Theory of Computation
CS3102

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www.cs.virginia.edu/robins/theory
Problem: Can 5 test tubes be spun simultaneously in a 12-hole centrifuge in a balanced way?

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Theory of Computation (CS3102) - Textbook

Textbook:


Good Articles / videos:

www.cs.virginia.edu/~robins/CS_readings.html
Theory of Computation (CS3102)

Required reading:

How to Solve It, by George Polya (MIT), Princeton University Press, 1945

• A classic on problem solving
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<tr>
<td>Anthrop</td>
<td>301</td>
</tr>
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<td>126</td>
</tr>
<tr>
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<td>104</td>
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<td>326</td>
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<td>217</td>
</tr>
<tr>
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<td>221</td>
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<td>204</td>
</tr>
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<td>Paleont.</td>
<td>113</td>
</tr>
<tr>
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<td>312</td>
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<tr>
<td>Psych</td>
<td>209</td>
</tr>
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<td>Toxic</td>
<td>307</td>
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Theory of Computation (CS3102) - Syllabus

A brief history of computing:

• Aristotle, Euclid, Archimedes, Eratosthenes
• Abu Ali al-Hasan ibn al-Haytham
• Fibonacci, Descartes, Fermat, Pascal
• Newton, Euler, Gauss, Hamilton
• Boole, De Morgan, Babbage, Ada Agusta
• Venn, Carroll, Cantor, Hilbert, Russell
• Hardy, Ramanujan, Ramsey
• Godel, Church, Turing, von Neumann
• Shannon, Kleene, Chomsky
Theory of Computation Syllabus (continued)

Fundamentals:
• Set theory
• Predicate logic
• Formalisms and notation
• Infinities and countability
• Dovetailing / diagonalization
• Proof techniques
• Problem solving
• Asymptotic growth
• Review of graph theory
Theory of Computation Syllabus (continued)

Formal languages and machine models:

- The Chomsky hierarchy
- Regular languages / finite automata
- Context-free grammars / pushdown automata
- Unrestricted grammars / Turing machines
- Non-determinism
- Closure operators
- Pumping lemmas
- Non-closures
- Decidable properties
Computability and undecidability:

- Basic models
- Modifications and extensions
- Computational universality
- Decidability
- Recognizability
- Undecidability
- Church-Turing thesis
- Rice’s theorem
Theory of Computation Syllabus (continued)

NP-completeness:

- Resource-constrained computation
- Complexity classes
- Intractability
- Boolean satisfiability
- Cook-Levin theorem
- Transformations
- Graph clique problem
- Independent sets
- Hamiltonian cycles
- Colorability problems
- Heuristics

\[
\begin{align*}
\text{NP-complete} & \quad \text{SAT} \\
\text{P} & \quad \text{P-complete} \quad \text{LP} \\
\text{co-NP} & \quad \text{co-NP-complete} \quad \text{TAUT}
\end{align*}
\]
Theory of Computation Syllabus (continued)

Other topics (as time permits):

• Generalized number systems
• Oracles and relativization
• Zero-knowledge proofs
• Cryptography & mental poker
• The Busy Beaver problem
• Randomness and compressibility
• The Turing test
• AI and the Technological Singularity
Generalized Numbers

Theorem: some real numbers are not finitely describable!
Theorem: some finitely describable real numbers are not computable!
Overarching Philosophy

- Focus on the “big picture” & “scientific method”
- Emphasis on problem solving & creativity
- Discuss applications & practice
- A primary objective: have fun!
Prerequisites

• Some **discrete math & algorithms knowledge**

• Ideally, should have taken CS2102

• Course will “**bootstrap**” (albeit quickly) from **first principles**

• Critical: **Tenacity, patience**
Course Organization

• **Exams**: probably take home
  – Decide by vote
  – Flexible exam schedule

• **Problem sets**:  
  – Lots of problem solving  
  – **Work in groups!**
  – Not formally graded
  – Most exam questions will come from these sets!

• **Readings**: papers / videos / books

• **Extra credit problems**
  – In class & take-home
  – Find mistakes in slides, handouts, etc.

