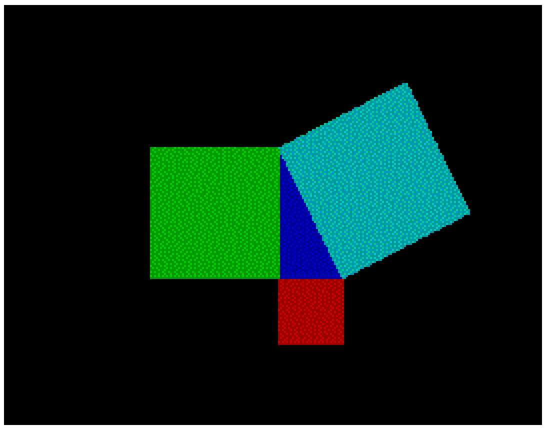
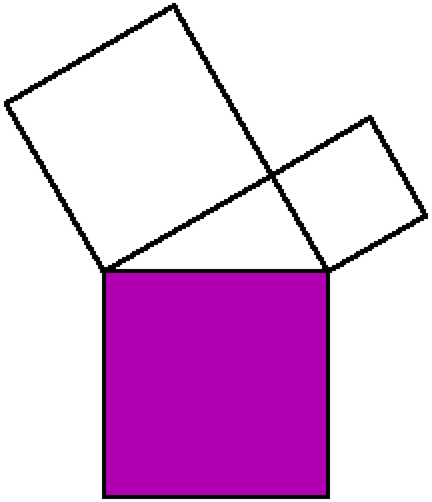
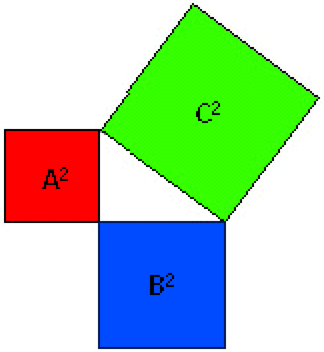
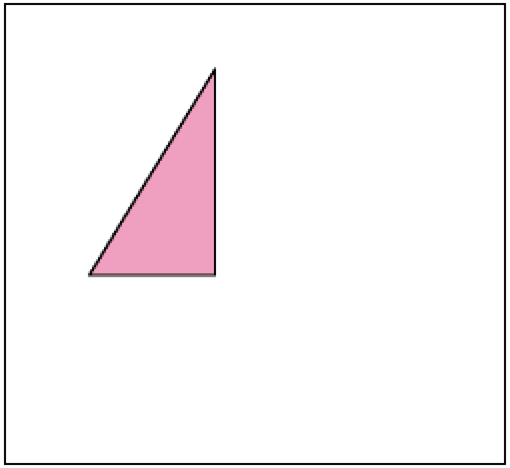
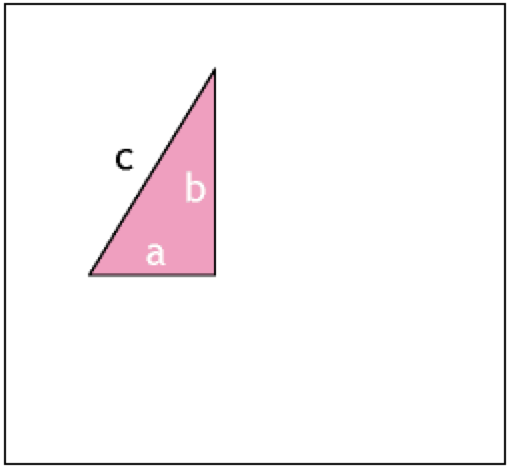
Problem: Give as many proofs as you can for the Pythagorean Theorem. i.e., $a^2 + b^2 = c^2$ holds for any right triangle with sides a & b and hypotenuse c.



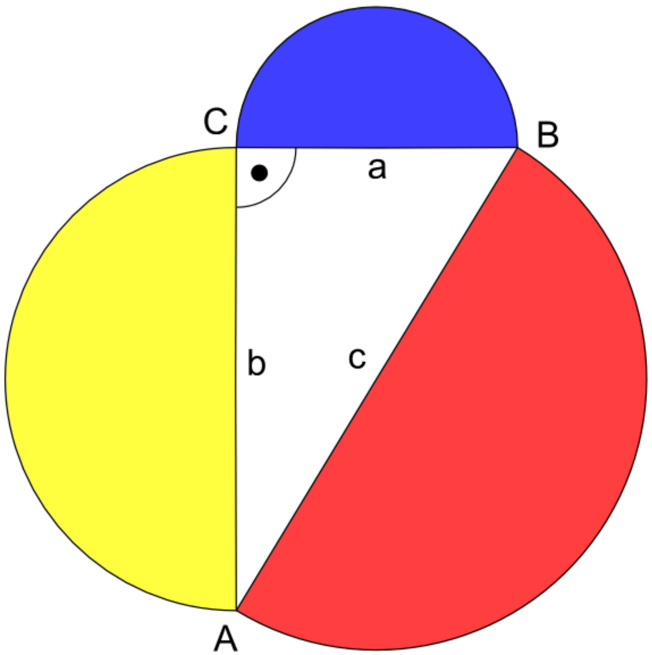


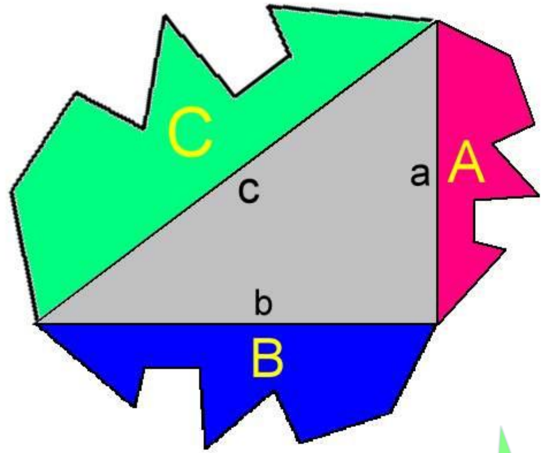


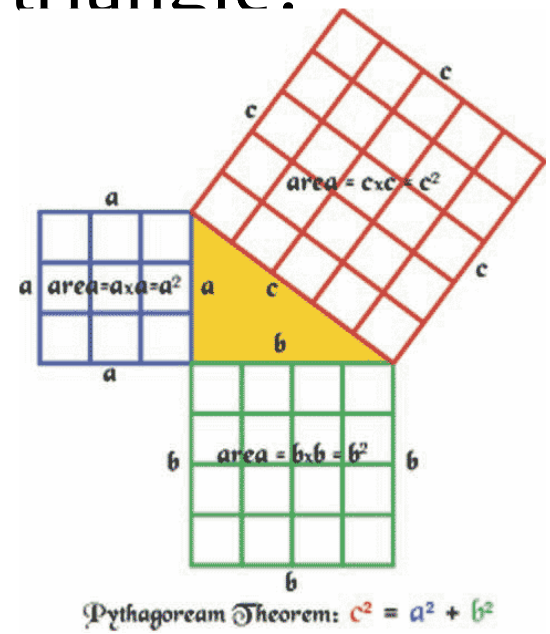


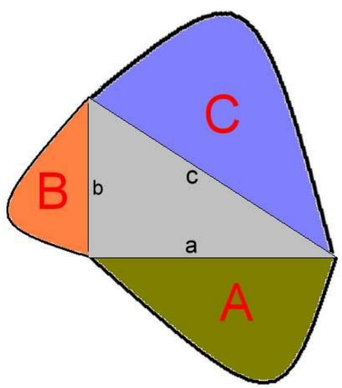


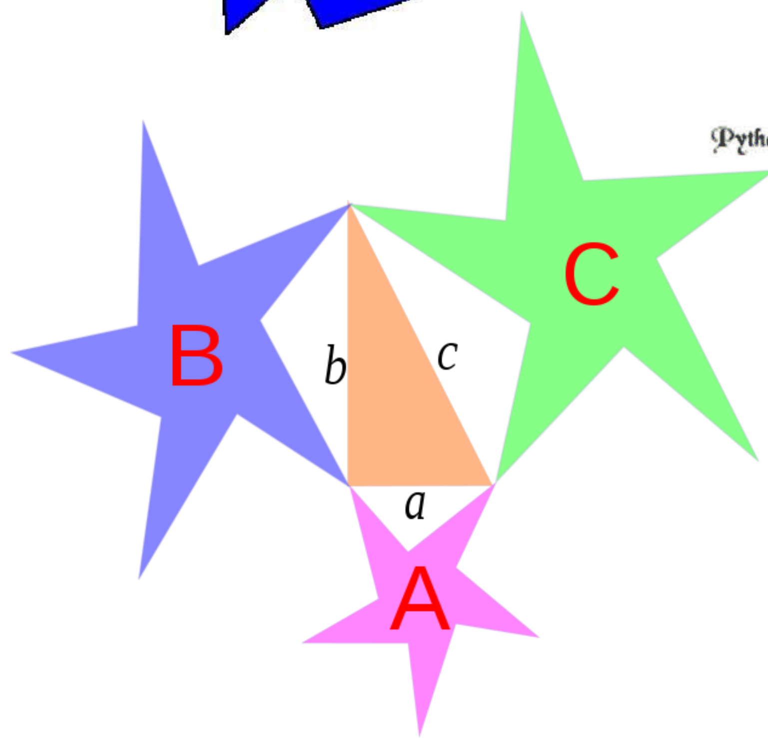
Problem: Does the Pythagorean theorem generalize to arbitrary figures on the sides of a right triangle?









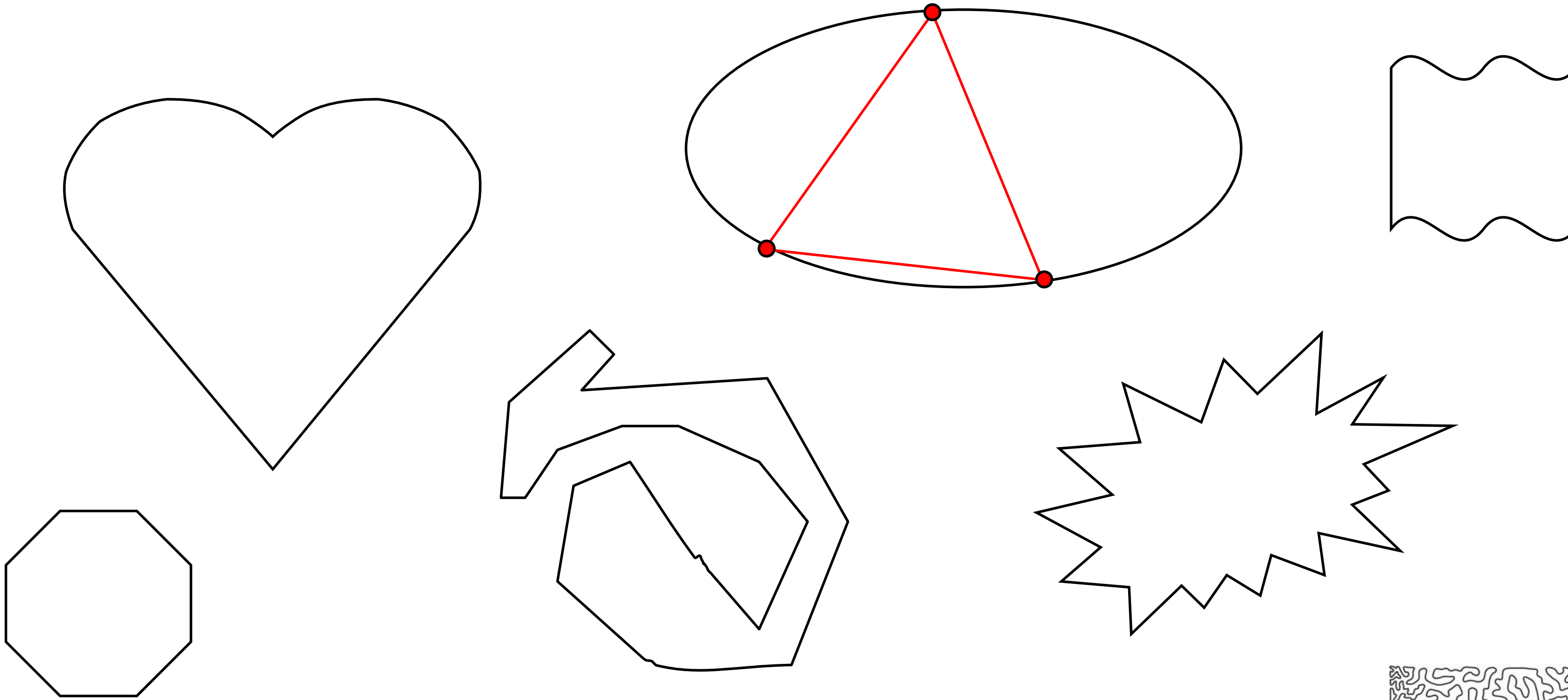


Problem: compute 111111111^2 in your head.

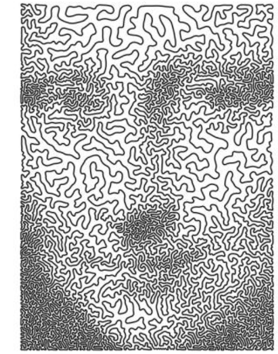
Problem: What is the approximate value of:

$$(1+9^{\wedge}(-(4^{\wedge}(7*6))))^{\wedge}(3^{\wedge}(2^{\wedge}85)) \approx ?$$

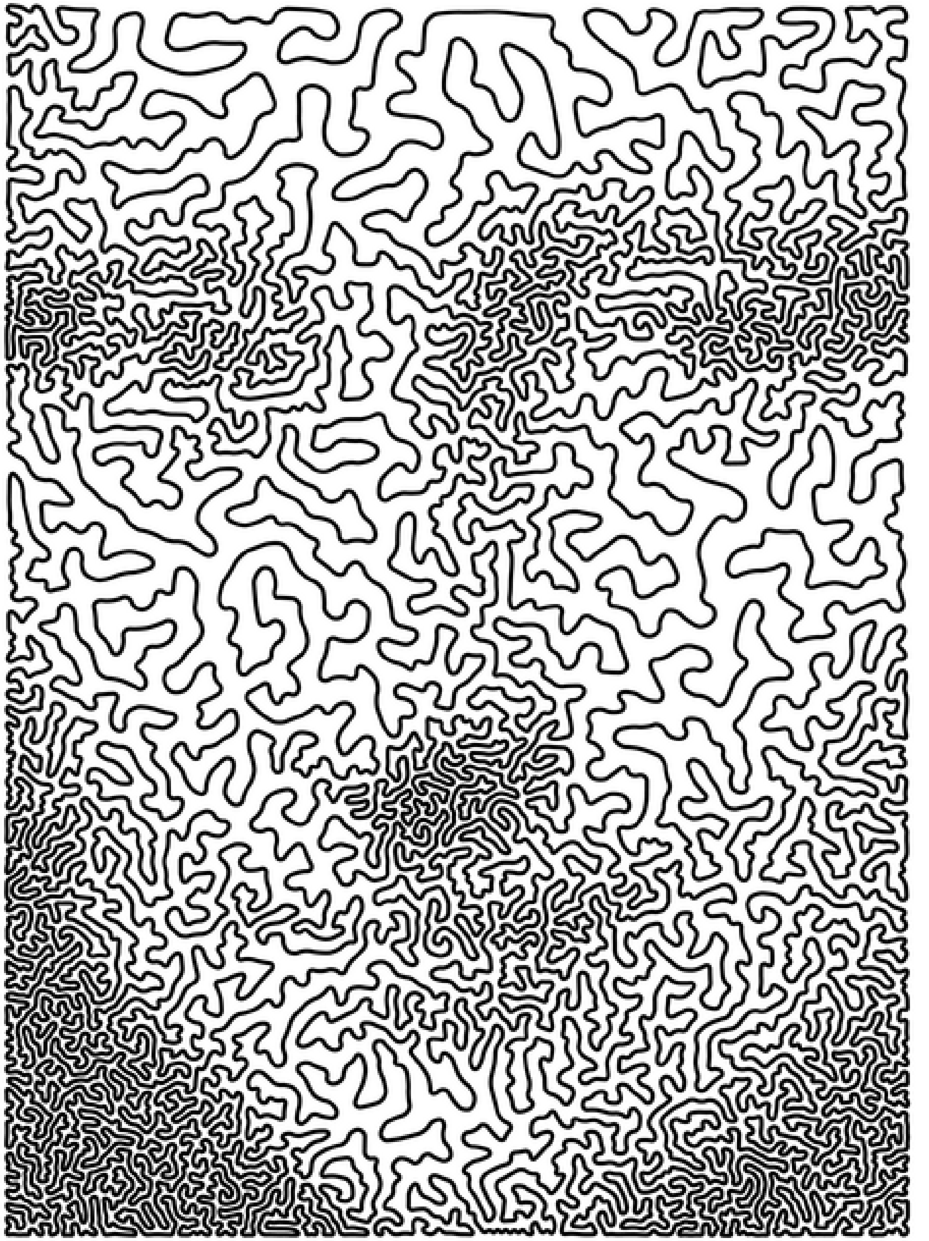
Problem: Does every closed simple curve contain the vertices of an equilateral triangle?



