

# CS3102 Theory of Computation

## Problem Set 2, Spring 2012

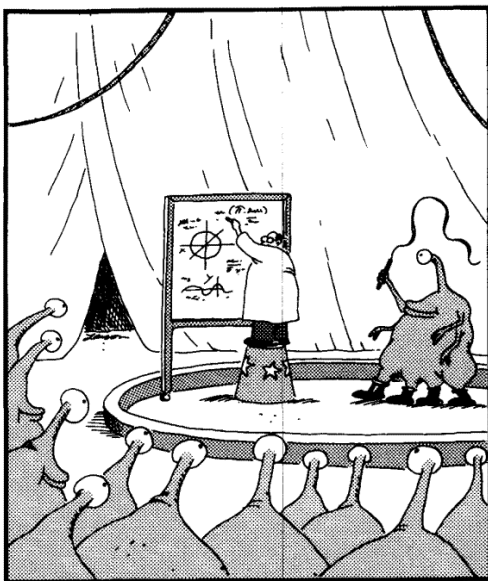
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Please start solving these problems immediately, and work in study groups. Please prove all your answers; informal arguments are acceptable, but please make them precise / detailed / convincing enough so that they can be easily made rigorous. To review notation and definitions, please read the "Basic Concepts" summary posted on the class Web site, and also read the second chapter (on regular languages) from the Sipser textbook.

**Important note:** this is not a “due homework”, but rather a “pool of problems” meant to calibrate the scope and depth of the knowledge & skills in CS theory that you (eventually) need to have for the course exams, becoming a better problem-solver, thinking more abstractly, performing more effectively as a computer scientist, etc. You don’t necessarily have to completely solve every last question in this problem set (although it would be great if you did!). Rather, please solve as many of these problems as you can, and use this problem set as a resource to improve your problem-solving skills, abstract thinking, and to find out what topics you need to further focus & improve on. Recall that most (and perhaps even all) of the midterm and final exam questions in this course will come from these problem sets, so your best strategy of studying for the exams in this course is to solve (in study groups) as many of these problems as possible, and the sooner the better!

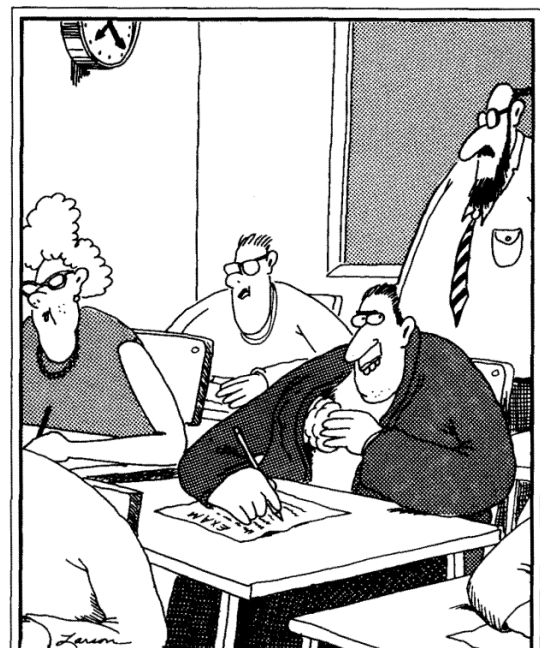
**Advice:** Please try to solve the easier problems first (where the meta-problem here is to figure out which are the easier ones. ☺) Please don’t spend too long on any single problem without also attempting (in parallel) to solve other problems as well; this way, the easiest problems (at least to you) will reveal themselves much sooner (think about this as a “hedging strategy”).



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.



Unbeknownst to most historians, Einstein started down the road of professional basketball before an ankle injury diverted him into science.



Midway through the exam, Allen pulls out a bigger brain.