• **Course materials posted on Web site**
  www.cs.virginia.edu/robins/theory
Contact Information

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Phone:  (434) 982-2207
Email:  robins@cs.virginia.edu
Web:  www.cs.virginia.edu/robins
www.cs.virginia.edu/robins/theory

Office hours: after class
• Any other time
• By email (preferred)
• By appointment
• Q&A blog posted on class Web site
Grading Scheme

- Midterm 35%
- Final 35%
- Readings 30%
- Extra credit 10%

Best strategy:

- Solve lots of problems!
- Do lots of readings / EC!
Course Readings
www.cs.virginia.edu/robins/CS_readings.html

Goal: broad exposure to lots of cool ideas & technologies!

• Required: total of at least 50 items over the semester

• Diversity: minimums in each of 3 categories:
  1. Minimum of 20 videos
  2. Minimum of 20 papers / Web sites
  3. Minimum of 10 books

• More than 50 is even better! (extra credit)

• Some required items in each category
  o Remaining “elective” items should be a diverse mix

• Email all submissions to: homework.cs3102@gmail.com
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- **Required videos:**
  - *Last Lecture*, Randy Pausch, 2007
  - *Time Management*, Randy Pausch, 2007
  - *Powers of Ten*, Charles and Ray Eames, 1977
Required Reading
• “Scale of the Universe”, Cary and Michael Huang, 2012

- $10^{-24}$ to $10^{26}$ meters $\Rightarrow$ 50 orders of magnitude!
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

• More required videos:
  – Claude Shannon - Father of the Information Age, UCTV
  – The Pattern Behind Self-Deception, Michael Shermer, 2010
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- Required articles:
  - Decoding an Ancient Computer, Freeth, 2009
  - Alan Turing’s Forgotten Ideas, Copeland and Proudfoot, 1999
  - You and Your Research, Richard Hamming, 1986
  - Who Can Name the Bigger Number, Scott Aaronson, 1999
BENEDICT CUMBERBATCH IS OUTSTANDING

“THE BEST BRITISH FILM OF THE YEAR”

“AN INSTANT CLASSIC”

“A SUPERB THRILLER”

THE IMITATION GAME

BENEDICT CUMBERBATCH KEIRA KNIGHTLEY

BASED ON THE INCREDIBLE TRUE STORY

IN CINEMAS NOVEMBER 14

Extra credit!
Basic Concepts and Notation

Gabriel Robins

"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean -- neither more nor less."

A set is formally an undefined term, but intuitively it is a (possibly empty) collection of arbitrary objects. A set is usually denoted by curly braces and some (optional) restrictions. Examples of sets are \{1,2,3\}, \{hi, there\}, and \{k \mid k \text{ is a perfect square}\}. The symbol \(\in\) denotes set membership, while the symbol \(\notin\) denotes set non-membership; for example, \(7 \in \{p \mid p \text{ prime}\}\) states that 7 is a prime number, while \(q \notin \{0,2,4,6,\ldots\}\) states that q is not an even number. Some common sets are denoted by special notation:

The natural numbers:  \[ \mathbb{N} = \{1,2,3,\ldots\} \]

The integers:  \[ \mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} \]

The rational numbers:  \[ \mathbb{Q} = \left\{ \frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0 \right\} \]

The real numbers:  \[ \mathbb{R} = \{x \mid x \text{ is a real number}\} \]

The empty set:  \[ \emptyset = \{\} \]
JOHANNES KEPLER'S UPHILL BATTLE

...so, you see, the orbit of a planet is elliptical.

What's an orbit? What's a planet? What's 'elliptical'?
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- **Required books:**
  - “How to Solve It”, Polya, 1957
  - “Infinity and the Mind”, Rucker, 1995
  - “Godel, Escher, Bach”, Hofstadter, 1979
  - “What If”, Munroe, 2014
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

• Remaining videos / articles / books are “electives”
• Pacing: at least 3 submissions per week (due 5pm Monday)
  - Policy intended to help you avoid “cramming”
• Length: 1-2 paragraphs per article / video
  1-2 pages per book
• Books are worth more credit than articles / videos
• Email all submissions to: homework.cs3102@gmail.com
• Additional readings beyond 50 are welcome! (extra credit)
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Theory and Algorithms:**
  – *Who Can Name the Bigger Number*, Scott Aaronson, 1999
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Biological Computing:**
  – Big Lab on a Tiny Chip, Charles Choi, Scientific American, October 2007, pp. 100-103.