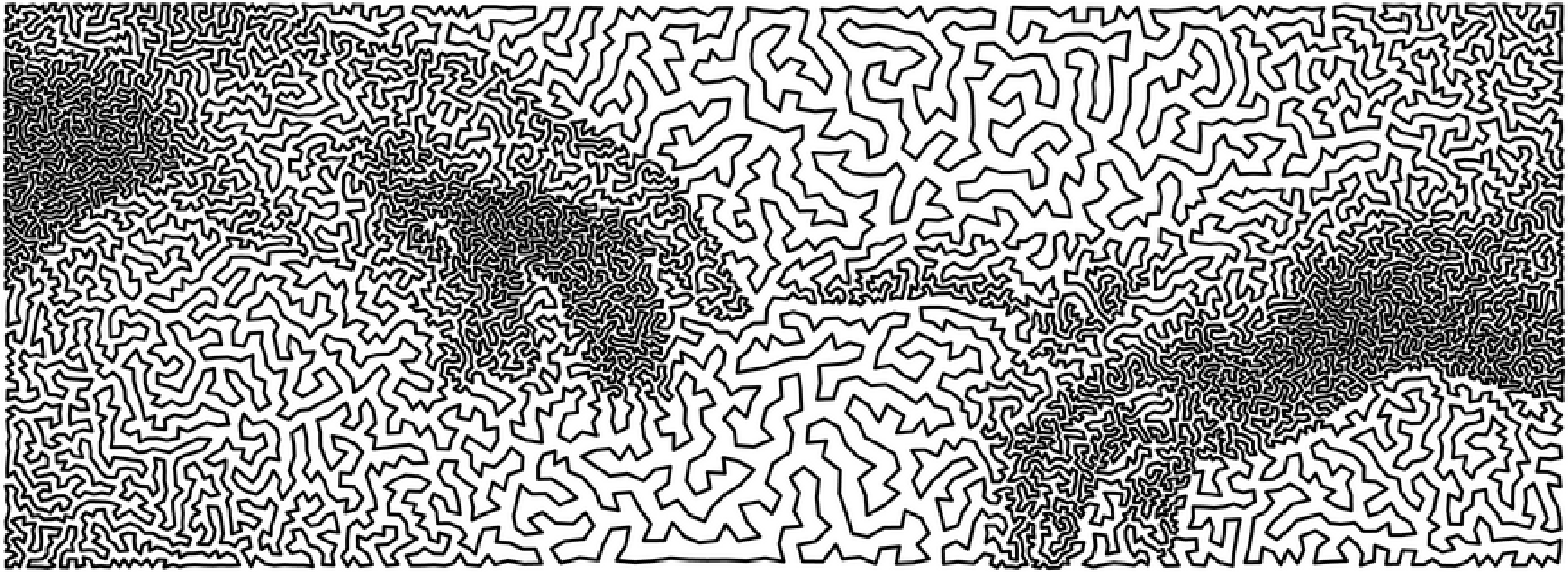
- What approaches fail?
- What techniques work and why?
- Lessons and generalizations



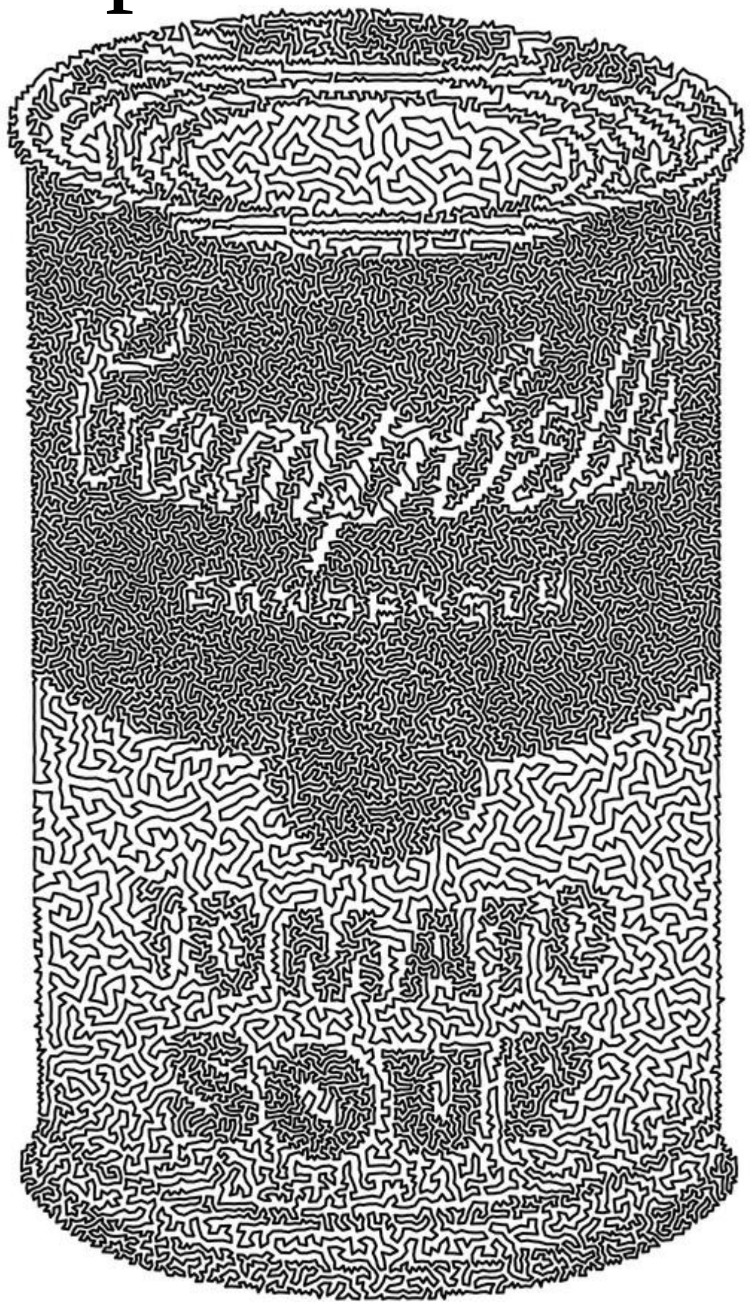
A Simple Closed Curve!



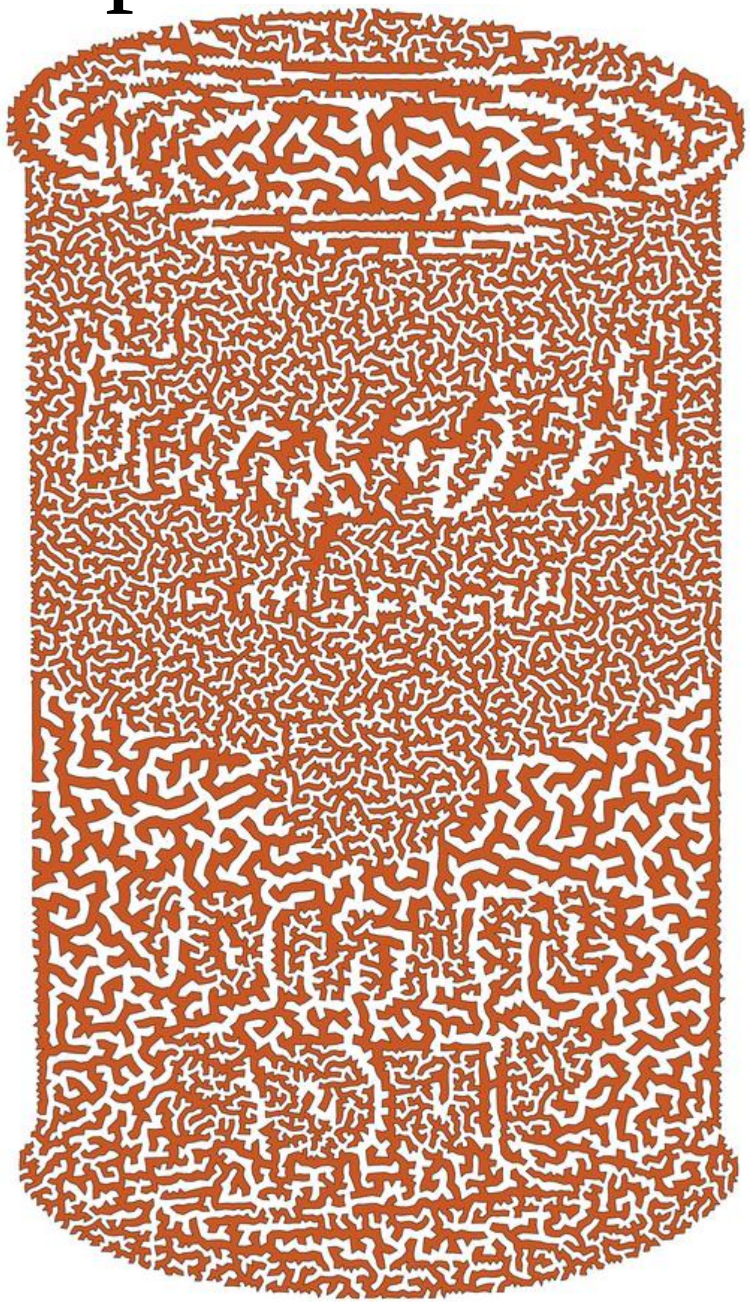
A Simple Closed Curve!



A Simple Closed Curve!

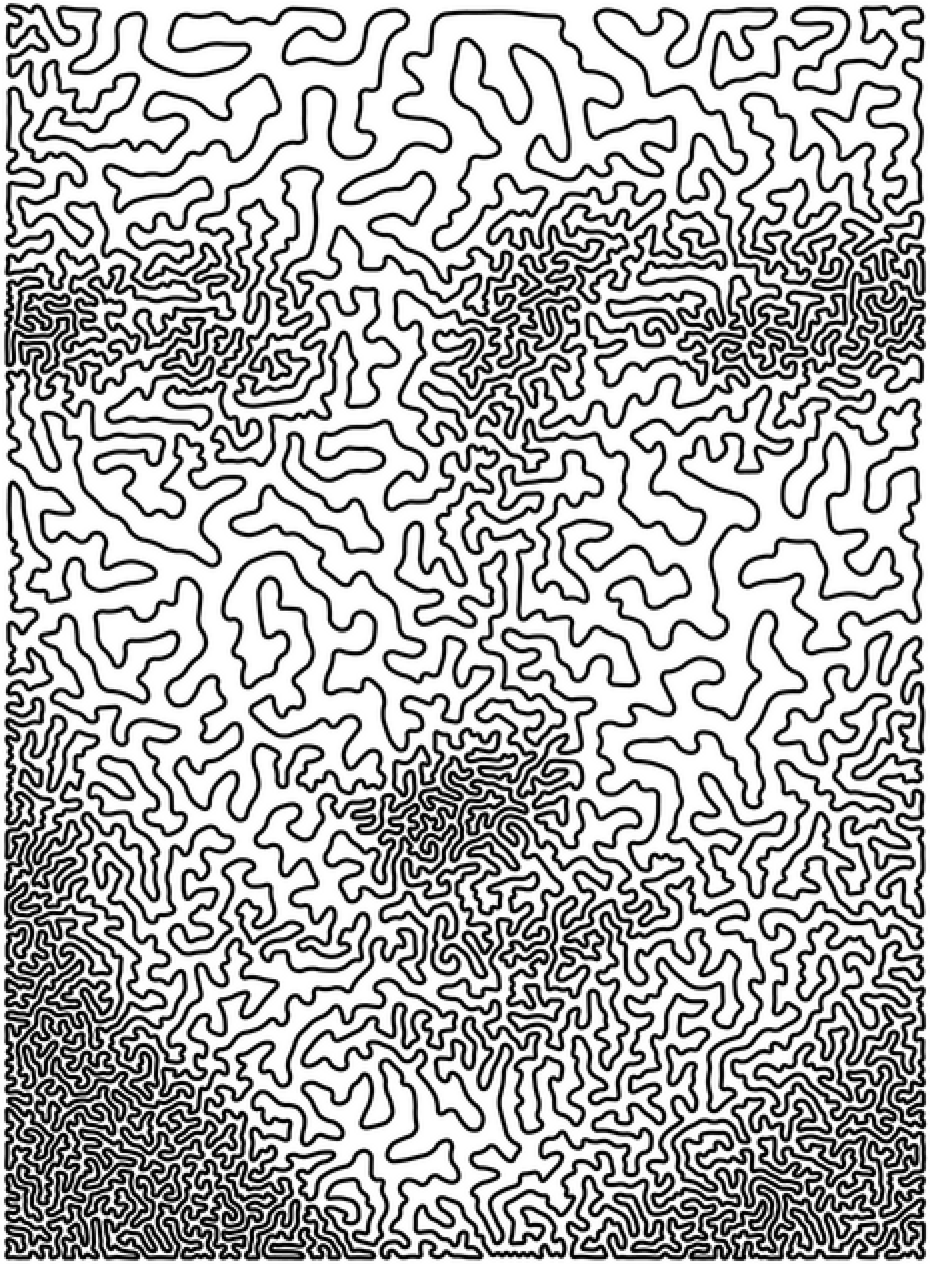


A Simple Closed Curve!



Project Idea: TSP Art

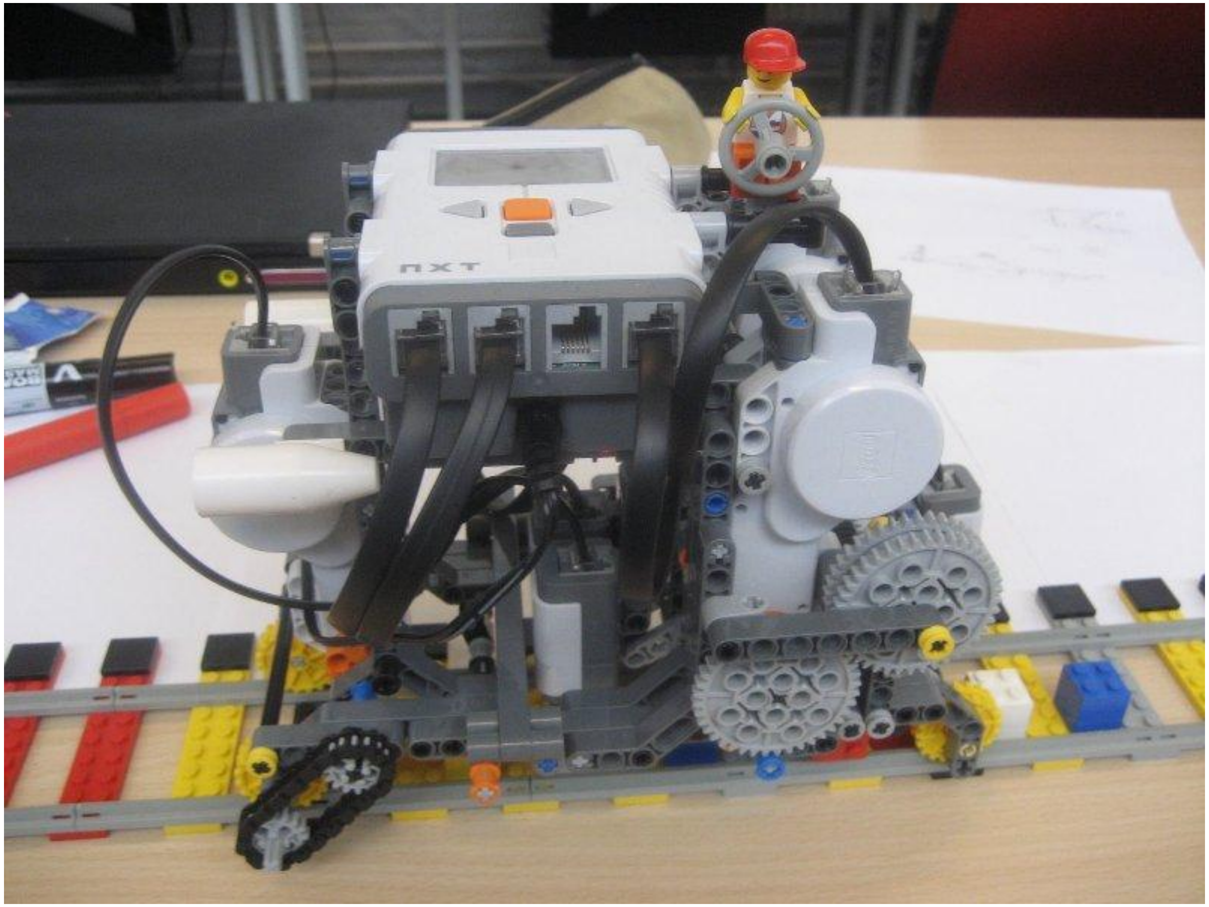
- Traveling Salesperson Tour
- Optimal is NP-complete
  So use heuristics
- Convert image to B&W
- Sample image density
  to obtain a pointset
- Run TSP heuristics
- Can use minimum spanning
  trees (easy to compute)
- Can also use minimum
  matchings (easy to compute)
- What about colors?



