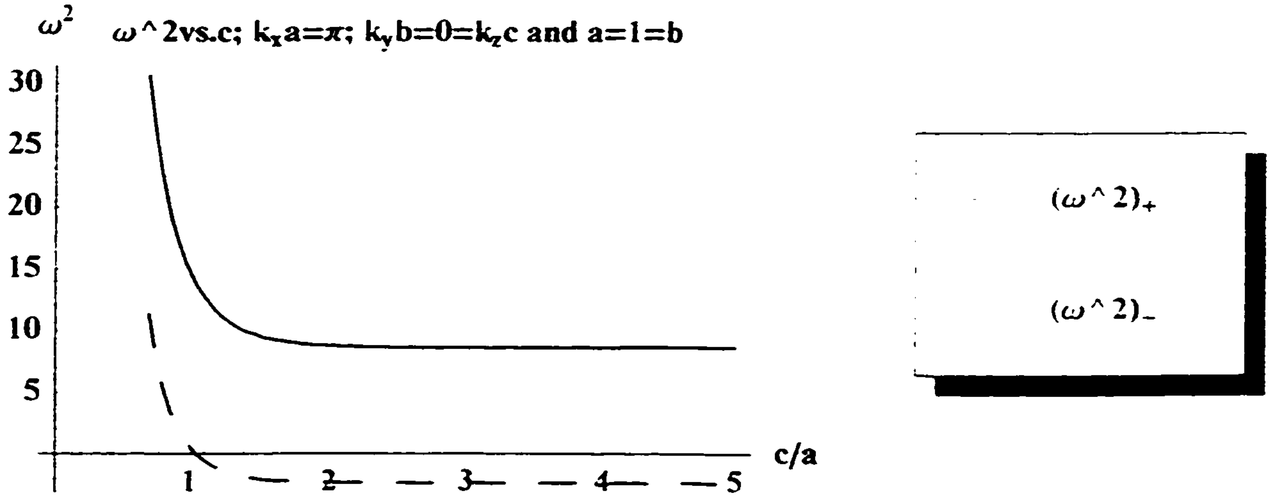
Variation in ω^2 with lattice parameters

Three dimensional model

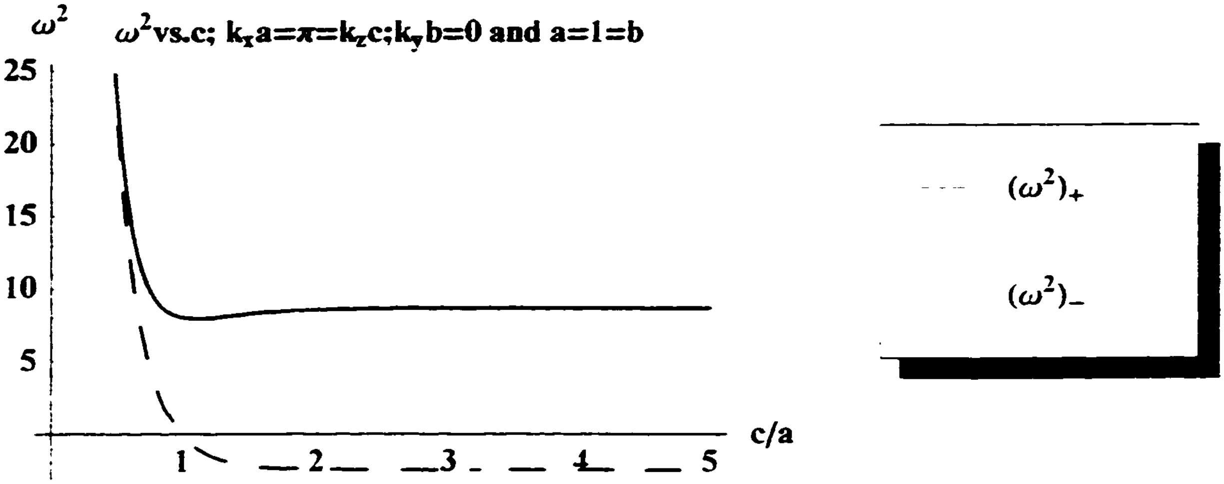
Case 1: $k_x a = 0 = k_y b = k_z c$; $a = b$



