# CS6160 Theory of Computation
# Problem Set 1
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Please start solving these problems immediately, and work in study groups. Please prove all your answers; informal arguments are acceptable, but please make them precise / detailed / convincing enough so that they can be easily made rigorous.

*"Begin at the beginning," said the King very gravely,*
*"and go on till you come to the end; then stop."*

0. Solve the following problems from [Sipser, Second Edition]. Page 27: 0.10, 0.11, 0.12.

1. True or false:
   a. $\emptyset \subseteq \emptyset$
   b. $\emptyset \subset \emptyset$
   c. $\emptyset \in \emptyset$
   d. $\{1,2\} \in 2^{\{1,2\}}$
   e. $\{1,2\} \subseteq 2^{\{1,2\}}$
   f. $\{x,y\} \in \{\{x,y\}\}$

2. Write the following set explicitly: $2^{\{1,\,2\}} \times \{v,w\}$

3. Prove without using induction that for an arbitrary finite set S, the sets $2^S$ and $\{0,1\}^{|S|}$ have the same number of elements.

4. Which of the following sets are closed under the specified operations?
   a) $\{x \mid x$ is an odd integer$\}$, multiplication
   b) $\{y \mid y=2n$, n some integer$\}$, subtraction
   c) $\{2m+1 \mid m$ some integer$\}$, division
   d) $\{z \mid z=a+bi$ where a and b are real, $|a||b| > 0$, and $i=\sqrt{-1}\ \}$, exponentiation

5. Is the transitive closure of a symmetric closure of a binary relation necessarily reflexive? (Assume that every element of the "universe" set participates in at lease one relation pair.)

6. True or false: a countable union of countable sets is countable.

7. True or false: if T is countable, then the set $\{S \mid S \subseteq T, S$ finite$\}$ is also countable.

8. Give a simple bijection for each one of the following pairs of sets:
   a) the integers, and the odd integers.
   b) the integers, and the positive integers.
   c) the naturals, and the rationals crossed with the integers.

*"Why," said the Dodo, "the best way to explain it is to do it."*

9. Is there a bijection between $\{x \mid x \in \mathbf{R}, 0 \le x \le 1\}$ and $\mathbf{R}$?

10. Generalize $|S| < |2^S|$ to arbitrary infinite sets (not necessarily countable ones).

11. What is the cardinality of each of the following sets ?
   a. The set of all polynomials with rational coefficients.
   b. The set of all functions mapping reals to reals.
   c. The set of all possible Java programs.
   d. The set of all finite strings over the alphabet $\{0,1,2\}$.
   e. The set of all 5x5 matrices over the rationals.
   f. The set of all points in 3-dimensional Euclidean space.
   g. The set of all valid English words.
   h. $\{\varnothing, \mathbf{N}, \mathbf{Q}, \mathbf{R}\}$
   i. $\mathbf{N} \times \mathbf{Z} \times \mathbf{Q}$
   j. $\mathbf{R} - \mathbf{Q}$

*"And thick and fast they came at last, and more, and more, and more"*

12. Prove without using induction that $n^4-4n^2$ is divisible by 3 for all $n\ge 0$.

13. How many distinct boolean functions on N variables are there? In other words, what is the cardinality of $|\{f \mid f:\{0,1\}^N \rightarrow \{0,1\}\}|$ ?

14. How many distinct N-ary functions are there from finite set A to finite set B? Does this generalize the previous question?

15. Show that in any group of people, there are at least two people with the same number of acquaintances within the group. Assume the "acquaintance" relation is symmetric but non-reflexive.

16. Show that in any group of six people, there are either 3 mutual strangers or 3 mutual acquaintances.

17. A clique in a graph is a complete subgraph (i.e., all nodes are connected with edges). Show that every graph with N nodes contains a clique or the complement of a clique, of size at least $\frac{1}{2} \log_2 N$.

18. Show that the set difference of an uncountable set and a countable set is uncountable.

19. Show that the intersection of two uncountable sets can be empty, finite, countably infinite, or uncountably infinite.

20. For an arbitrary language L, prove or disprove each of the following:
   a) $(L^*)^* = L^*$
   b) $L^+ = L^* - \{\wedge\}$

*"Is that all?" Alice timidly asked.*