Project Idea: Turing Machine Simulator

Ex 1: Using software (with a GUI)

Ex 2: Using Lego!

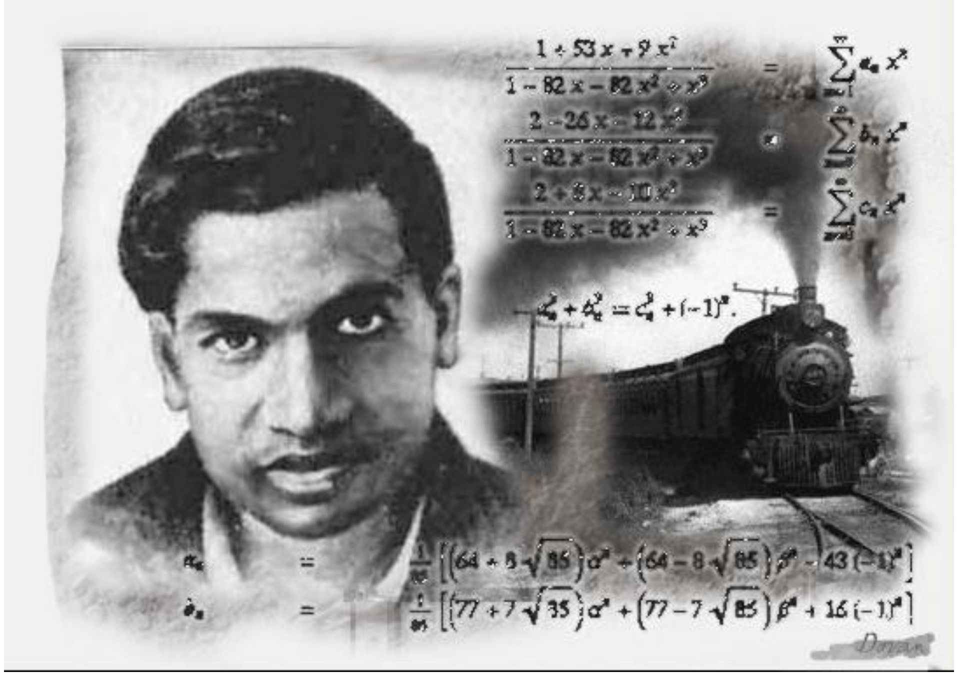


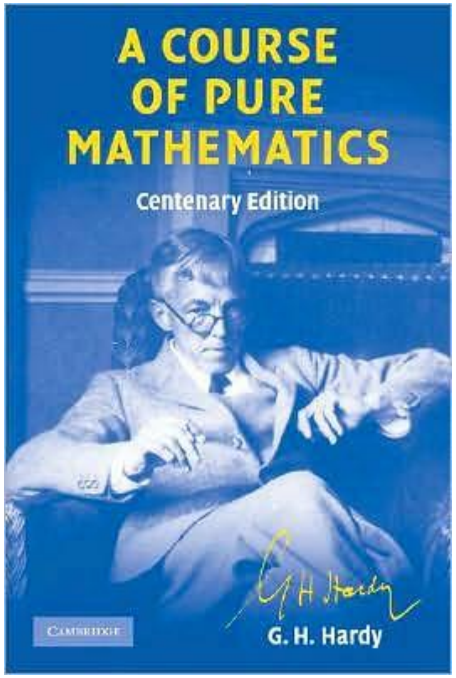
See: http://www.youtube.com/watch?v=cYw2ewoO6c4

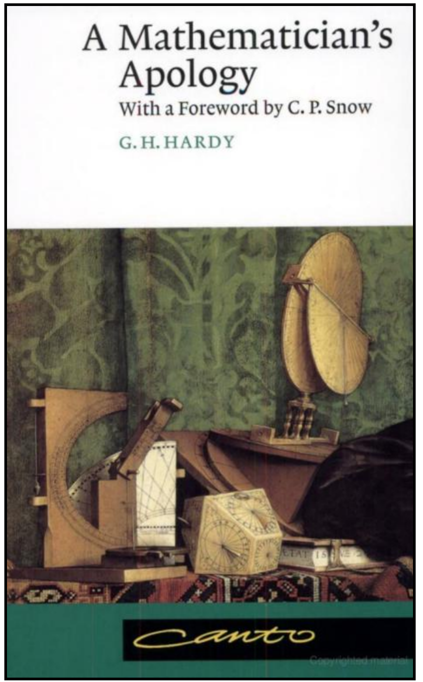
Historical Perspectives

Godfrey Hardy (1877-1947)

- Mathematician: contributed to analysis, number theory, physics, and genetics
- Wrote "A Mathematician's Apology" which popularized mathematics
- Discovered & mentored Ramanujan



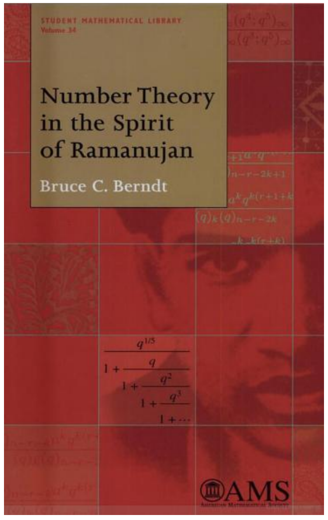


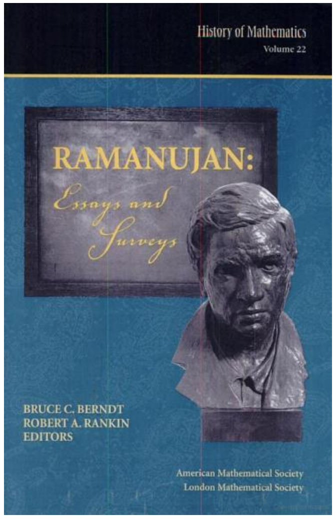


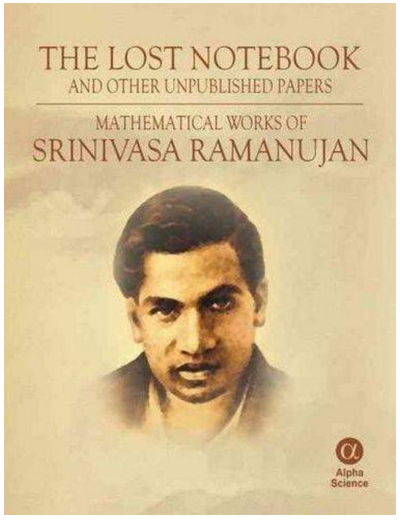
Historical Perspectives

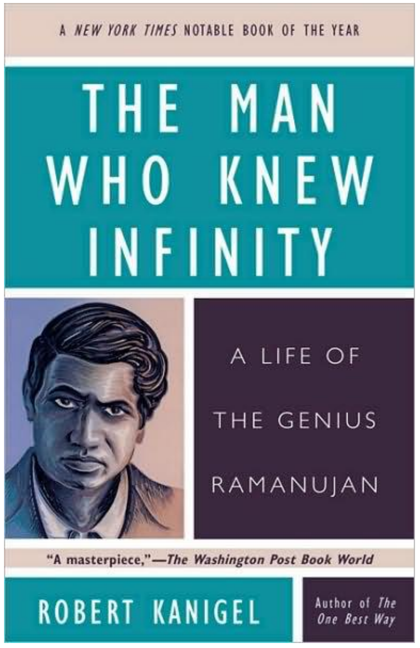
Srinivasa Ramanujan (1887-1920)

- Mathematician: contributed to number theory, analysis, infinite series & continued fractions
- Studied math on his own in isolation
- Proved 3,900 theorems
- Influenced many other fields, including physics
- Inspired generations of mathematicians
- Entire mathematical societies and journals are devoted to his work!









Volume 19 No.4 December 2004 ISSN 0970-1249

Journal of The RAMANUJAN Mathematical Society

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VOLUME 15 No. 4
MARCH 2006

ISSN 0971-1694

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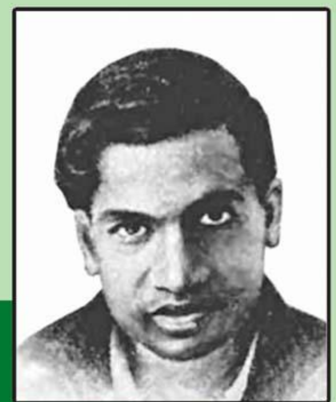

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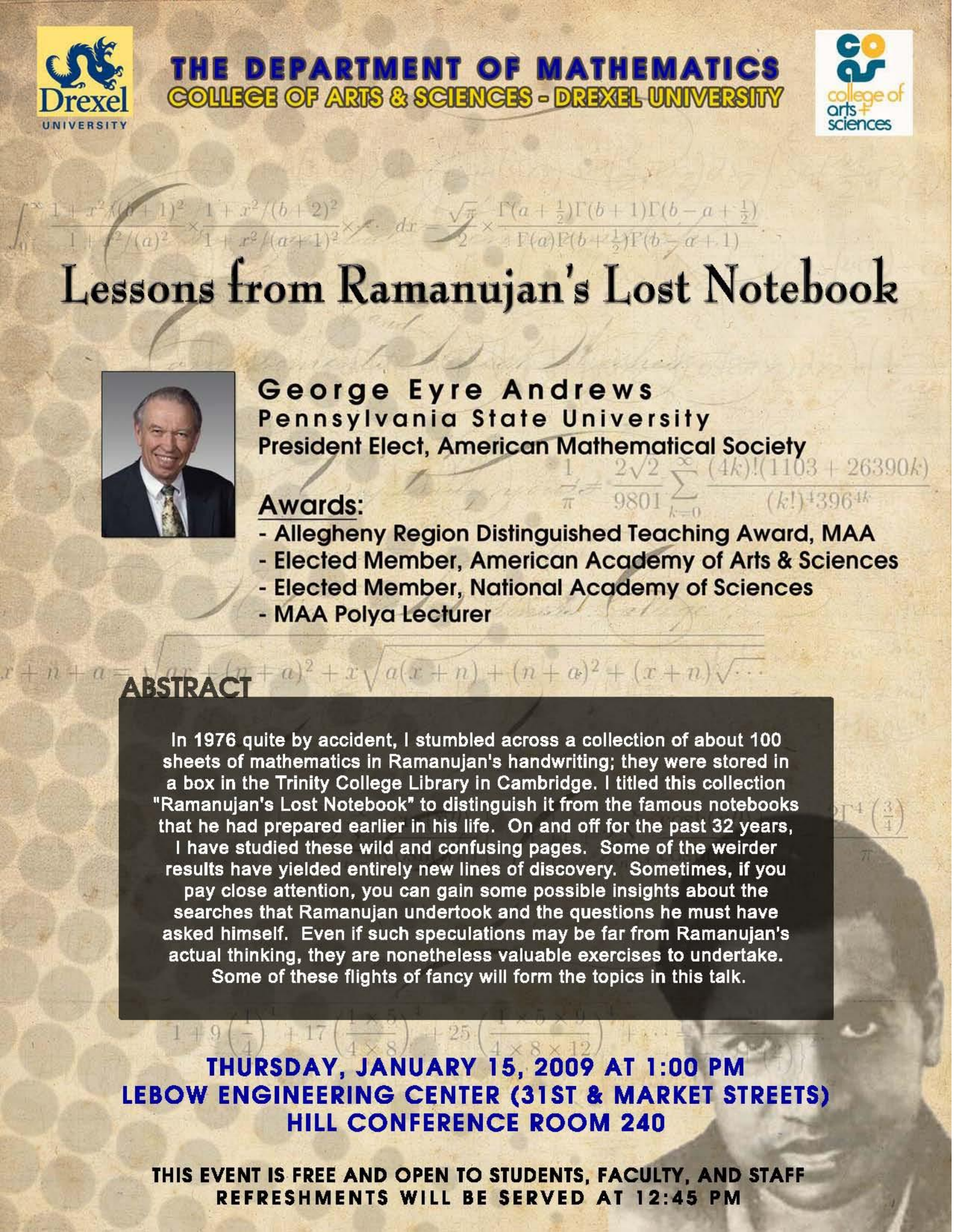
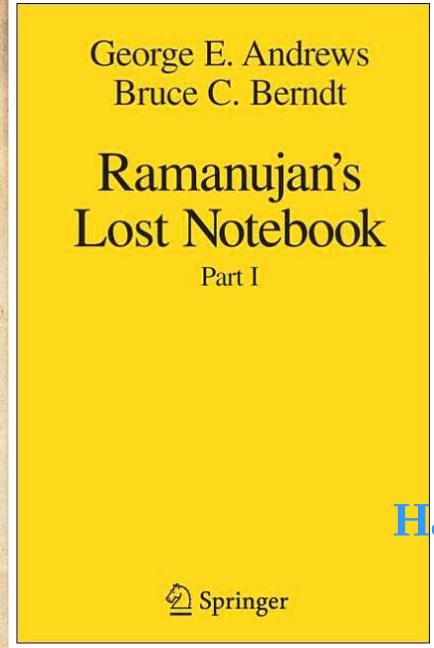




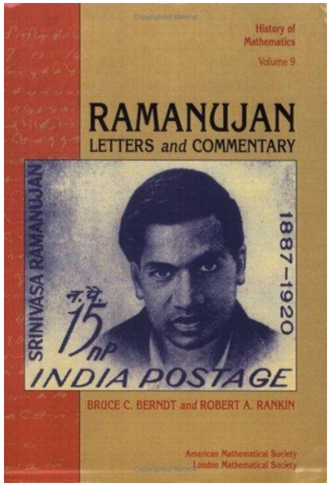
"The Hardy-Ramanujan Number"

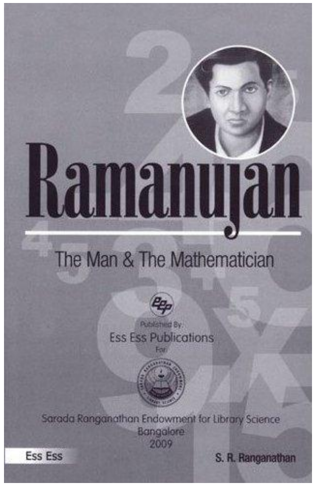
G. H. Hardy on Ramanujan:

"I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' "

A Fermat "near-miss":

$$1729 = 9^3 + 10^3 = 12^3 + 1^3$$





"My greatest contribution to mathematics was discovering Ramanujan." - G. H. Hardy

"Ramanujan's theorems must be true, because, if they were not true, no one would have the imagination to invent them."
- G. H. Hardy, upon first seeing Ramanujan's results

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}.$$

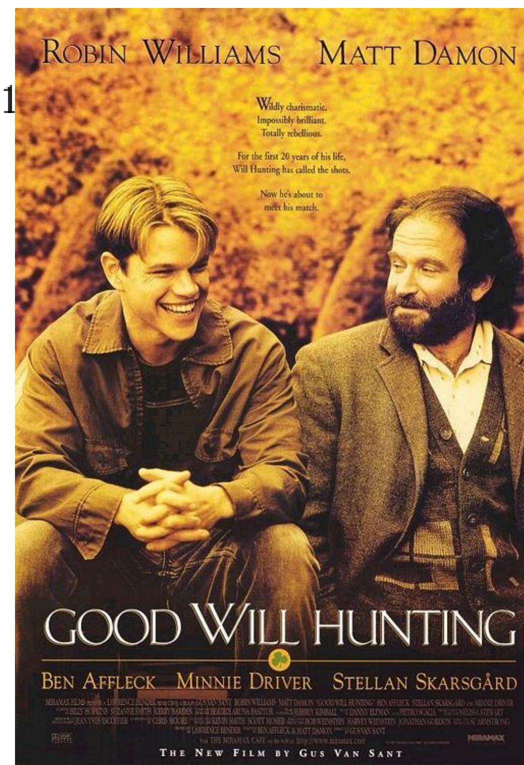
$$\int_0^\infty \frac{1+x^2/(b+1)^2}{1+x^2/(a)^2} \times \frac{1+x^2/(b+2)^2}{1+x^2/(a+1)^2} \times \cdots \; dx = \frac{\sqrt{\pi}}{2} \times \frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)}.$$