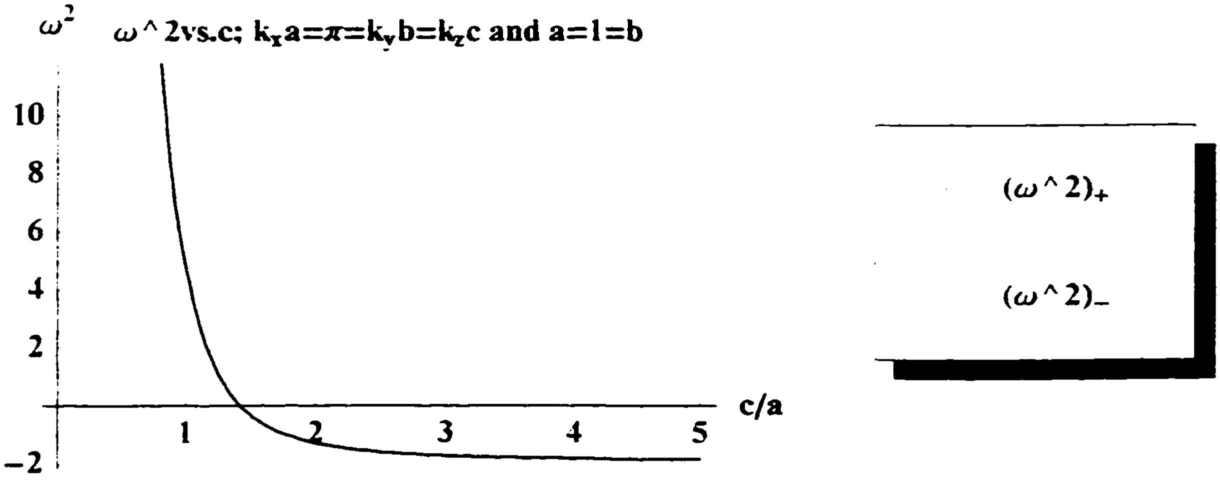
Case 2: $k_x a = \pi$, $k_y b = 0 = k_z c$; $a = b$



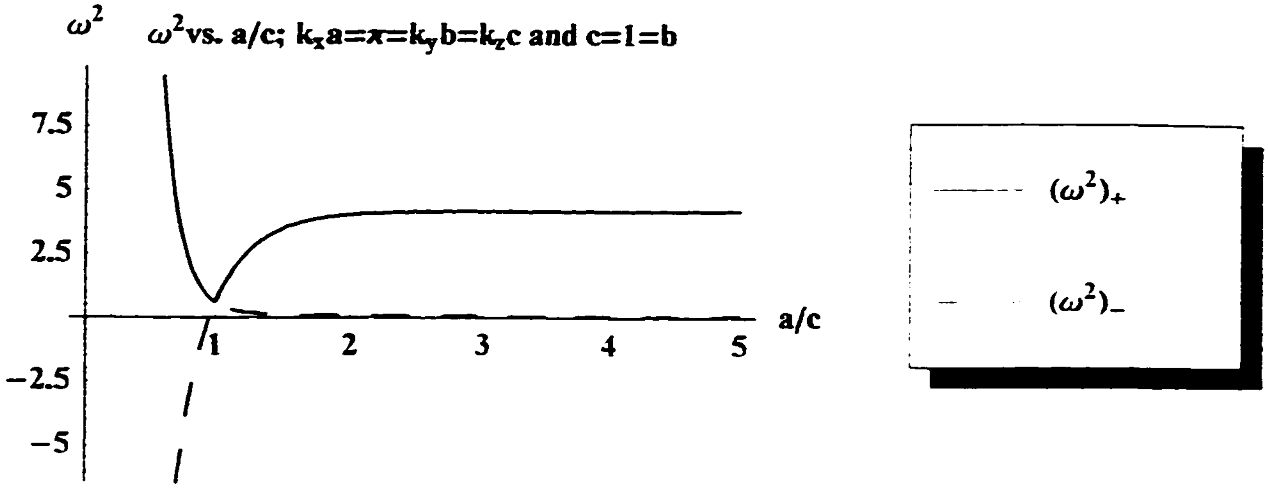
Case 3: $k_x a = \pi, k_y b = 0, k_z c = \pi$