1. Solve the following problems from the [Sipser, Second Edition] textbook:

Pages 83-93: 1.4, 1.5, 1.6, 1.7, 1.11, 1.12, 1.13, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.23, 1.29, 1.30, 1.31, 1.36, 1.38, 1.40, 1.41, 1.42, 1.43, 1.45, 1.46, 1.48, 1.57, 1.58, 1.60, 1.61, 1.63, 1.64

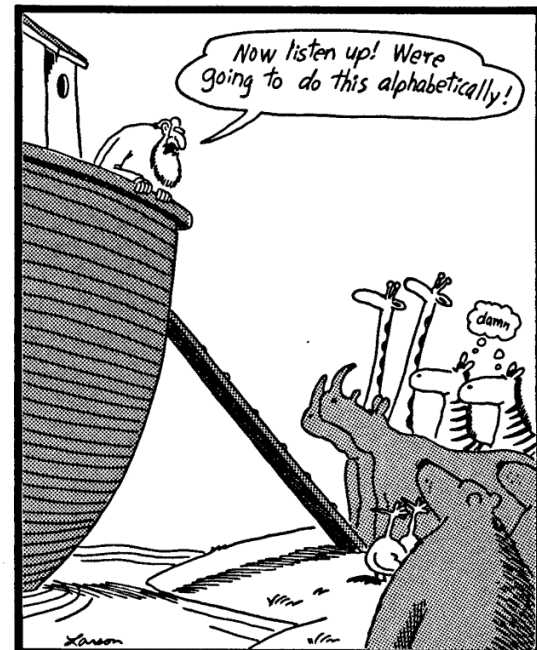
2. Prove or disprove: a countable set of parabolas (arbitrarily oriented and placed) can completely cover (every point inside) the unit square in the plane (i.e., the interior and boundary of a square of side 1)
3. Prove or disprove: an uncountable set of pairwise-disjoint line segments can completely cover (every point in) the unit disk in the plane (i.e., the interior and boundary of a circle of diameter 1). What if the segments could intersect each other, but must all have unique slopes?
4. What is the cardinality of the set of all finite-sized matrices with rational entries?
5. What is the cardinality of the set of all infinite matrices (i.e., matrices with a countably-infinite number of rows and columns) with Boolean entries?
6. Does every regular language have a proper regular subset?  
Does every regular language have a proper regular superset?
7. Is every subset of a regular language necessarily regular?  
Is every superset of a regular language necessarily non-regular?
8. Are the regular languages closed under infinite union? Infinite intersection?
9. Is a countable union of regular languages necessarily regular? Decidable?  
Is a countable union of decidable languages necessarily decidable?
10. Prove or disprove: every regular language is countable.
11. Prove or disprove: the set of all regular languages is countable.
12. Prove or disprove: the set of all languages is countable.



Final page of the Medical Boards



"Yes! That's right! The answer is 'Wisconsin!' Another 50 points for God, and ... uh-oh, looks like Norman, our current champion, hasn't even scored yet."



13. Determine as precisely as possible, for a language  $L$ , when is the following true:  $L^+ = L^* - \{\epsilon\}$
14. What is the infinite union of all the regular languages? Decidable languages?  
What is the infinite intersection of all the regular languages? Decidable languages?
15. Are the decidable languages closed Kleene closure? Concatenation? Union? Complementation?  
Are the non-decidable languages closed Kleene closure? Concatenation? Union?
16. Are the non-finitely-describable languages closed under concatenation? Kleene closure?  
Complementation? Union?
17. Can a non-computable number be rational? Must an irrational number be non-computable?
18. Is the set of non-finitely-describable real numbers closed under addition? Squaring?
19. Let  $YESNO(L) = \{xy \mid x \in L \text{ and } y \notin L, x, y \in \Sigma^*\}$ . Does YESNO preserve regularity?  
Decidability?
20. Let  $PALI(L) = \{w \mid w \in L \text{ and } w^R \in L\}$ . Does PALI preserve regularity? Decidability?
21. Describe an algorithm that determines for a given pair of regular expressions whether they denote the same language. What is the time complexity of your algorithm?
22. True or false: for any given regular language, there exists a linear-time algorithm for testing whether an arbitrary input string is a member of that language.
23. What is the smallest language, closed under concatenation, containing the languages  $L_1$  and  $L_2$ ?
24. Give a sufficient condition (but as general as possible) for  $L_1^* + L_2^* = (L_1 + L_2)^*$  to hold.
25. Given two arbitrary languages  $S$  and  $T$ , find a new language  $R$  (in terms of  $S$  and  $T$ ) so that the equation  $R = SR + T$  holds.
26. Describe exactly what happens if we apply the “powerset construction” to a finite automaton that is already deterministic?



“Frankly, I have a lot of trouble relating to all this symbolism.”



“It’s hard to tell you this, Melnik, but you’re being sent down to the Institute for Pretty Hard Thinking.”