Email all submissions to: homework.cs3102@gmail.com
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Quantum Computing:**


  – Black Hole Computers, Seth Lloyd and Jack Ng, Scientific American, November 2004, pp. 52-61.


Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• **History of Computing:**

• **Security and Privacy:**
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Future of Computing:**
  – Ballbots, Ralph Hollis, Scientific American, October 2006, pp. 72-77.
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• The Web:

• The Wikipedia Computer Science Portal:
  – Theory of computation and Automata theory
  – Formal languages and grammars
  – Chomsky hierarchy and the Complexity Zoo
  – Regular, context-free & Turing-decidable languages
  – Finite & pushdown automata; Turing machines
  – Computational complexity
  – List of data structures and algorithms

Email all submissions to: homework.cs3102@gmail.com
Other “Elective” Readings
www.cs.virginia.edu/robins/CS_readings.html

• The Wikipedia Math Portal:
  – Problem solving
  – List of Mathematical lists
  – Sets and Infinity
  – Discrete mathematics
  – Proof techniques and list of proofs
  – Information theory & randomness
  – Game theory

• Mathematica's “Math World”

Email all submissions to: homework.cs3102@gmail.com
THE PROBLEM WITH WIKIPEDIA:

TACOMA NARROWS BRIDGE

SUSPENSION BRIDGE

STRUCTURAL COLLAPSE

[THREE HOURS OF FASCINATED CLICKING]

WIKIFRIENDS:

I REALLY LIKED THAT MOVIE.

I HATED THAT MOVIE.

ME TOO.

WILLIAM HOWARD TAFT

LESBIANISM IN EROTICA

FATAL HILARITY

TAYLOR HANSON

COTTON T-SHIRT

WET T-SHIRT CONTEST
Good Advice

• Ask questions ASAP
• Solve problems ASAP
• Work in study groups
• Do not fall behind
• “Cramming” won’t work
• Do lots of extra credit
• Attend every lecture
• Visit class Website often
• Solve lots of problems
Goal: Become a more effective problem solver!

Email all submissions to: homework.cs3102@gmail.com
Problem: Can 5 test tubes be spun simultaneously in a 12-hole centrifuge in a balanced way?

- What does “balanced” mean?
- Why are 3 test tubes balanced?
- Symmetry!
- Can you merge solutions?
- Superposition!
- Linearity! \( f(x + y) = f(x) + f(y) \)
- Can you spin 7 test tubes?
- Complementarity!
- Empirical testing…

No vector calculus/trig! No equations! Truth is guaranteed! Fundamental principles exposed! Easy to generalize! High elegance/beauty!
Problem: \[ 1 + 2 + 3 + 4 + \ldots + 100 = ? \]

Proof: Induction...

\[
\begin{align*}
1 & \quad 2 \quad 3 \quad \ldots \quad 99 \quad 100 \\
100 & \quad 99 \quad 98 \quad \ldots \quad 2 \quad 1 \\
\hline
101 & \quad 101 \quad 101 \quad \ldots \quad 101 \quad 101
\end{align*}
\]

\[ 101 + 101 + 101 + \ldots + 101 + 101 = 100 \times 101 \]

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
**Drawbacks of Induction**

- You must *a priori* know the formula / result
- Easy to make *mistakes* in inductive proof
- Mostly “mechanical” – *ignores intuitions*
- *Tedious* to construct
- *Difficult* to check
- *Hard* to understand
- *Not very convincing*
- Generalizations *not obvious*
- Does not “*shed light on truth*”
- *Obfuscates* connections

**Conclusion**: only use induction as a *last resort!* (i.e., *rarely*)
Problem: \(1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = ?\)

\[\sum_{i=1}^{n} i^3 = ?\]

Extra Credit: find a short, geometric, induction-free proof.

Email all submissions to: homework.cs3102@gmail.com
Problem: \( (1/4) + (1/4)^2 + (1/4)^3 + (1/4)^4 + \ldots = ? \)

\[
\sum_{i=1}^{\infty} \frac{1}{4^i} = ?
\]

Extra Credit:
Find a short, geometric, induction-free proof.

Email all submissions to: homework.cs3102@gmail.com
Problem: \((1/8) + (1/8)^2 + (1/8)^3 + (1/8)^4 + \ldots = ?\)

\[
\sum_{i=1}^{\infty} \frac{1}{8^i} = ?
\]

Extra Credit:
Find a short, geometric, induction-free proof.