Case 4: $k_x a = \pi, k_y b = \pi, k_z c = \pi$

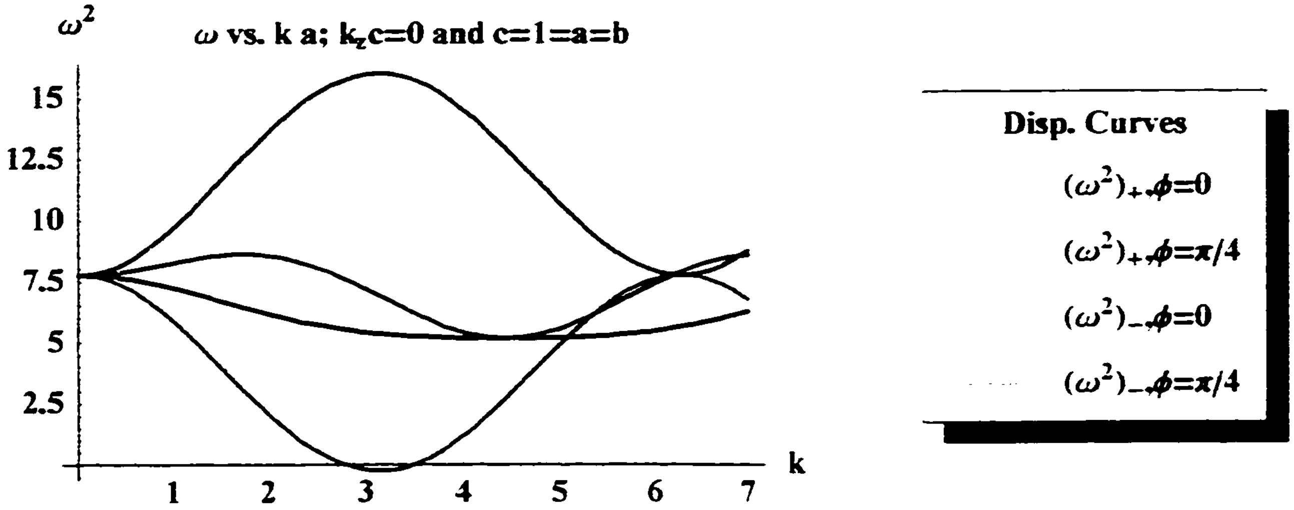


Case 5 : $k_x a = \pi = k_y b = k_z c$; $c = b$

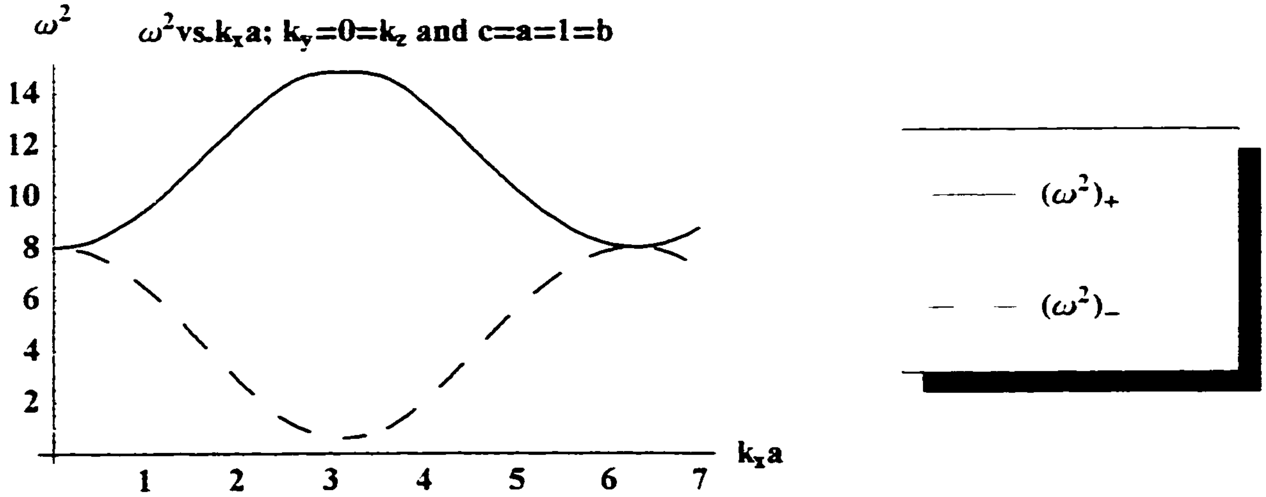


Dispersion curves for three dimensional model

(ω^2 in units of $p^2 / 2 I a^3$)