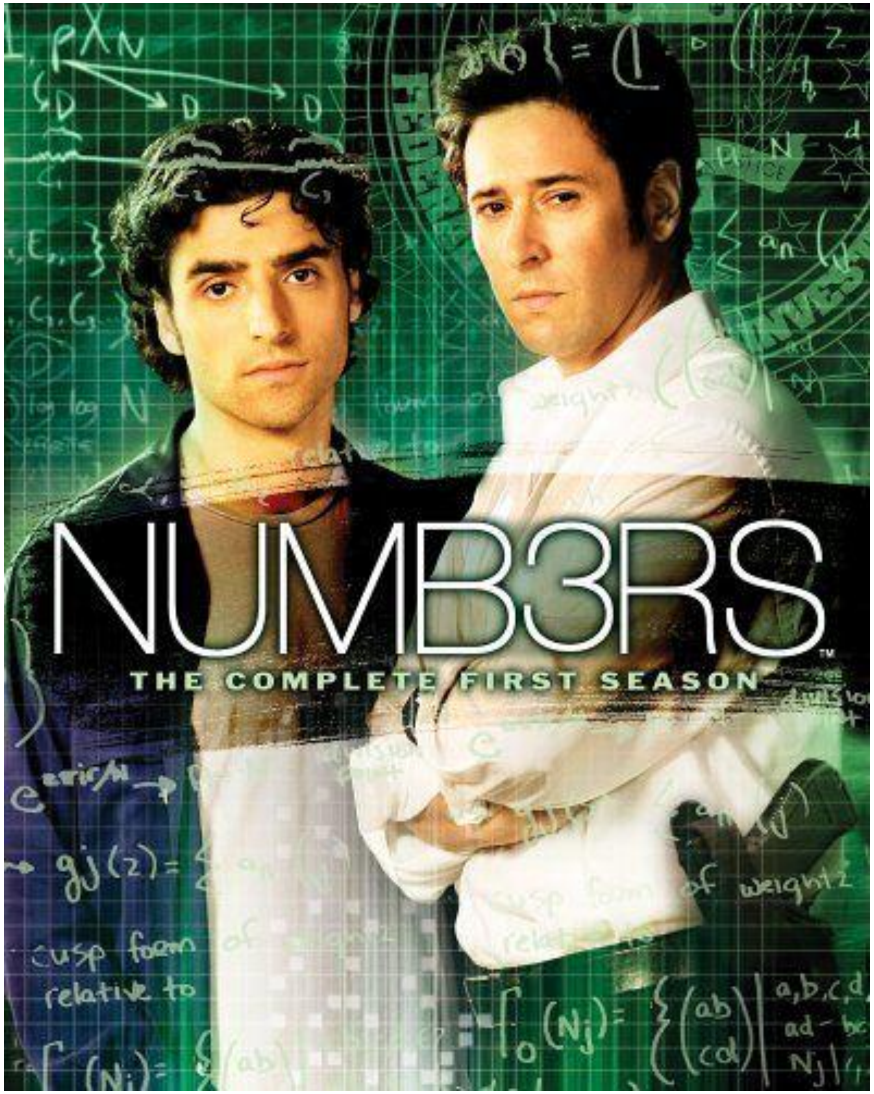
$$\frac{1}{1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\ldots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5} = e^{2\pi/5}\left(\sqrt{\varphi\sqrt{5}} - \varphi\right) = 0.9981$$

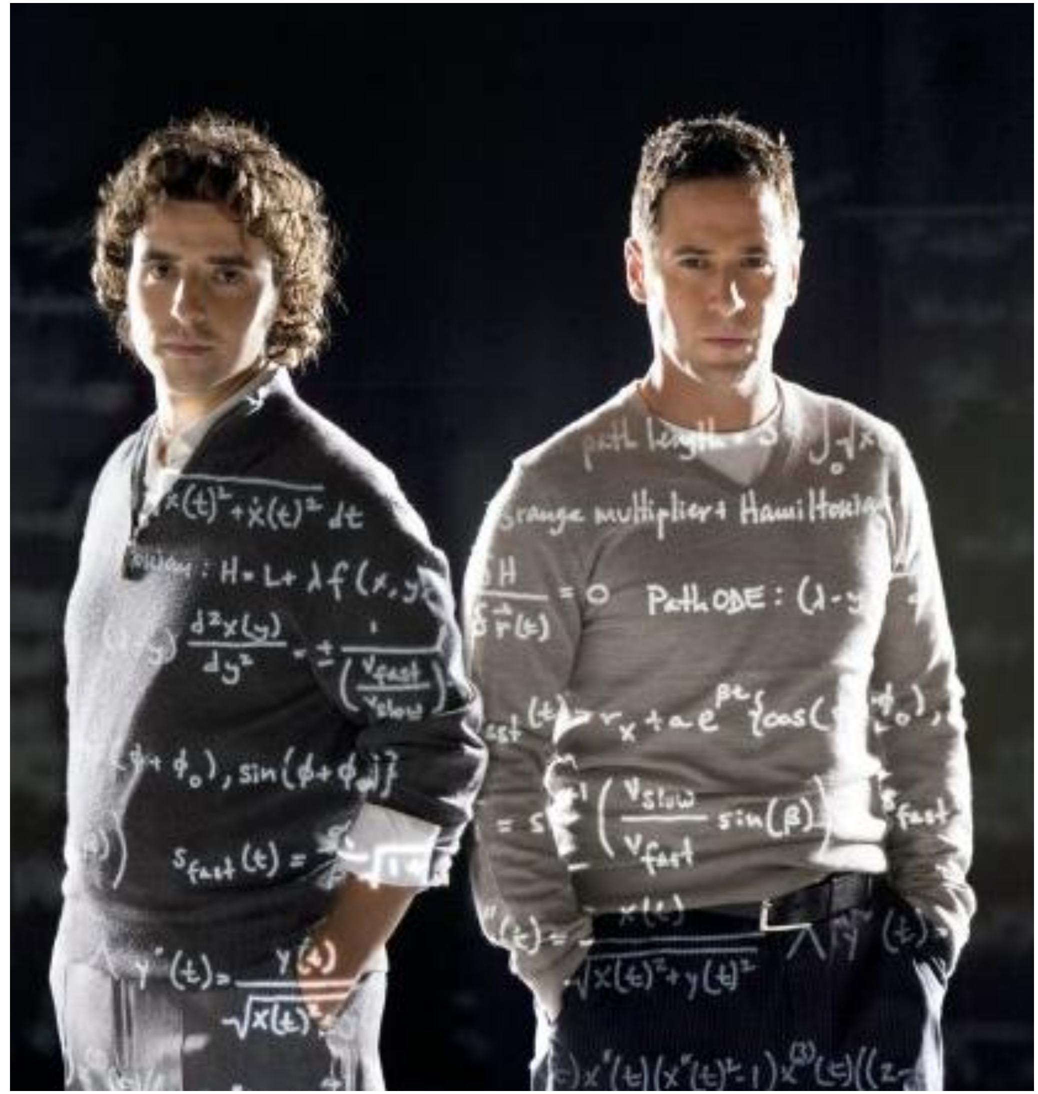
$$1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\times 5}{4\times 8}\right)^4 + 25\left(\frac{1\times 5\times 9}{4\times 8\times 12}\right)^4 + \cdots = \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\Gamma^2\left(\frac{3}{4}\right)}.$$

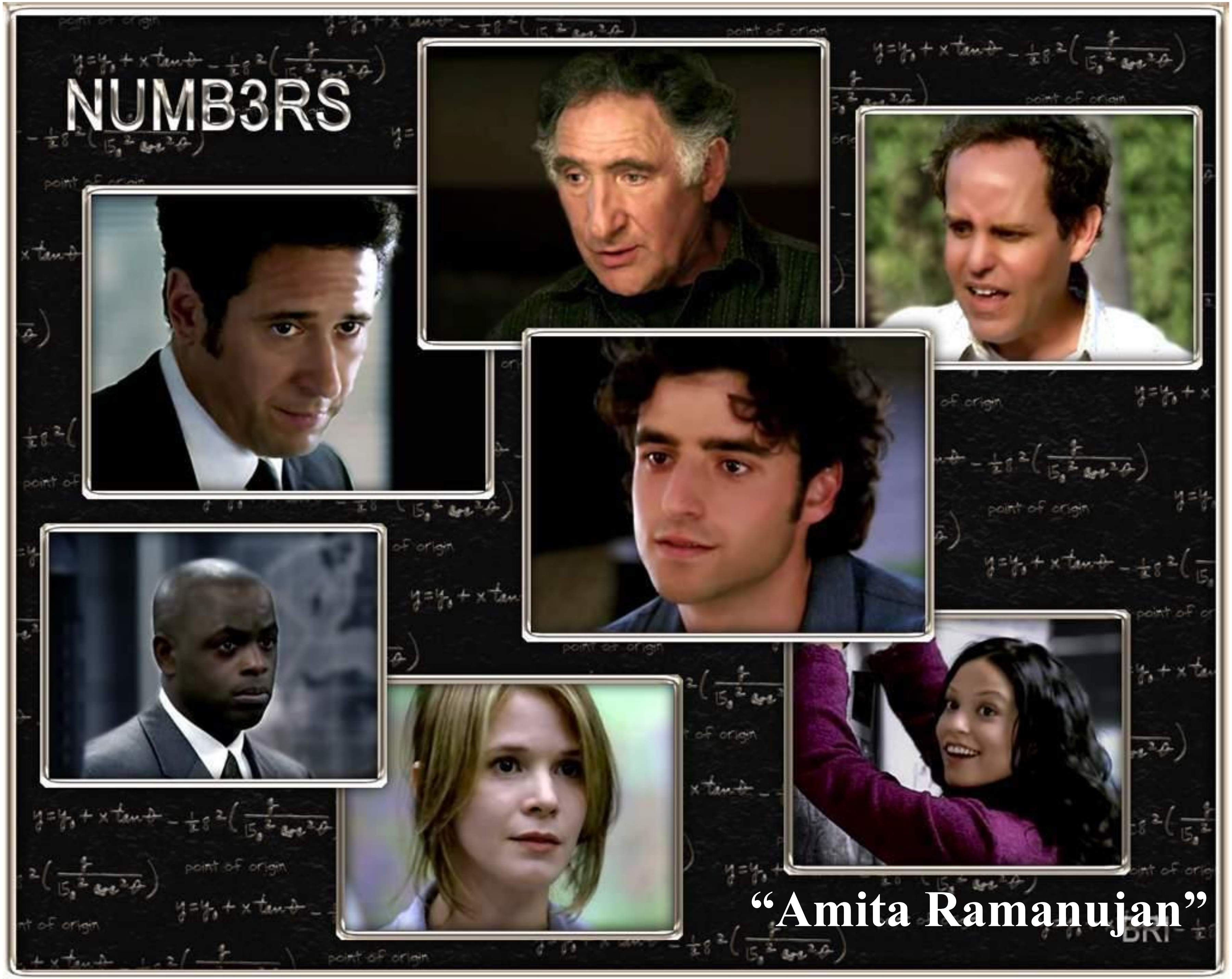
$$\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{\cosh(n\pi)}\right]^{-2} + \left[1 + 2\sum_{n=1}^{\infty} \frac{\cosh(n\theta)}{\cosh(n\pi)}\right]^{-2} = \frac{2\Gamma^4\left(\frac{3}{4}\right)}{\pi}$$

$$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\times 3}{2\times 4}\right)^3 - 13\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^3 + \cdots = \frac{2}{\pi}$$




NUMB3RS
THE COMPLETE FIRST SEASON

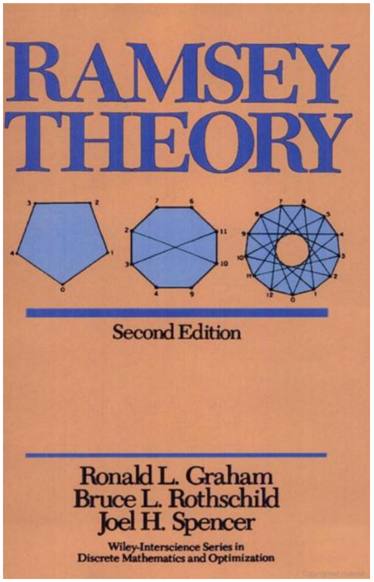


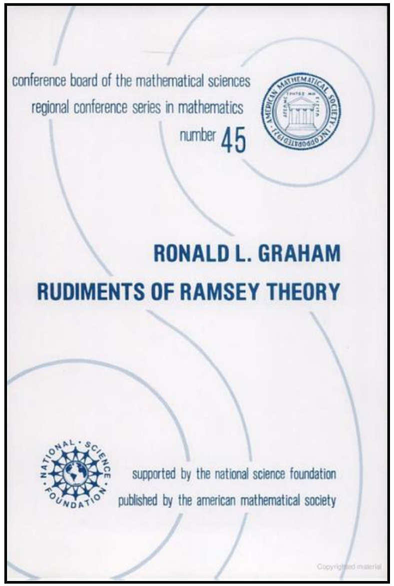

NUMB3RS
“Amita Ramanujan”

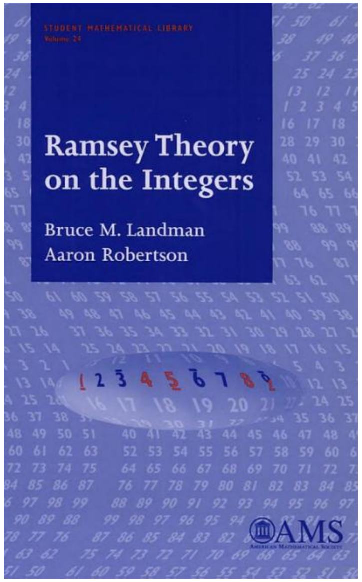
Historical Perspectives

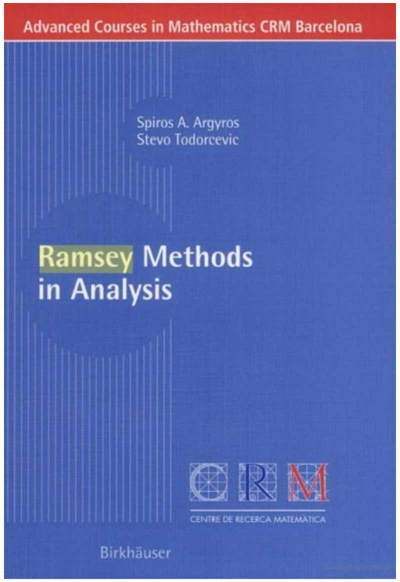
Frank Ramsey (1903-1930)

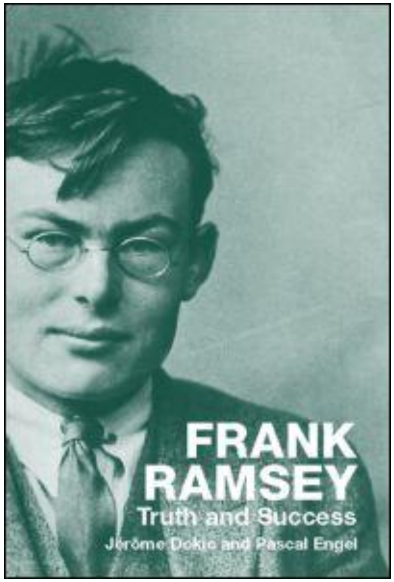
- Contributed to mathematics, decision theory, game theory, logic, philosophy, economics
- Pioneered Ramsey theory
- Was Wittgenstein's Ph.D. advisor
- Influenced Church, von Neumann, Keynes
- Died at age 26





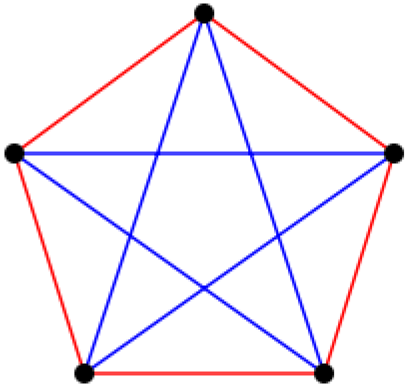






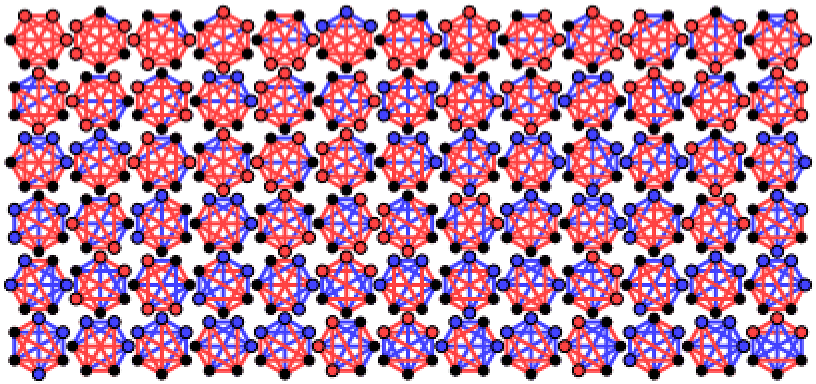
Problem: Show that any group of six people contains either 3 mutual friends or 3 mutual strangers.

