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Discrete Mathematics 306 (2006) 3011 – 3013
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Transversals in long rectangular arrays

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Received 21 November 2003; received in revised form 19 January 2004; accepted 5 February 2004
Available online 24 July 2006

Abstract

In this paper it is shown that every $m \times n$ array in which each symbol appears at most $(mn-1)/(m-1)$ times has a transversal, when $n \geqslant 2m^3$.

MSC: 05B15; 05D15

Keywords: Transversal; Rectangular array

1. Introduction

An $m \times n$ *array* $(m \leqslant n)$ is a table of m rows and n columns and therefore mn cells, where each cell contains exactly one symbol. A *partial transversal* in an array is a set of cells in which no two cells are in the same row or column and no two cells contain the same symbol. A *transversal* in an $m \times n$ array is a partial transversal of size m. We define $L(m, n)$ as the largest integer z such that every $m \times n$ array in which no symbol appears more than z times has at least one transversal. A *row latin rectangle* is an array in which no symbol occurs more than once in any row. A *column latin rectangle* is defined in a similar way. A *latin square* is an $n \times n$ row-column latin rectangle that contains exactly n different symbols. Ryser [7] conjectured that every latin square of odd order has a transversal. Moreover, Brualdi [3] conjectured that every latin square of order n has a partial transversal of size at least $n-1$. These conjectures have remained unsettled.

The problems on the existence of large partial transversals in latin squares and rectangular arrays are among the most beautiful problems in combinatorics. There are many theorems and conjectures in this area. The lower bounds of $n-\sqrt{n}$ [1, p. 256] and $n-5.53(\log n)^2$ [8] for the size of the largest partial transversal in latin squares of order n are well-known. Drisko [4] proved that if $n \geqslant 2m-1$, then any $m \times n$ column latin rectangle has a transversal. Stein [10] showed that $L(m, n) \leqslant n-1$ for $m \leqslant n \leqslant 2m-2$ by a simple construction and conjectured that $L(n-1, n) = n-1$. Clearly, if this conjecture is true, then Brualdi's Conjecture is also true. A result due to Hall [6] supports this conjecture: Any $(n-1) \times n$ array constructed from $n-1$ (not necessarily distinct) rows of the group table of an abelian group of order n, has a transversal. Stein [9] showed that in an $n \times n$ array where each symbol appears exactly n times there is a partial transversal with length at least approximately $(0.63)n$. Also Erdős and Spencer [5] proved that $L(n, n) \geqslant (n-1)/16$.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.02.027

2. Result

In [10] it is shown that $L(m,n) \leqslant (mn-1)/(m-1)$. To see this, suppose, on the contrary, that $L(m,n) > (mn-1)/(m-1)$. Then $(m-1)L(m,n) \geqslant mn$, and we could assign $m-1$ different symbols to the cells of A such that each symbol appears at most $L(m,n)$ times. Obviously, $m-1$ symbols is not sufficient for a transversal, showing that $L(m,n) \leqslant (mn-1)/(m-1)$. In [10] it is shown that $L(3,n) = \lfloor (3n-1)/2 \rfloor$, for $n \geqslant 5$.

The following theorem shows that when n is large enough in comparison to m, then the above upper bound on $L(m,n)$ is tight.

Theorem. *If $m \geqslant 2$ and $n \geqslant 2m^3$, then $L(m,n) = \lfloor (mn-1)/(m-1) \rfloor$.*

Proof. Define $f(m) = 2m^3 - 6m^2 + 6m - 1 \leqslant 2m^3$ and $g(m,n) = \lfloor (mn-1)/(m-1) \rfloor$. By the above discussion, we have $L(m,n) \leqslant g(m,n)$. Hence we just need to show that for $n \geqslant f(m)$, if each symbol appears at most $g(m,n)$ times in an $m \times n$ array A, then A has a transversal. We will prove this by applying induction on m. For $m=2$, it is clear that $L(2,n) = 2n-1$ when $n \geqslant 3$; since $f(2)=3$, the assertion holds for $m=2$. We may thus assume $m \geqslant 3$.

For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we use (i,j) to denote the cell at the intersection of row i and column j, and we refer to the symbol contained in that cell by $A(i,j)$. Note the distinction between cells (which are positions) and symbols (which are values assigned to the positions). Without loss of generality, we assume that symbols are positive integers. Consider the following three *primary operations* that can be applied to array A: (i) interchange of two rows; (ii) interchange of two columns; and (iii) permutation on symbols. Each of the above operations preserves the existence of transversals. Hence we may apply each of them wherever needed in the proof, and pretend that A has not been changed during the proof; that is, we will denote each of the arrays obtained from the original array, A, by the same symbol, A.

Let B be the $(m-1) \times n$ array consisting of the first $m-1$ rows of A. We have $n \geqslant f(m) \geqslant f(m-1)$ for $m \geqslant 2$, and each symbol appears at most $g(m,n) \leqslant g(m-1,n)$ times in B. Thus, by the induction hypothesis, B has a transversal T of size $m-1$. Without loss of generality and by using the primary operations, we may assume that T is on the main diagonal of B, and $A(i,i) = i$ for $1 \leqslant i \leqslant m-1$. Let $S = \{(m,j) \mid m \leqslant j \leqslant n\}$. If a cell in S has a symbol greater than $m-1$, then adding that cell to T creates a transversal, and we are done. Assume that k is the number of distinct symbols in S. Using the primary operations, without loss of generality we can assume that all symbols of S are less than or equal to k, $k \leqslant m-1$, and $A(m, m+i-1) = i$ for $1 \leqslant i \leqslant k$.