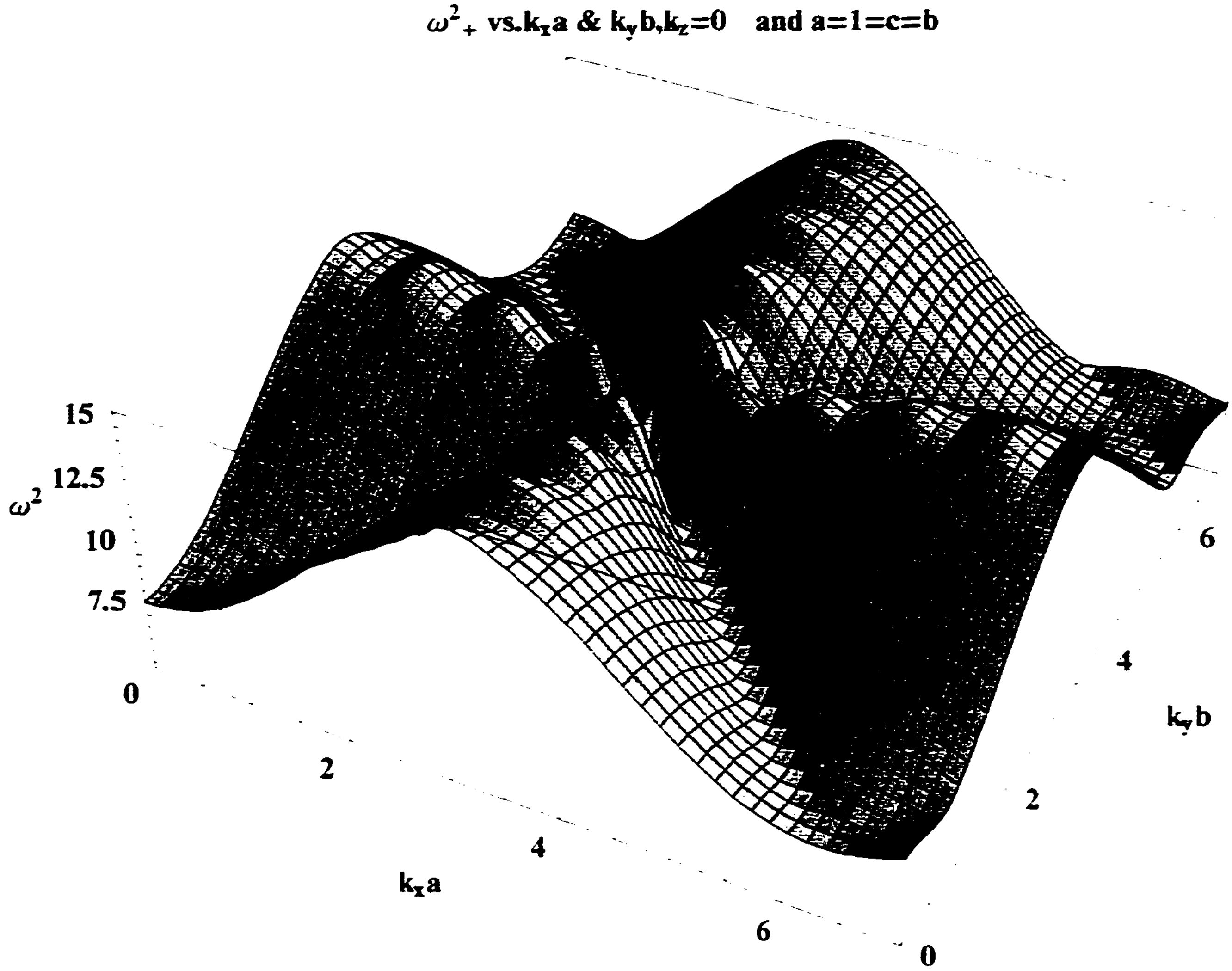
ω vs. k a; $k_z c=0$ and c=1=a=b



ω^2vs.$k_x a$; $k_y=0=k_z$ and c=a=1=b

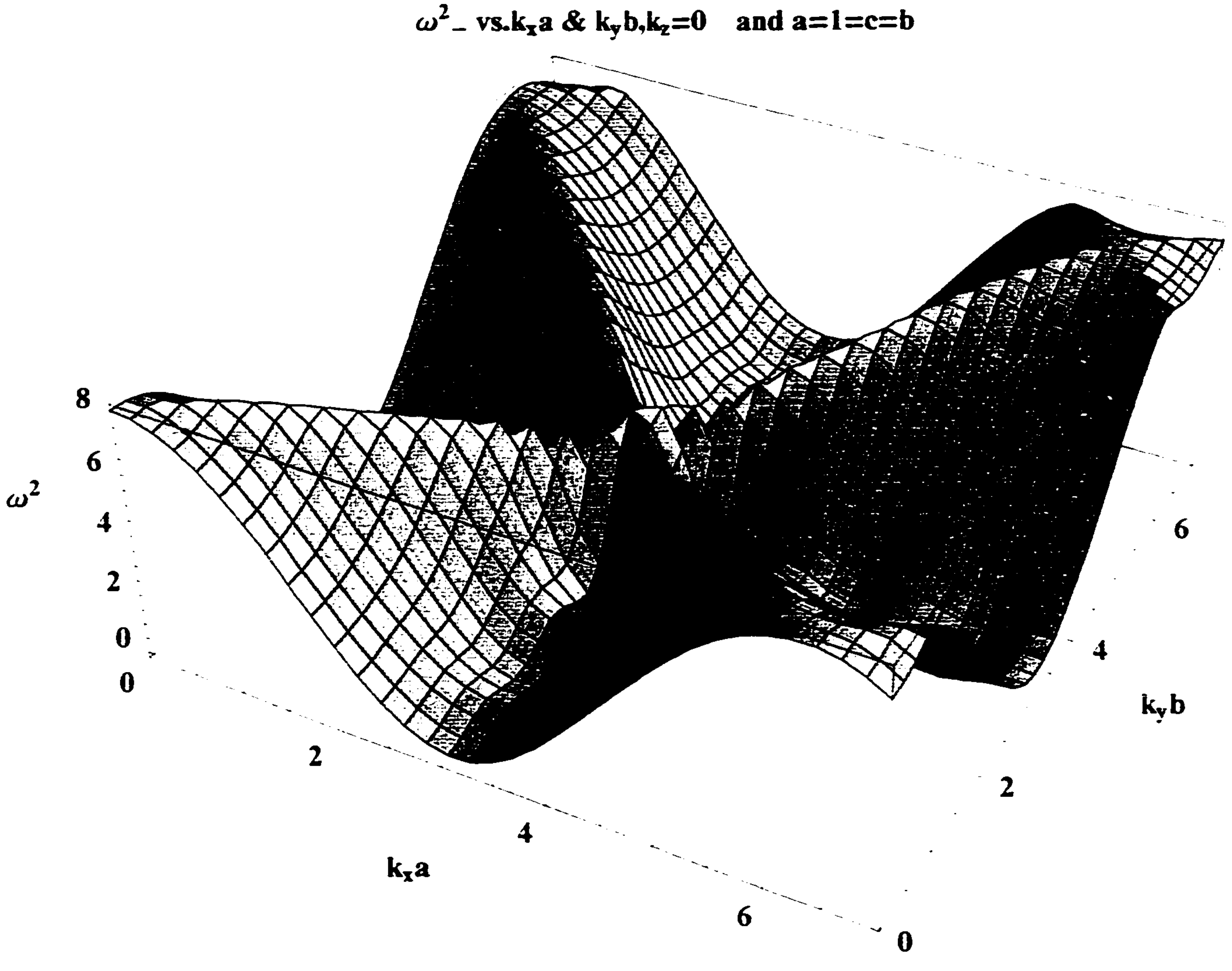
Transverse Mode

(ω^2 in units of $p^2 / 2 I a^3$)



Longitudinal Mode

(ω^2 in units of $p^2/2\,l\,a^3$)



Chapter 5

5.1 Conclusions

In reviewing the general features of the results, we find the following as main features:

(a). There exists finite frequency at k=0 even though the interaction is not a Coulomb interaction. This corresponds to jelly - like motion. The order of magnitude of this frequency for electric dipoles is estimated below:

$Z \sim 10{,}000$

$m \sim 10$ E -12 gm (assuming mass density of 1 gram/ cc, 1 micron grain size)

$a \sim 100$ microns

α_{eff} = Effective polarizability.

Therefore, $\omega^2 \sim 5\,(Z\,e)^2 /\; m\, a^3 \alpha_{eff} \sim 10$ E 8 Hz.

Furthermore, $\omega^2 \sim \omega_p^2$.

For the ferromagnetic dipoles, the frequency depends on the square of magnetic dipole moment, and in the planned experiment, it is expected to be close to plasma frequency.

(b). We get two modes of collective oscillation in a real lattice which can be identified as longitudinal and transverse modes. Near K=0, the frequency of the transverse mode increases with K and that of the longitudinal mode decreases with K. There are additional symmetry points where modes become identical.