Email all submissions to: homework.cs3102@gmail.com
Problem: Prove that $\sqrt{2}$ is irrational.

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Problem: Prove that there are an infinity of primes.

Extra Credit: Find a short, induction-free proof.

• What approaches fail?
• What techniques work and why?
• Lessons and generalizations

Email all submissions to: homework.cs3102@gmail.com
Problem: True or false: there are arbitrarily long blocks of consecutive composite integers.

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Problem: Are the complex numbers closed under exponentiation? E.g., what is the value of $i^i$?
Problem: Does exponentiation preserve irrationality? i.e., are there two irrational numbers $x$ and $y$ such that $x^y$ is rational?

Extra Credit: find a short, induction-free proof.

• What approaches fail?
• What techniques work and why?
• Lessons and generalizations
Historical Perspectives

The Unknown Scientist

(Who did some very important groundwork)
Historical Perspectives

- Knowing the “big picture” is empowering
- Science and mathematics builds heavily on past
- Often the simplest ideas are the most subtle
- Most fundamental progress was done by a few
- We learn much by observing the best minds
- Research benefits from seeing connections
- The field of computer science has many “parents”
- We get inspired and motivated by excellence
- The giants can show us what is possible to achieve
- It is fun to know these things!
“Standing on the Shoulders of Giants”

- Aristotle, Euclid, Archimedes, Eratosthenes
- Abu Ali al-Hasan ibn al-Haytham
- Fibonacci, Descartes, Fermat, Pascal
- Newton, Euler, Gauss, Hamilton
- Boole, De Morgan
- Babbage, Ada Lovelace
- Venn, Carroll
“Standing on the Shoulders of Giants”

- Cantor, Hilbert, Russell
- Hardy, Ramanujan, Ramsey
- Gödel, Church, Turing
- von Neumann, Shannon
- Kleene, Chomsky
- Hoare, McCarthy, Erdos
- Knuth, Backus, Dijkstra

Many others…
Making Philosophy Accessible: Pop-up Plato
Aristotle (384BC-322BC)
- Founded Western philosophy
- Student of Plato
- Taught Alexander the Great
- “Aristotelianism”
- Developed the “scientific method”
- One of the most influential people ever
- Wrote on physics, theatre, poetry, music, logic, rhetoric, politics, government, ethics, biology, zoology, morality, optics, science, aesthetics, psychology, metaphysics, …
- Last person to know everything known in his own time!

“Almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine.” – Bertrand Russell
“Wit is educated insolence.”
- Aristotle (384-322 B.C.)
“The School of Athens” (by Raphael, 1483-1520)
THE UNKNOWN PHILOSOPHER
WHO FIRST REALIZED
LIFE IS NO PICNIC
Birds fly because they're lighter than air.
Some trees have different fruits each year.
At night, clouds rest on the ground.

Are you sure he's Aristotle?
“What I especially like about being a philosopher-scientist is that I don’t have to get my hands dirty.”
Historical Perspectives

Euclid (325BC-265BC)

- Founder of geometry & the axiomatic method
- “Elements” – oldest and most impactful textbook
- Unified logic & math
- Introduced rigor and “Euclidean” geometry
- Influenced all other fields of science: Copernicus, Kepler, Galileo, Newton, Russell, Lincoln, Einstein & many others
THE ELEMENTS
OF GEOMETRY
of the most auncient Philosopher
EUCLIDE
of Megara.

Faithfully (now first) translated into the English tongue, by
H. Billingsley, Citizen of London.
Whereunto are annexed certaine
Scholars, annotations, and Inventions,
of the best Mathematicians,
both of time past, and
in this our age.

With a very fruitful Preface made by M. I. Dec, specifying the chief Mathematical Sciences, what they are, and wherein commendation whereof all are dissembled certaine new Secrets Mathematicall, and several all, with their own diagrams, greatly designed.

Imprinted at London by Iohn Daye.
Chapter I
Call me Euclid.
Euclid’s Axioms

1: Any two points can be connected by exactly one straight line.

2: Any segment can be extended indefinitely into a straight line.

3: A circle exists for any given center and radius.

4: All right angles are equal to each other.