Q: Is this true for N=5?



No mono-chromatic triangles

Brute force approach?

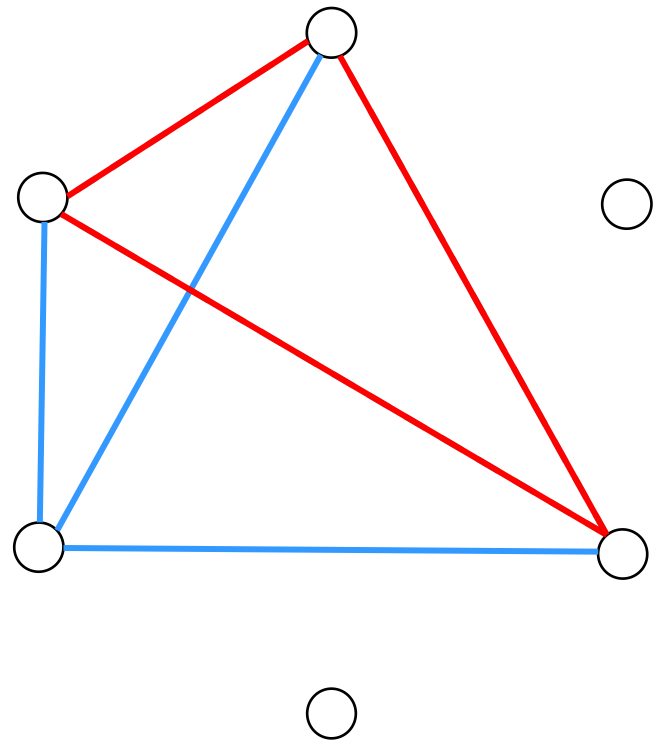


78 possible friends-strangers graphs with 6 nodes

A more elegant approach is needed!

Problem: Show that any group of six people contains either 3 mutual friends or 3 mutual strangers.

Pigeon-hole principle!



6 is said to be the "Ramsey number" R(3,3).

Theorem: any group of 18 people contains either 4 mutual friends or 4 mutual strangers. R(4,4)=18

Ramsey Theory

- R(3,3)=6 is the tip of a deep mathematical theory.

Theorem [Ramsey]: For any pair of positive integers b and r, there exists a least positive integer R(b,r) such that any complete graph over R(b,r) vertices, where each edge is colored either blue or red, contains a monochromatic clique of size b or r.

- Ramsey theory seeks "order" among "chaos": i.e., even "random" graphs / configurations still contain regular and predictable sub-structures.

- Pigeon-hole principle is a special case!

Other known Ramsey numbers (and bounds):

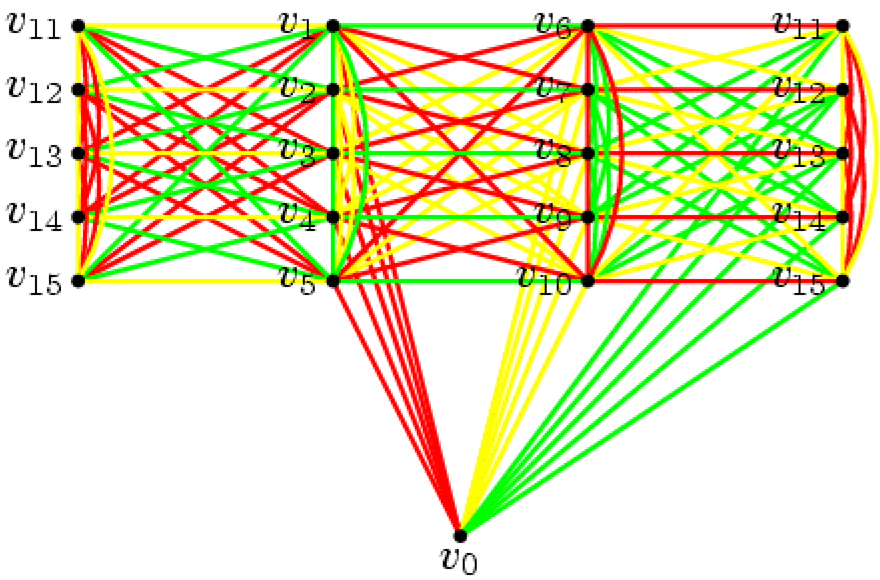
r,*s*	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6 =R(3,3)	9	14	18	23	28	36	40–43
4	1	4	9	18	25	35–41	49–61	56–84	73–115	92–149
5	1	5	14	25	43–49	58–87	80–143	101–216	125–316	143–442
6	1	6	18	35–41	58–87	102–165	113–298	127–495	169–780	179–1171
7	1	7	23	49–61	80–143	113–298	205–540	216–1031	233–1713	289–2826
8	1	8	28	56–84	101–216	127–495	216–1031	282–1870	317–3583	≤ 6090
9	1	9	36	73–115	125–316	169–780	233–1713	317–3583	565–6588	580–12677
10	1	10	40–43	92–149	143–442	179–1171	289–2826	≤ 6090	580–12677	798–23556

"Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5,5) or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6,6). In that case, we should attempt to destroy the aliens." – Paul Erdös (1913-1996)

Generalizations of Ramsey numbers

- Multi-colors: only known non-trivial exact value is R(3,3,3)=17

  E.g.: 16-node graph containing no mono-chromatic triangles:



Extra credit:
prove that
R(3,3,3)=17

- Hypergraphs (where "edges" can be vertex subsets of size > 2)

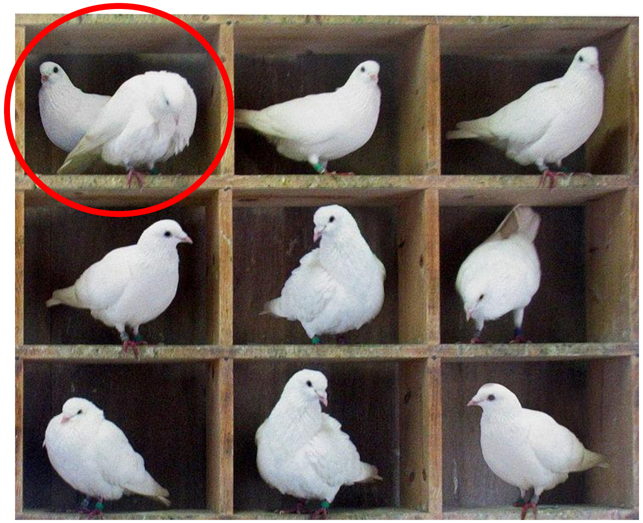
- Infinite graphs (which imply the finite cases as a corollary)

"Complete disorder is impossible."

– T. S. Motzkin (1908-1970)

Pigeon-Hole Principle

- J. Dirichlet (1834)
- "Drawer principle"
- "Shelf Principle"
- "Box principle"



Theorem (pigeon-hole): There is no injective (1-to-1) function from a finite set (domain) to a smaller finite set (range).

Generalization:

N objects placed in M containers; then:

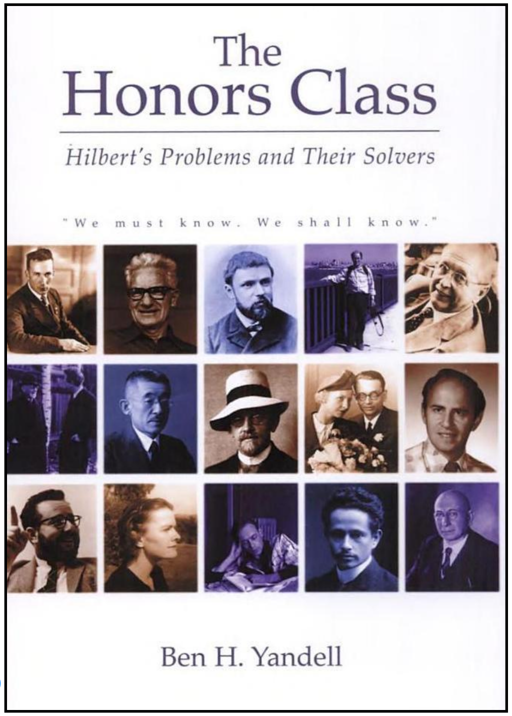
- at least 1 container must hold $\geq \left\lceil \frac{N}{M} \right\rceil$
- at least 1 container must hold $\leq \left\lfloor \frac{N}{M} \right\rfloor$

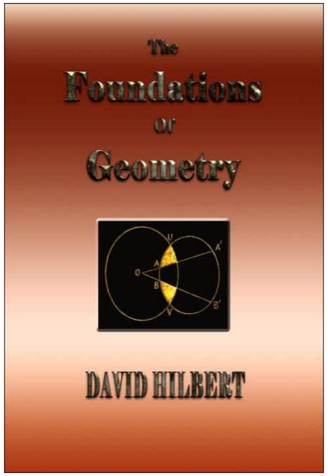
Historical Perspectives

David Hilbert (1862-1943)

- One of the most influential mathematicians
- Developed invariant theory, Hilbert spaces
- Axiomatized geometry, "Hilbert's axioms"
- Co-founded proof theory, mathematical logic, meta-mathematics, & formalist school
- Created famous list of 23 open problems that greatly impacted mathematics research
- Defended Cantor's transfinite numbers
- Contributed to relativity theory & physics
- Hilbert's students included Courant, Hecke, Lasker, Weyl, Ackermann, and Zarmelo
- Influenced Russell, Gödel, Church, & Turing

John von Neumann was Hilbert's assistant!

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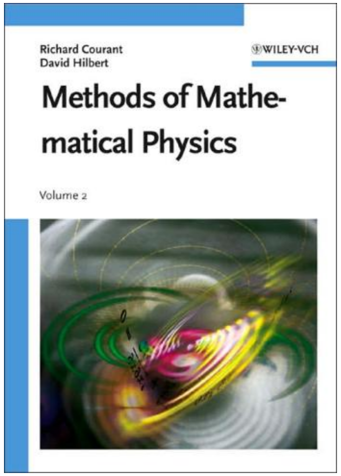
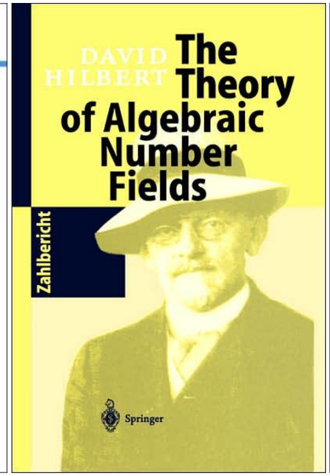
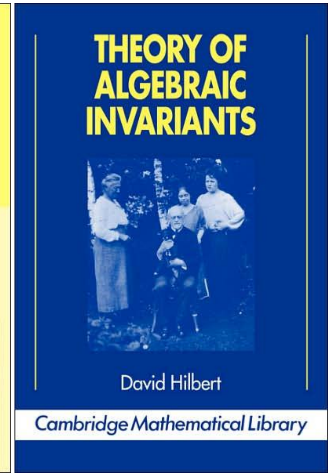
DAVID HILBERT

Richard Courant
David Hilbert
WILEY-VCH
Methods of Mathematical Physics
Volume 2

DAVID HILBERT
The Theory of Algebraic Number Fields
Zahlbericht
Springer

THEORY OF ALGEBRAIC INVARIANTS
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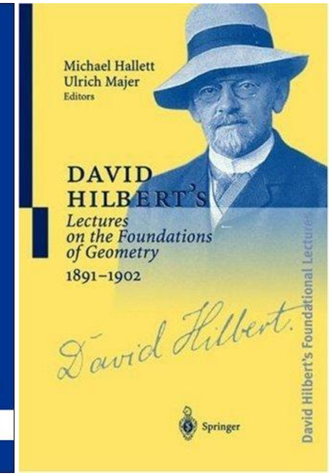
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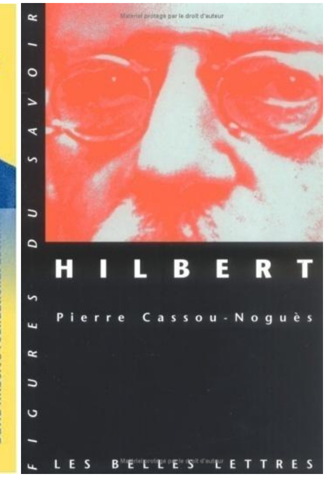
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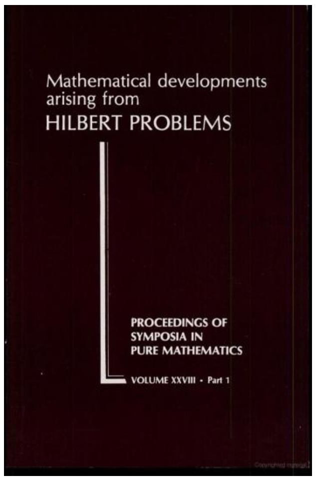
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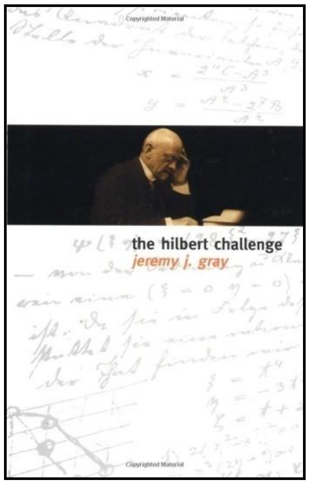
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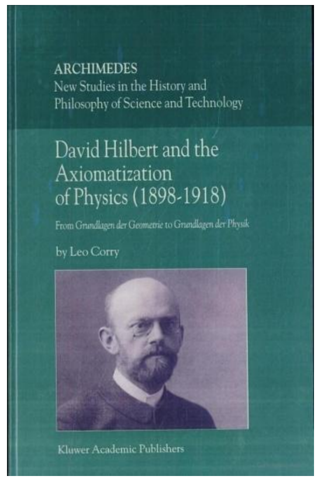
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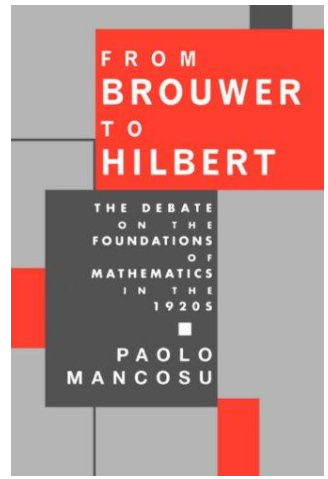
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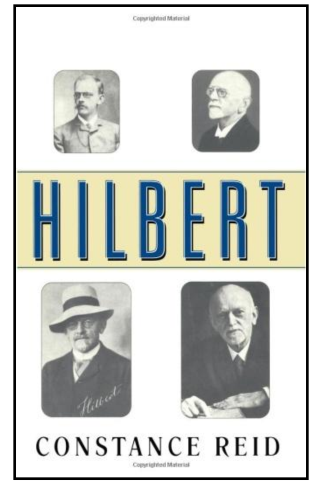
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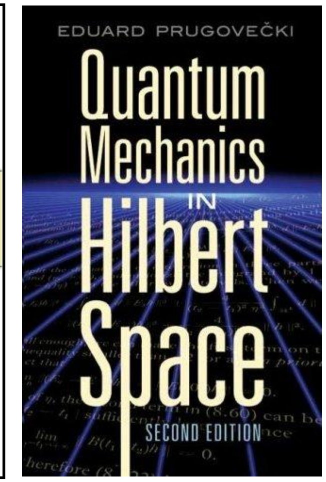
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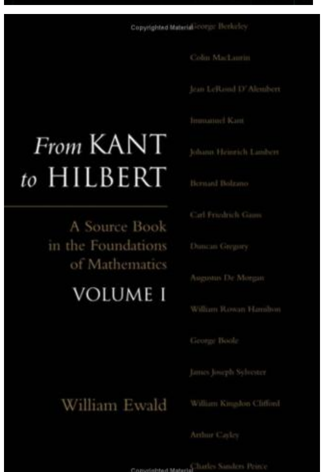
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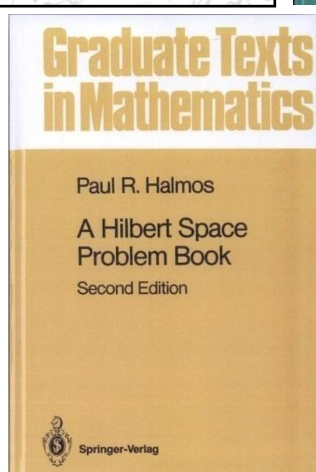
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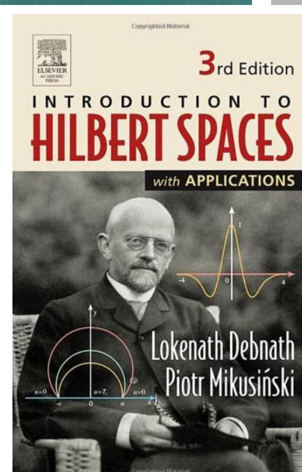
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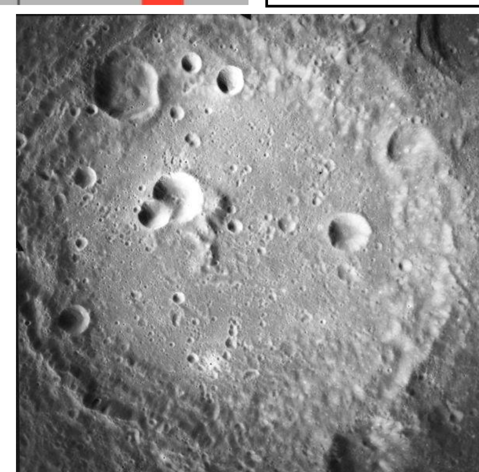
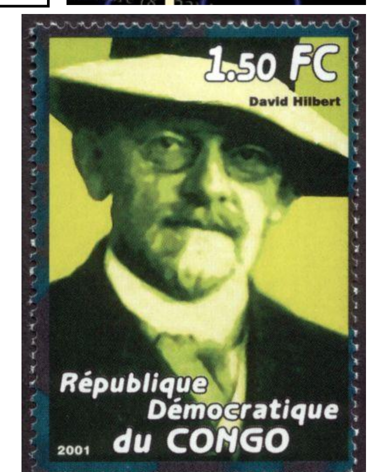
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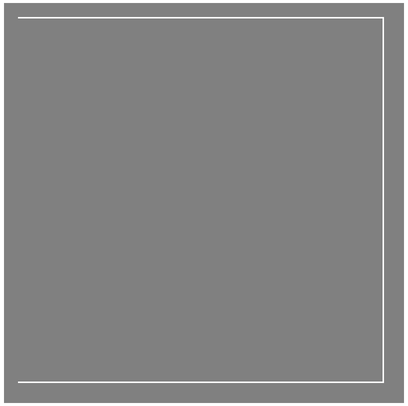
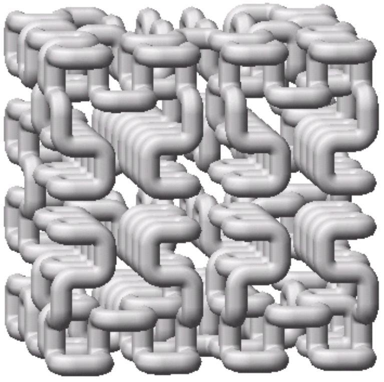
Hilbert's Impact