(c). There are instability domains which coincide with the boundaries of the minimum energy configurations. For square lattices the most unstable oscillations are out of phase oscillations (K_x a, K_z c) = (π, π) along x and z axes. For cubic lattices, for equal lattice constants along x and y axes, the most unstable oscillation corresponds to (K_x a, K_y b, K_z c) = (π, 0, π). However, one finds that the critical value of c/a is higher for (π, π, π) oscillations. The reason for this behavior is that the dipoles contained in adjacent x-z planes support their neighbors from being flipped. Additionally, only the longitudinal mode becomes unstable for (π, 0, π) oscillations but in the case of (π, π, π) oscillations both, longitudinal and transverse, modes become unstable.

(d). In the case of (π, 0, π) oscillations in cubic lattices, the system becomes quasi one-dimensional for sufficiently low c/a.

5.2 Outlook

Future work should address these problems.

(a). There is screening in the system. Yukawa type potential is the correct potential for electric dipoles. Magnetic dipoles are unscreened. As the result of screening, the importance of dipole-dipole interaction relative to Coulomb interaction increases because the former acquires exp(- κ_D a)/ r term.

(b). If the grains are electric dipoles, then the effect of external electric field used to levitate them is important. The electric field will polarize the grains and the effect of induced dipole moment is important. Additionally, there is interaction between the field and the dipoles which tries to align the dipoles along field lines.

(c). There are charge fluctuations in the grain chrge and dipole moment.

(d). The collisions with neutrals damps the motion of dust grains.

(e). Grains are not identical in reality. Depending on their shapes and other properties, they acquire different charges, which leads to additional degrees of freedom.

(f). Plasma oscillation could probably excite wobbling modes. In that case, these modes would be coupled to plasma oscillations.

(g). If external magnetic field is present, then the effect of B on the motion of the plasma particles will affect the wobbling motion of grains. Induced currents due to the external magnetic field will affect the dielectric properties of the plasma and the effective dipole – dipole interaction.

(h). The planned future experiments with ferromagnetic grains would provide an exciting opportunity to verify our results.

(i). For $1 << \Gamma_{eff} < \Gamma_{cr}$, the system is in liquid state and the wobbling modes could be studied through the quasi localization of charge [Kalman & Golden, 1990] developed for this situation.

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