27. Define a new operation  $\clubsuit$  on languages as follows:  $\clubsuit(L) = \{w \mid \exists w \in \Sigma^*, ww^R \in L\}$ , where  $w^R$  denotes the "reverse" of the string  $w$ . Does  $\clubsuit$  preserve regularity?
28. Let  $L = \{0^n 1^n \mid n \geq 0\}$ . Is  $\bar{L}$  (i.e. the complement of  $L$ ) a regular language?
29. Let  $L = \{0^i 1^j \mid i \neq j\}$ . Is  $L$  a regular language?
30. Are there two non-regular languages whose concatenation is regular? Are there a countably infinite number of such examples? Are there an uncountable number of such examples?
31. Show that the intersection of two sets of languages can be empty, finite (of arbitrarily large cardinality), countably infinite, or uncountably infinite.
32. Define a new operation  $\diamond$  on languages as follows:  $\diamond(L) = \{w \mid \exists z \in \Sigma^* \ni |w|=|z| \wedge wz \in L\}$ . Does the operation  $\diamond$  preserve regularity?
33. Define an "infinite automata" similarly to finite automata, but where the state set  $Q$  is no longer restricted to be finite. Characterize precisely the class of languages accepted by deterministic infinite automata. Is the characterization any different for non-deterministic infinite automata?
34. We define the SHUFFLE of two strings  $v, w \in \Sigma^*$  as:

$$\text{SHUFFLE}(v, w) = \{v_1 w_1 v_2 w_2 \dots v_k w_k \mid v = v_1 v_2 \dots v_k, w = w_1 w_2 \dots w_k,$$

$$\text{and for some } k \geq 1, v_i, w_i \in \Sigma^*, 1 \leq i \leq k\}$$

For example,  $212\underline{ab}1\underline{baa}2\underline{b}22 \in \text{SHUFFLE}(\underline{abbaab}, 2121222)$

Extend the definition of SHUFFLE to two languages  $L_1, L_2 \subseteq \Sigma^*$  as follows:

$$\text{SHUFFLE}(L_1, L_2) = \{w \mid w_1 \in L_1, w_2 \in L_2, w \in \text{SHUFFLE}(w_1, w_2)\}$$

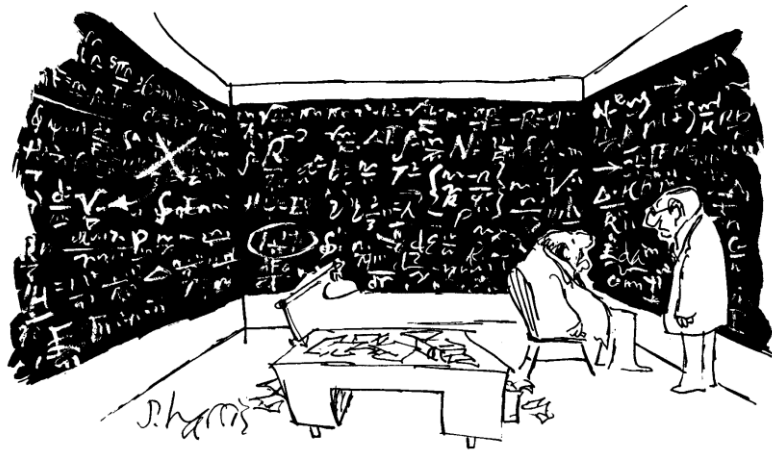
- a) Is the SHUFFLE of two regular languages necessarily regular?
- b) Is the SHUFFLE of two decidable languages necessarily decidable?



35. Which of the following modifications / restrictions to finite automata would change the class of languages accepted relative to "normal" finite automata?
- a) The ability to move the read head backwards (as well as forwards) on the input.
  - b) The ability to write on (as well as read from) the input tape.
  - c) Both a) and b) simultaneously.
  - d) Having 2 read-heads moving (independently, left-to-right) over the input.
  - e) Having one billion or less different states.