- Hilbert's axioms
- Hilbert class field
- Hilbert C*-module
- Hilbert cube
- Hilbert symbol
- Hilbert function
- Hilbert inequality
- Hilbert matrix
- Hilbert metric
- Hilbert number
- Hilbert polynomial
- Hilbert's problems
- Hilbert's program
- Hilbert–Poincaré series
- Hilbert space
- Hilbert spectrum

- Hilbert transform
- Hilbert's Arithmetic of Ends
- Hilbert's constants
- Hilbert's irreducibility theorem
- Hilbert's Nullstellensatz
- Hilbert's hotel paradox
- Hilbert's theorem
- Hilbert's syzygy theorem
- Hilbert-style deduction system
- Hilbert–Pólya conjecture
- Hilbert–Schmidt operator
- Hilbert–Smith conjecture
- Hilbert–Speiser theorem
- Einstein–Hilbert action
- Hilbert curve

Hilbert curve:

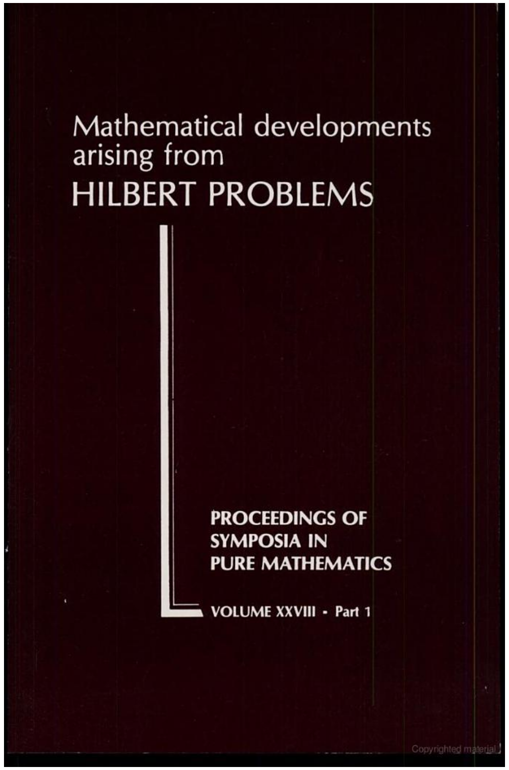


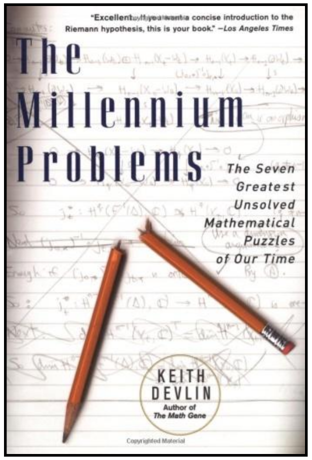


Hilbert's Problems

International Congress of Mathematics, Paris, 1900

- David Hilbert proposed 23 open problems
- Most successful open problems compilation ever
- Set the direction for 20^{th} century mathematics
- Hilbert's problems received much attention to date
- Several have been resolved (e.g., Continuum hypothesis)
- Others still open (e.g., Riemann hypothesis)
- Catalyzed other open problems lists:
  - Clay Institute's Millennium Prize problems
  - DARPA Mathematical Challenges, 2009

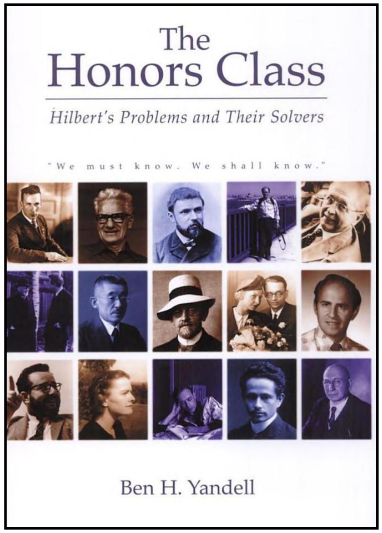




Introduction from Hilbert's Lecture

"**Who of us would not be glad to lift the veil behind which the future lies hidden**; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? **What particular goals will there be toward which the leading mathematical spirits of coming generations will strive?** What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

History teaches the continuity of the development of science. **We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones.** If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled questions pass before our minds and look over the problems which the science of today sets and whose solution we expect from the future. To such a review of problems the present day, lying at the meeting of the centuries, seems to me well adapted. For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future.



The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as every human undertaking pursues certain objects, so also mathematical research requires its problems. **It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.**

It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem. Nevertheless we can ask whether there are general criteria which mark a good mathematical problem. An old French mathematician said: **"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."** This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for **what is clear and easily comprehended attracts, the complicated repels us**.

Occam's Razor!

Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to **hidden truths**, and ultimately a reminder of our **pleasure** in the successful solution."

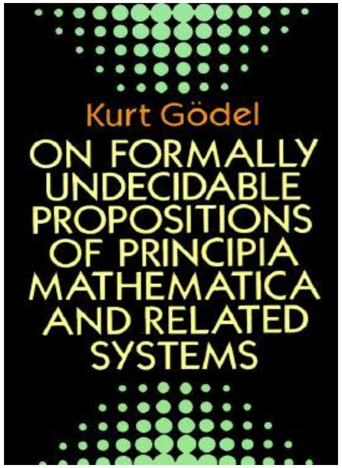
Hilbert's Problems

Problem 1: The continuum hypothesis (conjectured by Georg Cantor: there is no set whose cardinality is strictly between those of the integers and the reals)

Status: The continuum hypothesis was proven by Gödel (1939) and Cohen (1963) to be independent of (i.e., impossible to prove or disprove) Zermelo–Frankel set theory. Related open questions remain (e.g., regarding the generalized continuum hypothesis), and there is still much active research in this area.

Problem 2: Prove the axioms of arithmetic are consistent.

Status: Gödel (1931) proved that the consistency of Peano arithmetic can not be proven within Peano arithmetic itself. Gödel also proved that every consistent formal axiomatic system is incomplete. Gentzen (1936) proved the consistency Peano arithmetic within a different system (that is weaker than set theory).



Hilbert's Problems

Problem 3: Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?

Status: Proved via counter-example to be impossible by Hilbert's student Dehn (1901). The Wallace-Bolyai–Gerwien theorem (1807): in 2D this is always possible for polygons of equal areas.

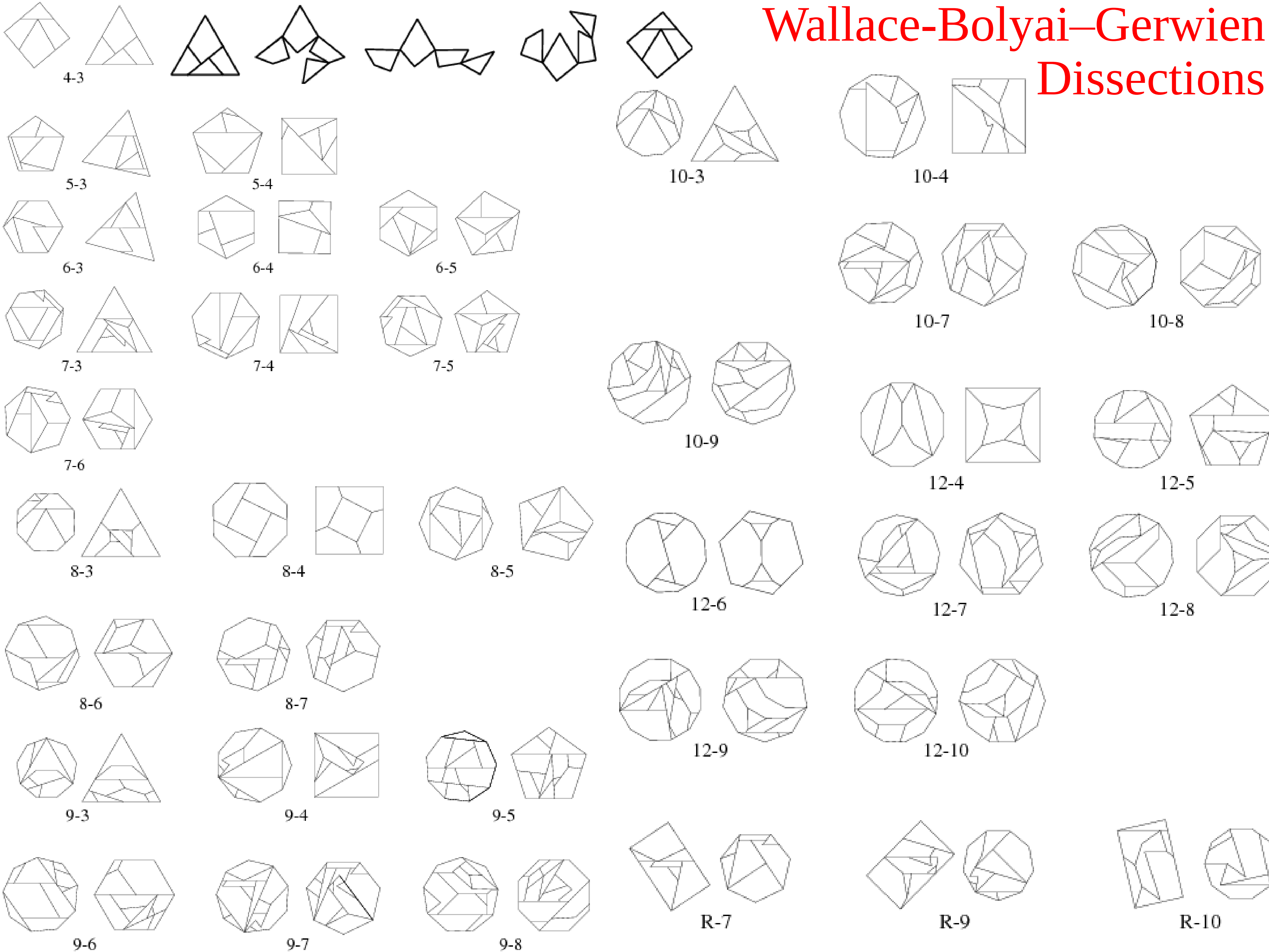
Problem 4: Construct all metrics where lines are geodesics.

Status: Too vague for a definite answer.

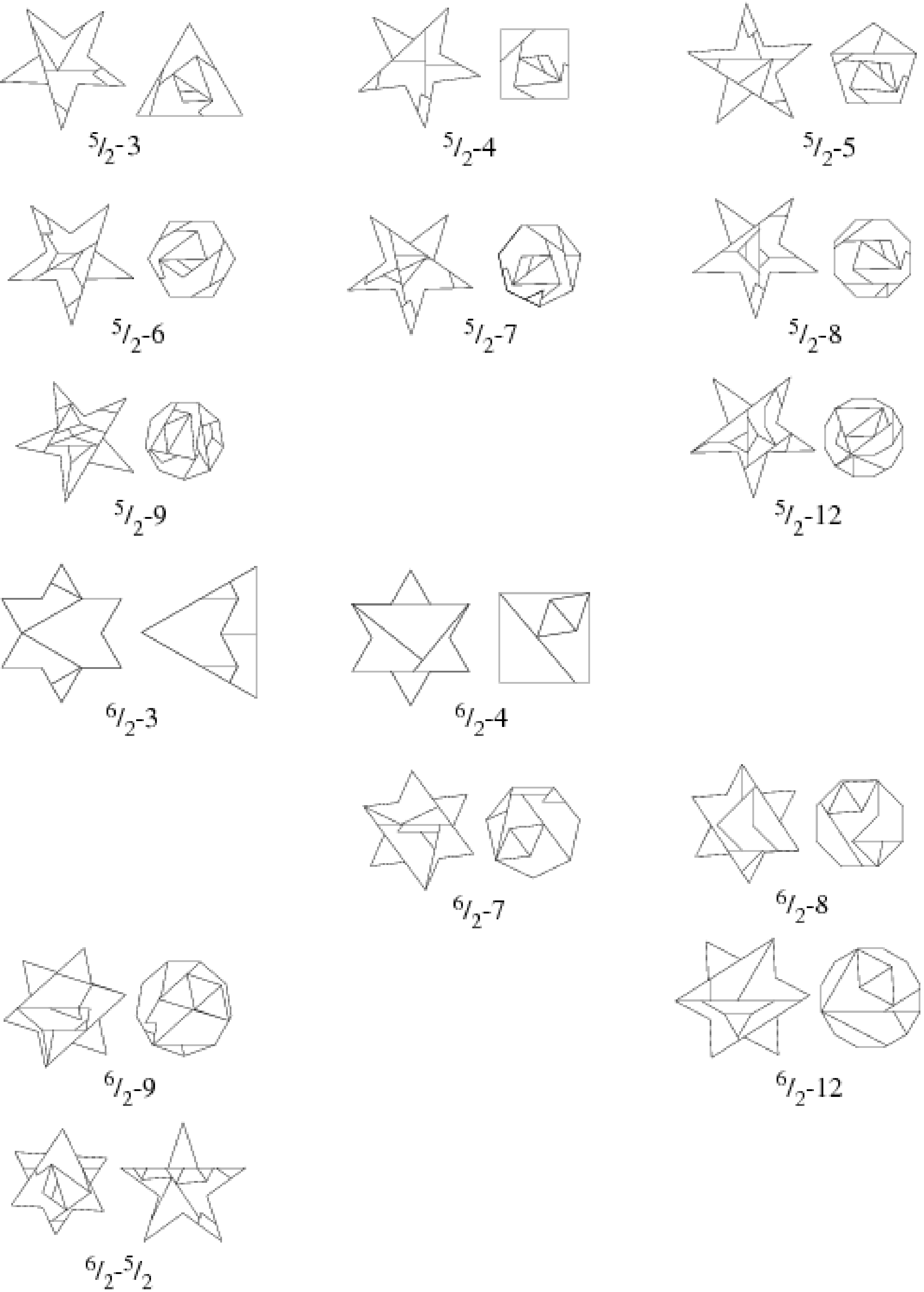
Problem 5: Are continuous groups automatically differential groups?