We will apply a sequence of primary operations to A to construct a new array, which will still call A. In this array we will construct partial transversals $T_1, T_2, \ldots, T_{m-1}$, each consisting of $m-1$ cells, such that for every $1 \leqslant i \leqslant m-1$, the following conditions hold:

1. $A(i,i) = i$.
2. Each row, except row i, has a cell in T_i.
3. For every j, $i < j < m$, cell (j,j) is included in T_i.
4. The symbols in T_i are $1, 2, \ldots, m-1$.

For $1 \leqslant i \leqslant k$, we define T_i to be $\{(j,j) \mid 1 \leqslant j \leqslant m-1,\ j \neq i\} \cup \{(m, m+i-1)\}$. It is easy to verify that $T_1, T_2, \ldots, T_k$ satisfy Conditions 1–4.

Now suppose that we have constructed $T_1, T_2, \ldots, T_p$ for $p \geqslant k$. We will show how to construct T_{p+1}, when $p \leqslant m-2$.

For each i, $1 \leqslant i \leqslant p$, let $X_i = \{(i,j) : T_i$ has no cell in column j$\}$. Each X_i has $n-(m-1)$ elements and the sets $X_1, X_2, \ldots, X_p$ are disjoint. Let X be their union. We claim that at least one symbol of X is greater than p. If this does not hold, then all symbols in X and S are less than or equal to p. Since X_i's are pairwise disjoint we have $|X| = p(n-m+1)$ and $|S| = n-m+1$. Hence at least one symbol appears at least $(n-m+1)(p+1)/p \geqslant (n-m+1)(m-1)/(m-2)$ times in A. An elementary calculation shows that $f(m) > m^3 - 3m^2 + 2m + 1$, which implies $(n-m+1)(m-1)/(m-2) > g(m,n)$. But this requires that a symbol appears more than $g(m,n)$ times in A, which is a contradiction. Consequently, there is $(r,s) \in X \cup S$, hence in X, such that $A(r,s) = t > p$.

None of the cells of T_r is in row r. Besides, T_r has no cell from column s since $(r,s) \in X$. It follows that if $t \geqslant m$, then $T_r \cup \{(r,s)\}$ is a transversal, and we are done. Thus one may assume that $t \leqslant m-1$. We will show how to make $A(r,s) = p+1$ in case t is not already $p+1$: Cells (t,t) and $(p+1, p+1)$ have rows and columns different from the row and column of the cell (r,s). Therefore, one way to ensure that $A(r,s) = p+1$ is by first interchanging symbols

t and $p+1$, next interchanging rows t and $p+1$, and finally interchanging columns t and $p+1$. To see why this works, we note that the last two primary operations (i.e. interchanges of rows and columns t and $p+1$) swap the values of $A(t,t)$ and $A(p+1,p+1)$. Furthermore, the above primary operations preserve Conditions 1–4. This is because $T_1,\ldots,T_p$ all have the entries $A(i,i)=i$ for $p<i<m$, and the above operations only act on rows, columns, and symbols $p+1,\ldots,m-1$. Now that we have $A(r,s)=p+1$, it is not hard to check that $T_{p+1}=T_r\cup\{(r,s)\}\backslash\{(p+1,p+1)\}$ does not violate any of Conditions 1–4. By this method, all $T_1,\ldots,T_{m-1}$ are constructed.

Assume that $T_1,\ldots,T_{m-1}$ are as we introduced before. For $1\leqslant i\leqslant m-1$, let $C_i=\{j\mid 1\leqslant j\leqslant m-1$ or column j intersects $T_1\cup\cdots\cup T_i\}$. By the above construction we have $|C_k|=m-1+k$. Furthermore we have $|C_{i+1}|\leqslant|C_i|+1$ for $k\leqslant i<m-1$, since T_{i+1} does not introduce more than one new cell (see definition of T_{p+1}). Hence we have $|C_{m-1}|\leqslant(m-1+k)+(m-1-k)=2m-2$. Let Q be the set of all cells residing in the columns of C_{m-1}, and let Q' be the complement of Q relative to A. Since each symbol appears at most $g(m,n)$ times, one of the following two cases always happens:

Case 1: There exists $(x,y)\in Q'$ such that $A(x,y)\geqslant m$. It is clear that $x\neq m$ and since T_x does not intersect column y, $T_x\cup\{(x,y)\}$ is a transversal.

Case 2: All symbols appearing in Q' are less than m. Hence there exists $(x,y)\in Q$ such that $A(x,y)\geqslant m$. Let Z be an $(m-1)\times(n-2m+2)$ array that is obtained from Q' by removing row x and $2m-2-|C_{m-1}|$ arbitrary columns. This is possible since $n-|C_{m-1}|\geqslant 2m-2-|C_{m-1}|$, that is, $n\geqslant 2m-2$. To show that Z has a transversal by induction on m, we check that:

1. There are $(n-2m+2)\geqslant f(m)-2m+2\geqslant f(m-1)$ columns in Z.
2. Every symbol appears at most $g(m-1,n-2m+2)$ times in Z, because $n\geqslant f(m)$ implies $g(m,n)\leqslant g(m-1,n-2m+2)$.

Now since all symbols of Z are less than m, the transversal of Z can be extended to a transversal for A by adding (x,y) and the proof is complete. □

Remark. As the proof of theorem shows, if $n\geqslant f(m)=2m^3-6m^2+6m-1$, then $L(m,n)=\lfloor(mn-1)/(m-1)\rfloor$. Thus we obtain that $L(3,n)=\lfloor(3n-1)/2\rfloor$, for $n\geqslant 17$ and $L(4,n)=\lfloor(4n-1)/3\rfloor$, for $n\geqslant 55$.

Acknowledgements

The authors are indebted to the research council of Sharif University of Technology for support. Also the authors are grateful to the referees for their valuable comments.

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