5: The parallel postulate: Given a line and a point off that line, there is exactly one line passing through the point, which does not intersect the first line.

The first 28 propositions of Euclid’s Elements were proven without using the parallel postulate!

Theorem [Beltrami, 1868]: The parallel postulate is independent of the other axioms of Euclidean geometry.

The parallel postulate can be modified to yield non-Euclidean geometries!
Non-Euclidean Geometries

Hyperbolic geometry: Given a line and a point off that line, there are an infinity of lines passing through that point that do not intersect the first line.

- Sum of triangle angles is less than 180°
- Different triangles have different angle sum
- Triangles with same angles have same area
- There are no similar triangles
- Used in relativity theory
Non-Euclidean Geometries

Spherical / Elliptic geometry: Given a line and a point off that line, there are no lines passing through that point that do not intersect the first line.

- Lines are geodesics - “great circles”
- Sum of triangle angles is > 180°
- Not all triangles have same angle sum
- Figures can not scale up indefinitely
- Area does not scale as the square
- Volume does not scale as the cube
- The Pythagorean theorem fails
- Self-consistent, and complete
Founders of Non-Euclidean Geometry

János Bolyai (1802-1860)

Nikolai Ivanovich Lobachevsky (1792-1856)
Non-Euclidean Non-Orientable Surfaces

- Möbius strip: one side, one boundary!
- Klein bottle: one side, no boundary!
- Projective plane: one side, no boundary!

Images of each surface illustrate their unique properties.
THE GEOMETRY OF EVERYDAY LIFE

TUNA SANDWICH

SNEAKER

GRANDMA
**Problem**: A man leaves his house and walks one mile south. He then walks one mile west and sees a Bear. Then he walks one mile north back to his house. What color was the bear?

**Problem**: Is the house location unique?
Historical Perspectives

Archimedes of Syracuse (287-212 BC)

- Mathematician, physicist, engineer, inventor, astronomer
- Leading scientist of classical antiquity
- Originated hydrostatics, mechanics
  - Archimedean screw, spiral, lever
- Discovered Archimedes’ principle
- Used infinitesimals, approximated Pi
- Designed siege and naval weapons
- Invented large number notation
Archimedes’ “ostomachion” puzzle

How it works:
1. The water will flow under the mill into the tubes housing the Archimedean screws. The water pressure on the screw blades causes them to turn.

Each screw is approximately 35 feet long, with a diameter of six feet.

2. The screw is attached to a generator that produces electricity as it turns.

About 1.5 tons of water passes through each of the two screws every second.

The screws are designed to allow fish and eels to pass through safely.
"There goes Archimedes with his confounded lever again!"
"The periodic table."
Problem: \((1/4) + (1/4)^2 + (1/4)^3 + (1/4)^4 + \ldots = ?\)

Find a short, geometric, induction-free proof.

\[
\sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{3}
\]
Problem: \( (1/8) + (1/8)^2 + (1/8)^3 + (1/8)^4 + \ldots = ? \)

Find a short, **geometric**, induction-free proof.

\[
\sum_{i=1}^{\infty} \frac{1}{8^i} = \frac{1}{7}
\]
Problem: Are the complex numbers closed under exponentiation? E.g., what is the value of $i^i$?

$$i^i = \frac{1}{\sqrt{e^{\pi}}} = 0.207879...$$

$$i^i = \frac{1}{\sqrt{e^{\pi + 2k\pi}}}$$

$e^{ix} = \cos(x) + i \sin(x)$

$i^i$ is multi-valued!
Problem: Solve the following equation for $X$:

$$X^X^{X^{X^{X^{\ddots}}}} = 2$$

where the stack of exponentiated $x$’s extends forever.

• What approaches fail?
• What techniques work and why?
• Lessons and generalizations

“Mr. Osborne, may I be excused? My brain is full.”
Problem: For the given infinite ladder of resistors of resistance R each, what is the resistance measured between points x and y?

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
There's a certain type of brain that's easily disabled.

If you show it an interesting problem, it involuntarily drops everything else to work on it.

This has led me to invent a new sport: Nerd Sniping.

See that physicist crossing the road?

Hey!

On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?

It's... Hmm. Interesting. Maybe if you start with... No, wait. Hmm... You could--

I will have no part in this. C'mon, make a sign. It's fun! Physicists are two points, mathematicians three.