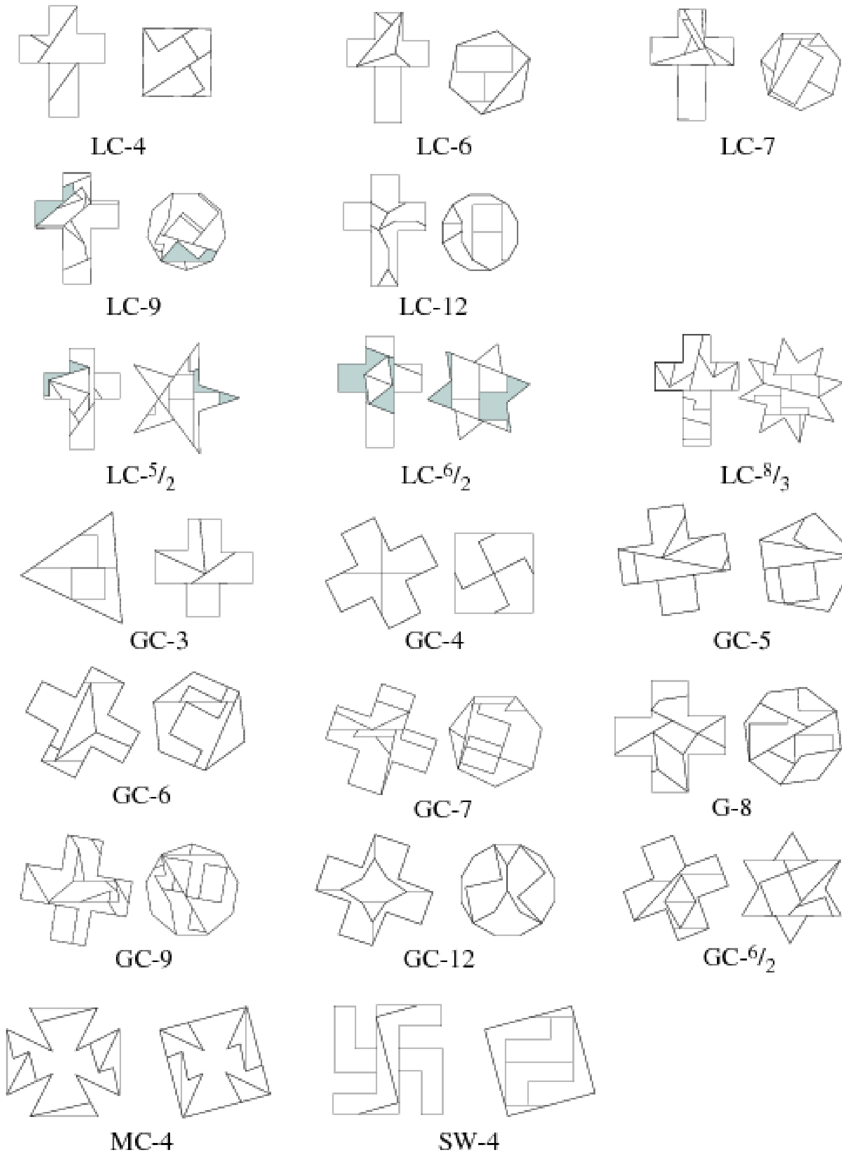
Status: Resolved in the negative by von Neumann (1929), Pontryagin (1934), Gleason-Montgomery-Zippin (1950's), and Yamabe (1953).

Wallace-Bolyai–Gerwien Dissections



Wallace-Bolyai–Gerwien Dissections

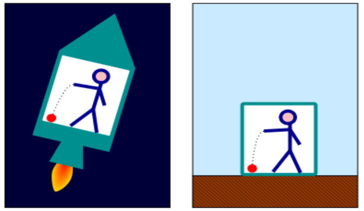


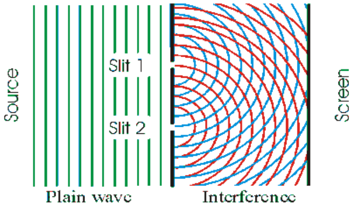


Hilbert's Problems

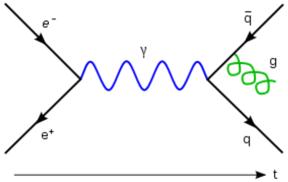
Problem 6: Axiomatize all of physics.

Status: Since Hilbert stated this problem in 1900, relativity theory was developed by Einstein (1905 and 1915), as was quantum mechanics by Dirac (1920's), followed by other more modern approaches, e.g. quantum field theory, the standard model, quantum gravity, and string theory. Hilbert himself made significant contributions to relativity and physics, but his original problem/goal of axiomatizing all of physics remains mostly open.





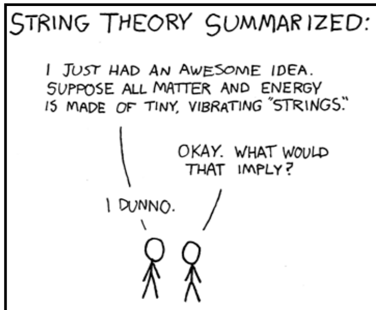
Problem 7: Is a^b transcendental, for algebraic $a \neq 0,1$ and irrational algebraic b ?

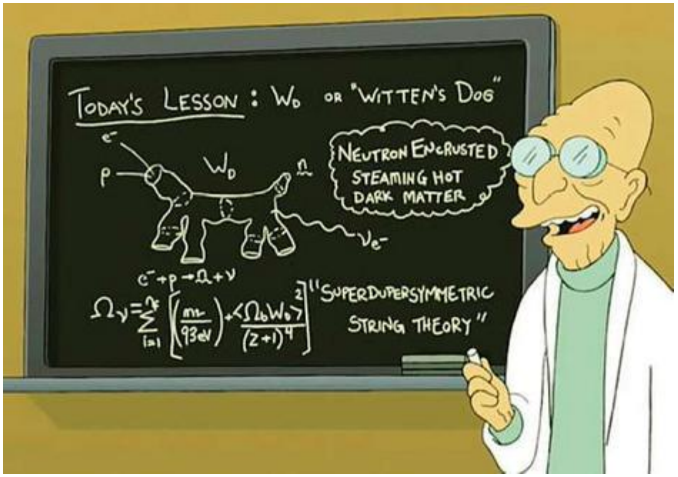


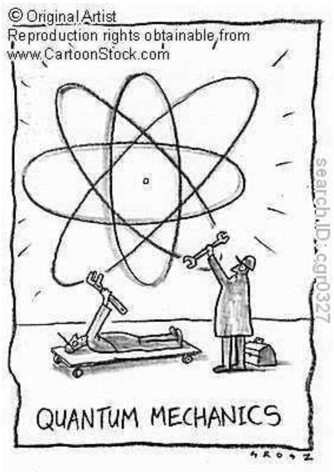
Status: Shown to be true by Gelfond and Schneider (1934), even for complex a and b. This proves that, e.g.,

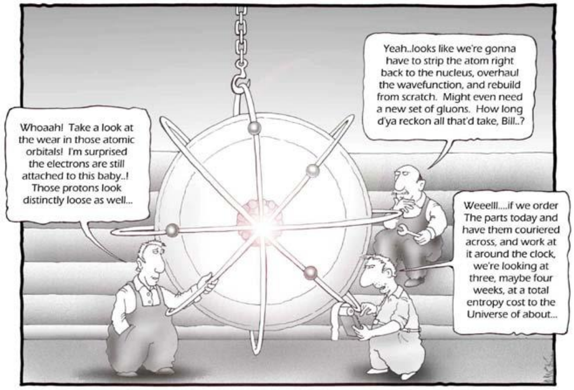
$$e^{\pi} \quad i^{i} \quad 2^{\sqrt{2}} \quad \sqrt{2}^{\sqrt{2}}$$

are all transcendental. But many similar problems remain open, such as the trancendance (or even the irrationality) of π^e, 2^e, or even $\pi + e$ and π / e.

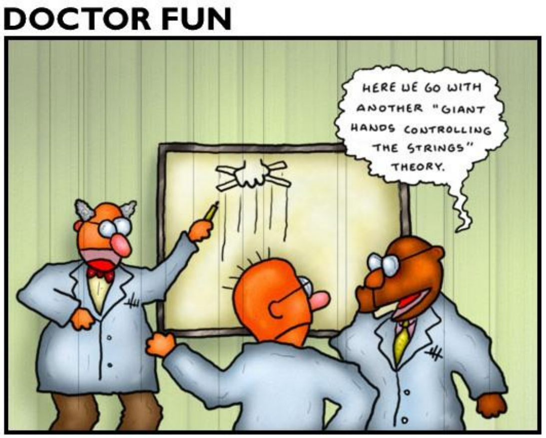




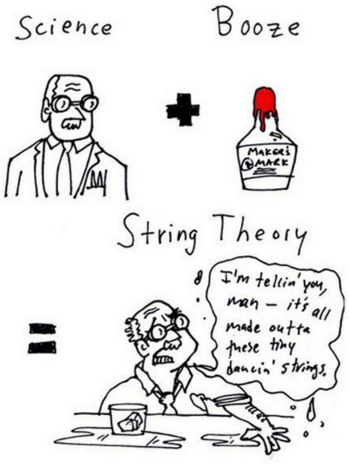


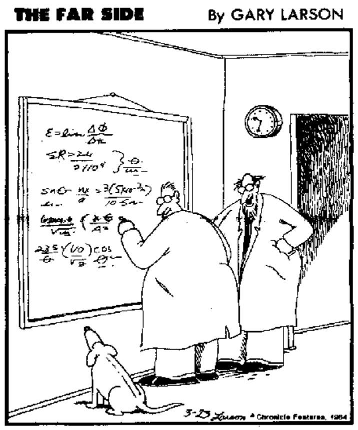


Quantum mechanics.



Puppet Theoretical Physics





"Ohhhhhhh . . . Look at that, Schuster . . . Dogs are so cute when they try to comprehend quantum mechanics."

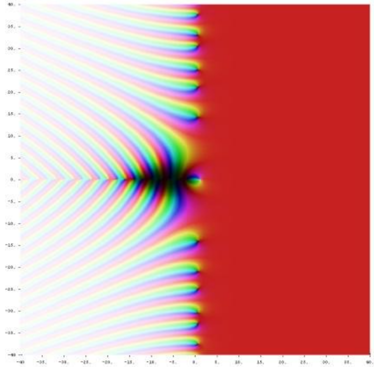
"What are you talking about? — how can you have *half* a quantum theory?"


BELIEVE
BROTHERS AND SISTERS, AT THE TIME OF 10^{-33} SECONDS AFTER THE BIG BANG, THE HEAT WAS ENORMOUS.
VERILY, IT WAS OVER 10^{32} DEGREES!
MATTER AND ANTI-MATTER AROSE!
HALLELUJAH- THEY ANNIHILATED EACH OTHER.
AND THE UNIVERSE WAS FILLED WITH PARTICLES...
AMEN! QUARKS
AND GLUONS.
YEA, LEPTONS
S. HARRIS

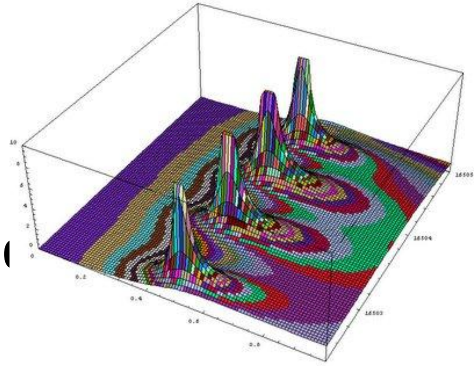
Hilbert's Problems

Problem 8: The Riemann hypothesis (the real part of any non-trivial zero of the Riemann zeta function is ½) and Goldbach's conjecture (every even number > 2 can be written as the sum of two primes).

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$



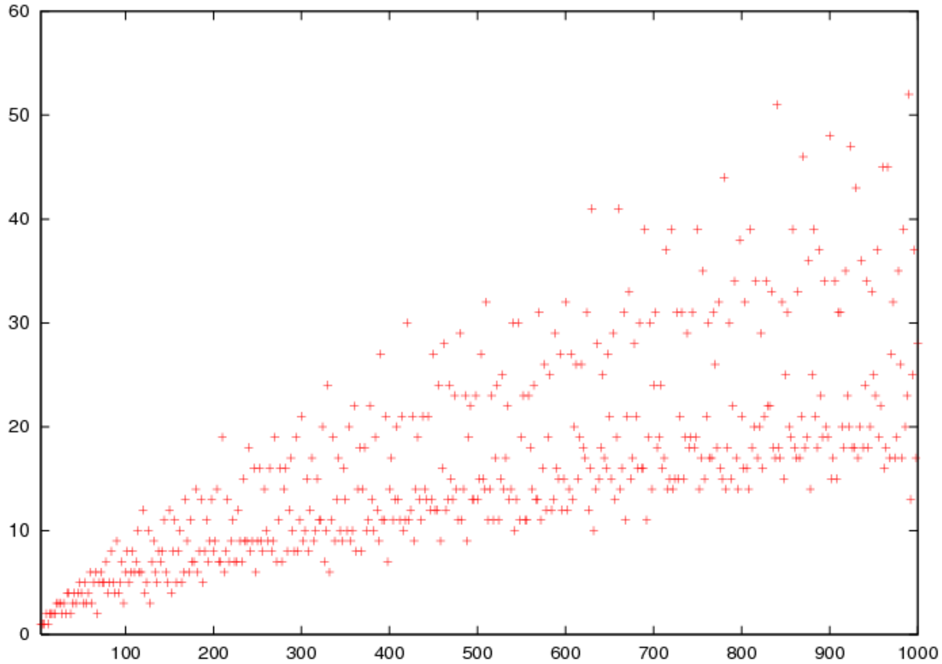
Status: Both the Reimann hypothesis (1859) and Goldbach's conjecture (1742) remain open to this day. The Reimann hypothesis has many far-reaching implications in mathematics, logic, and computer science. It was numerically verified for the first ten trillion zeroes, and appears on the Millennium Prize list ($1M bounty) as well as the ARPA Mathematical Challenges List. The Goldbach conjecture was verified for the first 10^{18} values.