36. Determine whether each of the following languages is regular:

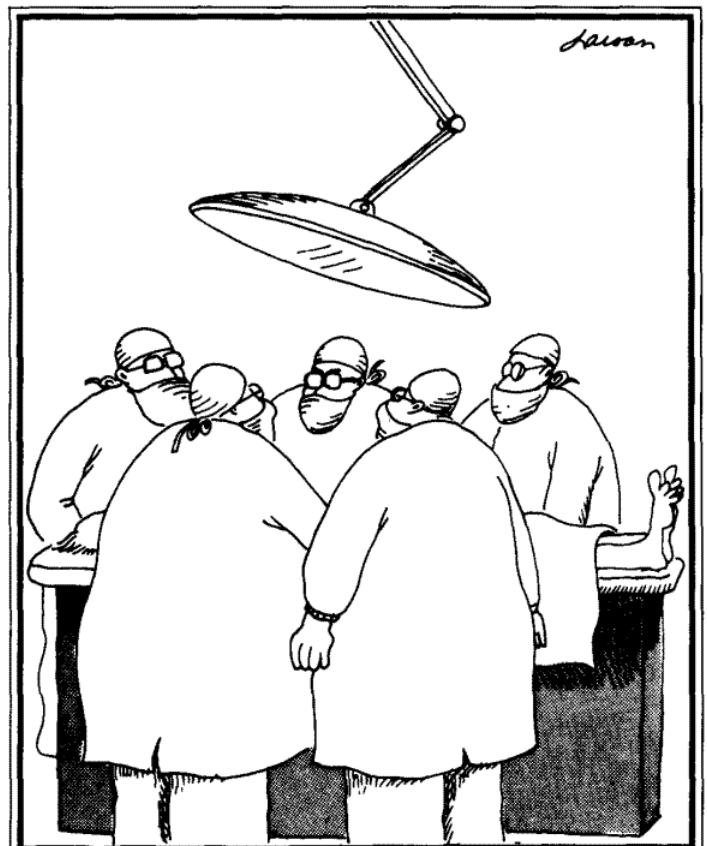
- a)  $\{a^n a^n a^n \mid n > 0\}$
- b)  $\{www \mid w \in \{x,y,z\}^*, |w| < 10^{100}\}$
- c)  $\{vw \mid v,w \in \{a,b\}^*\}$
- d)  $\{ww \mid w \in \{a\}^*\}$



"Whatever happened to elegant solutions?"



Unwittingly, Irwin has a brush with Death.



"Okay, Williams, we'll vote . . . how many here say the heart has four chambers?"

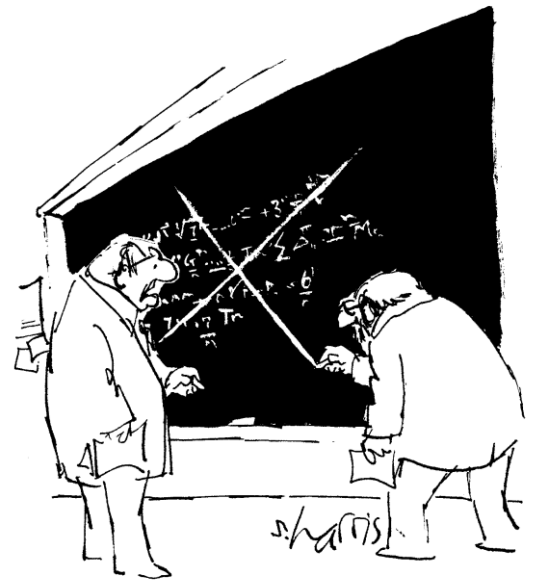
37. Given an arbitrary alphabet  $\Sigma = \{a_1, a_2, \dots, a_n\}$ , we can impose a total ordering on it in the sense that we can define  $<$  so that  $a_1 < a_2 < \dots < a_n$ . We now proceed to define a new operation called the SORT of a string  $w = w_1 w_2 \dots w_k \in \Sigma^*$  (where  $w_i \in \Sigma$  and  $k = |w|$ ) as:

$$\text{SORT}(w) = w_{\sigma(1)} w_{\sigma(2)} \dots w_{\sigma(k)} \quad \text{so that } w_{\sigma(i)} < w_{\sigma(i+1)} \text{ for } 1 \leq i \leq k-1$$

and  $\sigma$  is a permutation (i.e., a 1-to-1 onto mapping  $\sigma: [1..k] \rightarrow [1..k]$ )

For example,  $\text{SORT}(11210010120) = 00001111122$ . Now extend the definition of SORT to languages, so that  $\text{SORT}(L) = \{\text{SORT}(w) \mid w \in L\}$ . For each one of the following statements, state whether it is true or false and explain:

- a)  $\text{SORT}(\Sigma^*)$  is regular.
- b)  $\text{SORT}(L) \subseteq L$
- c)  $\text{SORT}(\text{SORT}(L)) = \text{SORT}(L)$
- d)  $\text{SHUFFLE}(L_1, L_2) = \text{SHUFFLE}(L_2, L_1)$
- e)  $\text{SORT}(\text{SHUFFLE}(L_1, L_2)) = \text{SORT}(L_1 L_2)$
- f)  $\exists L$  such that  $\text{SORT}(L) = \text{SHUFFLE}(L, L) = L$
- g) SORT preserves regularity.
- h) SORT preserves decidability
- i) SORT preserves non-decidability
- j) SORT preserves non-finite-describability



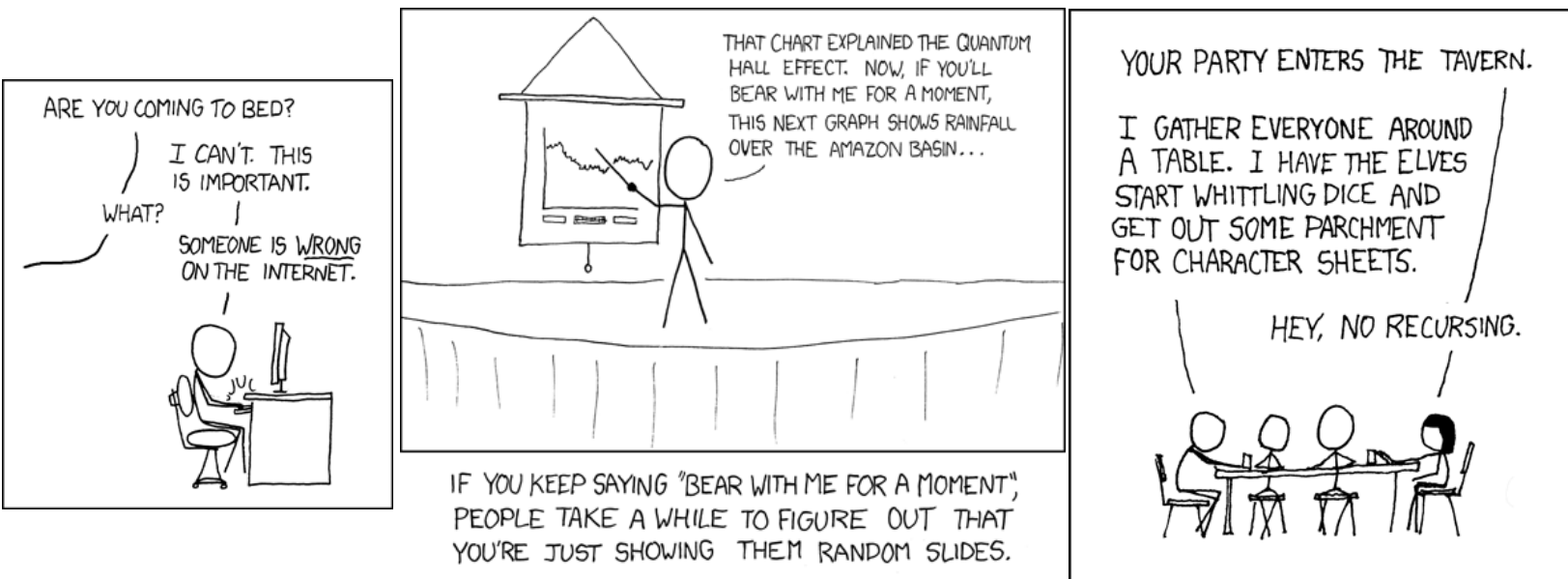
"That's it? That's peer review?"

(The definition of SHUFFLE operator is the same as above)

38. Define a DIVISION operator on languages as follows:

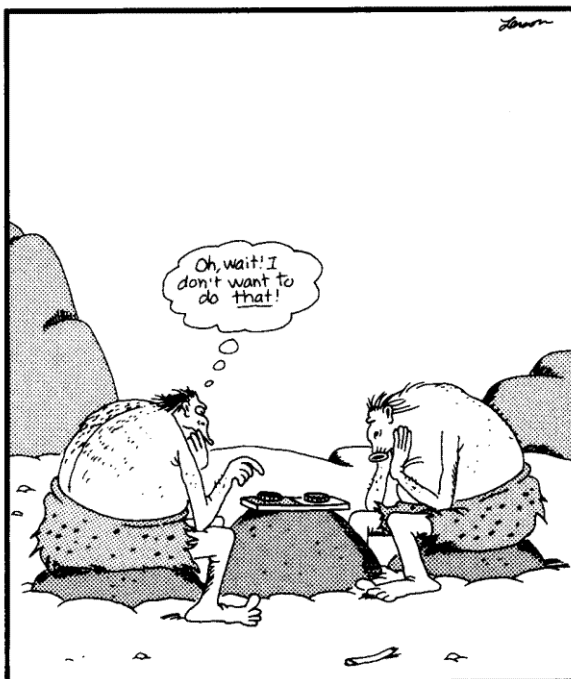
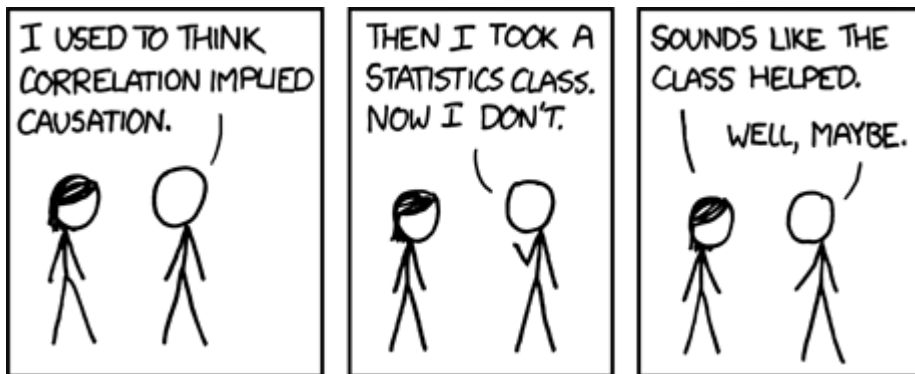
$$\frac{L_1}{L_2} = \{w \mid w \in \Sigma^* \text{ and } \exists v \in L_2 \exists wv \in L_1\}$$

Does DIVISION preserve regularity? Decidability? What if  $L_1$  is regular and  $L_2$  is arbitrary?

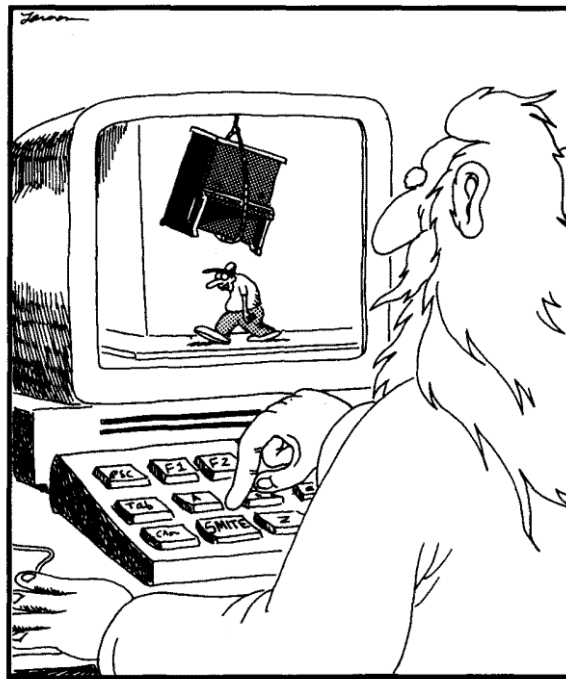


39. Give algorithms (i.e., a well-defined, deterministic, always-terminating decision procedures, and state their time complexities) to determine whether for a given finite automaton  $M$ ,  $L(M)$  is:
- a) countable
  - a) empty
  - b)  $\Sigma^*$
  - c) finite
  - d) infinite
  - e) co-finite (i.e., with a finite complement)
  - f) regular
  - g) also accepted by a smaller FA (i.e., with fewer states)

40. Give algorithms (and state their time complexities) to determine whether for a given pair of finite automata:
- a) they both accept the same language
  - b) the intersection of their languages is empty
  - c) the intersection of their languages is  $\Sigma^*$
  - d) the intersection of their languages is finite
  - e) the difference of their languages is finite



Early checkers



God at His computer