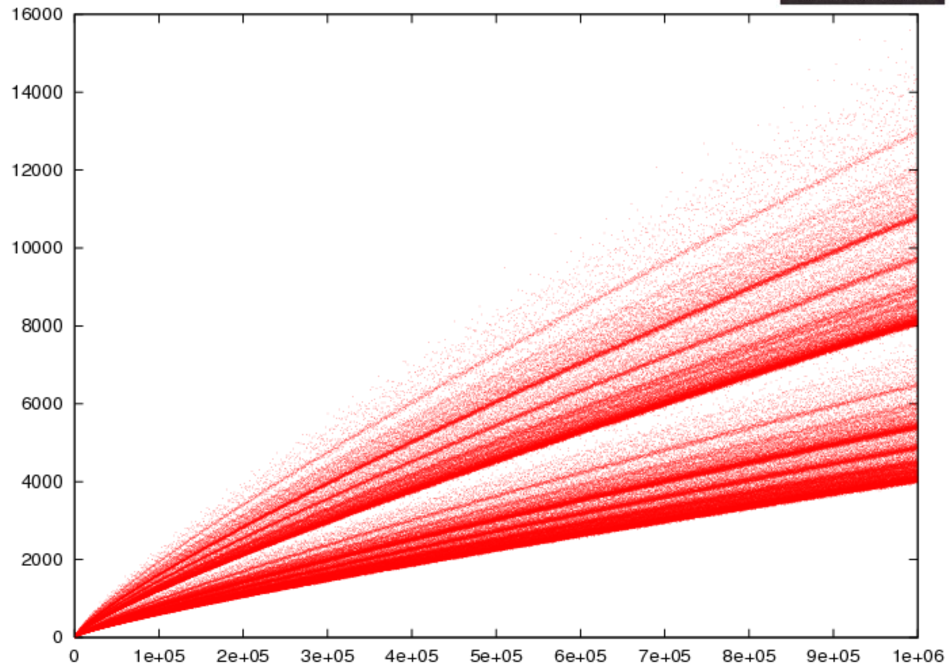


Problem 9: Find most general law of the reciprocity theorem in any algebraic number field.

Status: Partially solved by Artin (1924), Takagi & Hass and Shafarevich (1948); still some open issues.

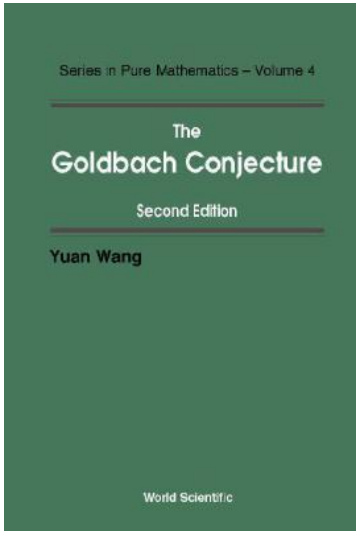
Evidence for Goldbach’s conjecture: the number of distinct ways to write an even number as the sum of two primes (computational data for 4 < n < 1,000,000):

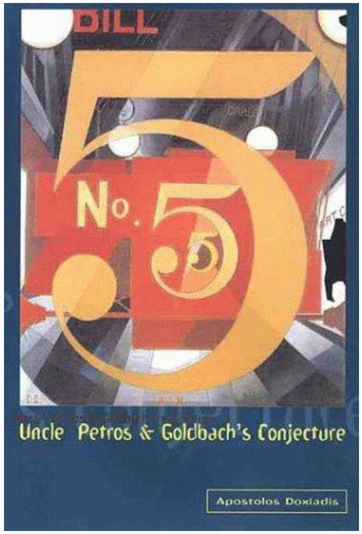




Theorem (Jingrun, 1973): Every sufficiently large even number can be written as either the sum of two primes, or the sum of a prime and a product of two primes.

Theorem (Ramaré, 1995): Every even number >2 is the sum of at most six primes.



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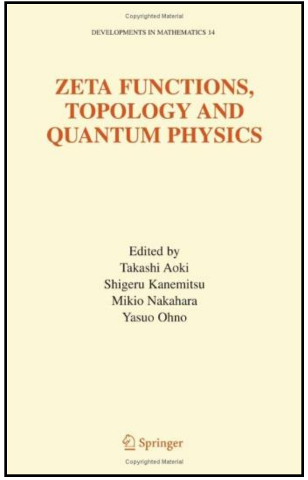
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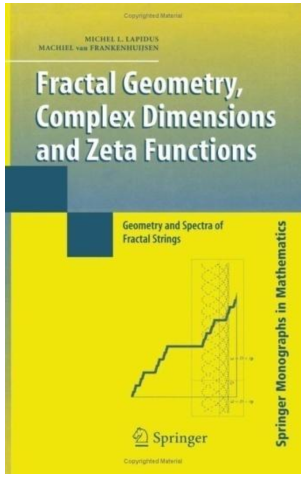
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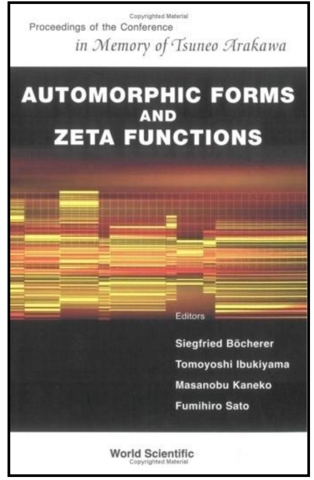
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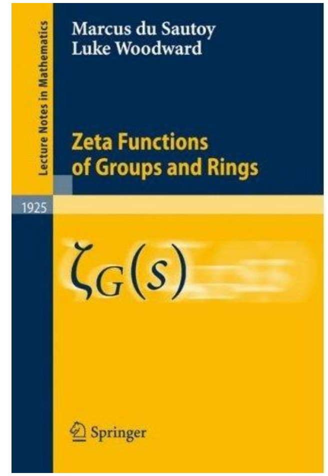
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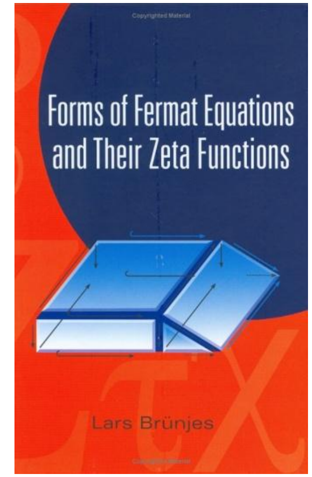
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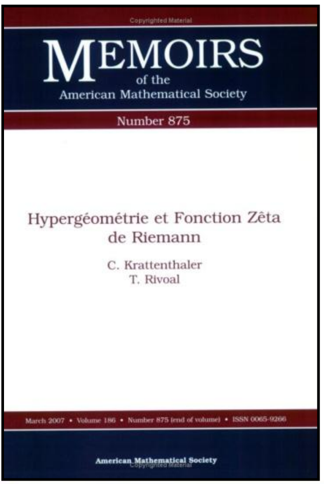
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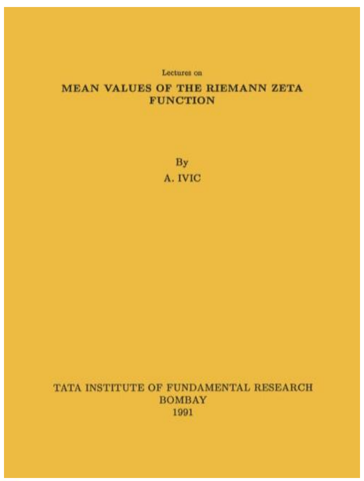
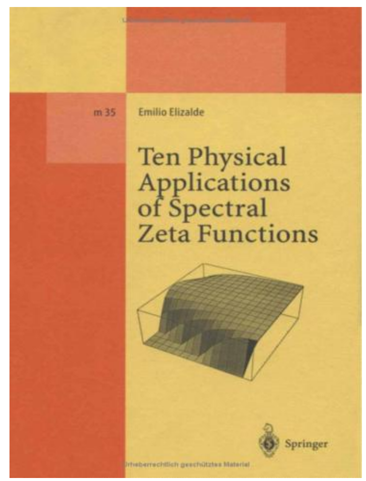
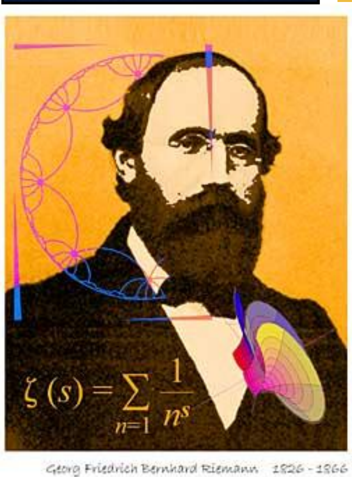
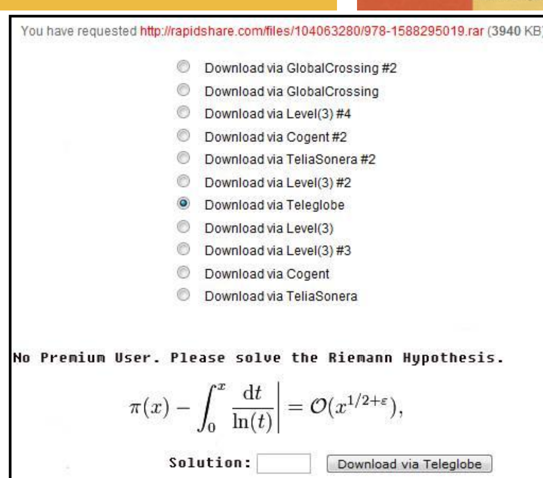

Lectures on
MEAN VALUES OF THE RIEMANN ZETA FUNCTION
By
A. IVIC
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY
1991


m 35
Emilio Elizalde
Ten Physical Applications of Spectral Zeta Functions
Springer


RIEMANN HYPOTHESIS IS PROBABLY TRUE
NOW STOP WORRYING AND GET A GIRLFRIEND


$\zeta(s) = \sum_{n=1} \frac{1}{n^s}$
Georg Friedrich Bernhard Riemann 1826 - 1866


You have requested http://rapidshare.com/files/104063280/978-1588295019.rar (3940 KB)
Download via GlobalCrossing #2
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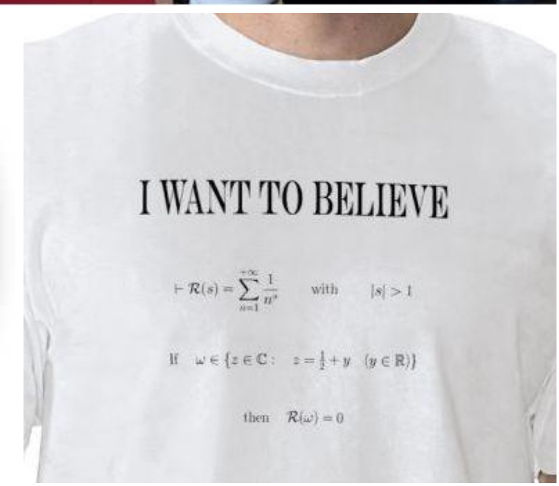
No Premium User. Please solve the Riemann Hypothesis.
$\pi(x) - \int_0^x \frac{dt}{\ln(t)} = \mathcal{O}(x^{1/2+\varepsilon}),$
Solution:
Download via Teleglobe


I HAVE A PRIME OBSESSION


I WANT TO BELIEVE

Clay Mathematics Institute

Dedicated to increasing and disseminating mathematical knowledge

Millennium Problems

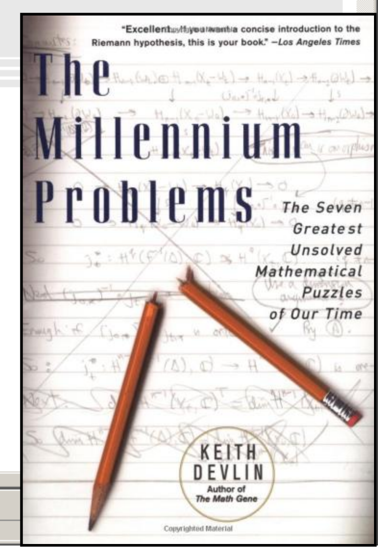
In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

One hundred years earlier, on August 8, 1900, David Hilbert delivered his famous lecture about open mathematical problems at the second International Congress of Mathematicians in Paris. This influenced our decision to announce the millennium problems as the central theme of a Paris meeting.

The rules for the award of the prize have the endorsement of the CMI Scientific Advisory Board and the approval of the Directors. The members of these boards have the responsibility to preserve the nature, the integrity, and the spirit of this prize.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos



Hilbert's Problems

Problem 10: Find an algorithm that determines whether a given Diophantine (i.e., multi-variable polynomial) equation has any integer solutions.

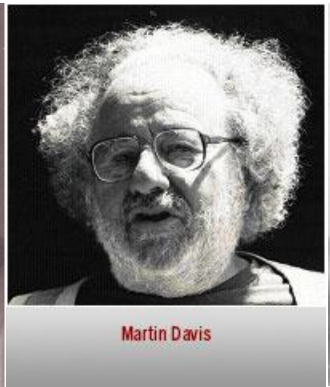
Ex: $x^2+y^2=z^2$ has many integer solutions
(Pythagorean theorem, e.g., $x=3$, $y=4$, $z=5$)

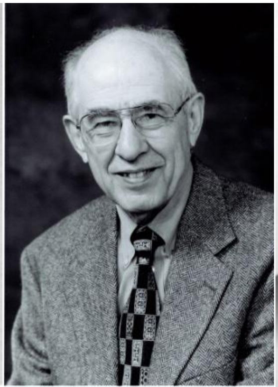
$x^9+y^9=z^9$ has no integer solutions (corollary of Fermat's Last Theorem, conjectured in 1637, proved in 1995 by Andrew Wiles)

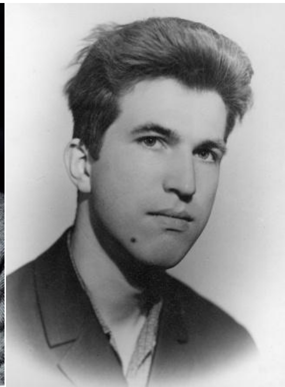
Many attempts at solution & partial results: Emil Post (1944), Martin Davis (1949), Julia Robinson (1950), Hilary Putnam (1959)

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobaea, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.







Hilbert's Tenth Problem

Solving even simple Diophantine equations is hard:

Q: ∃ an integer solution for $x^3 + y^3 + z^3 = 29$?
A: Yes: $x=3$, $y=1$, $z=1$

Q: ∃ an integer solution for $x^3 + y^3 + z^3 = 30$?
A: Yes: $x = 2220422932$, $y = -2218888517$, $z = -283059965$

Q: ∃ an integer solution for $x^3 + y^3 + z^3 = 33$?
A: still unknown!

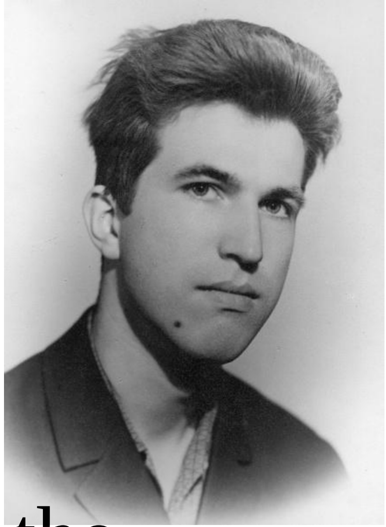
Q: Is $\{x^3 + y^3 + z^3 \mid x, y, z \in \mathbb{Z}\} = \mathbb{Z}$?
A: still unknown!

Q: Is $\{x^3 + y^3 + z^3 \mid x, y, z \in \mathbb{Z}\}$ Turing-decidable?
A: still unknown!

Theorem [Lagrange]: $\{w^2 + x^2 + y^2 + z^2 \mid w, x, y, z \in \mathbb{Z}\} = \mathbb{Z}$

http://www.asahi-net.or.jp/~KC2H-MSM/mathland/math04/matb0100.htm

Hilbert's Tenth Problem



Theorem [Matiyasevich, 1970]: Every Turing-enumerable set is Diophantine

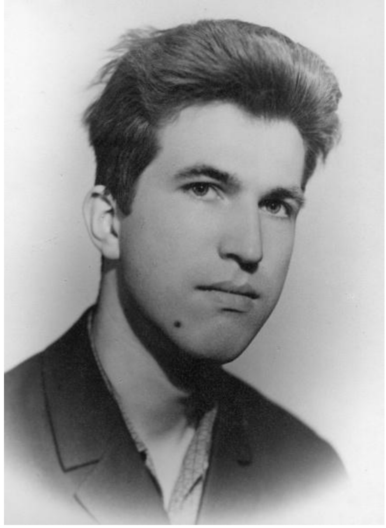
(i.e., the solutions of some polynomial)

Ex: the set of primes coincides exactly with the positive values of this 26-variable polynomial:

$$\begin{aligned}
&(k+2)(1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2\\
&-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[2n+p+q+z-e]^2\\
&-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2\\
&-[16r^2y^4(a^2-1)+1-u^2]^2-[n+l+v-y]^2-[(a^2-1)l^2+1-m^2]^2\\
&-[ai+k+1-l-i]^2-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1\\
&-(x+cu)^2]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2\\
&-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2\\
&-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2)
\end{aligned}$$

as a, b, c, … , z range over the nonnegative integers!

Hilbert's Tenth Problem

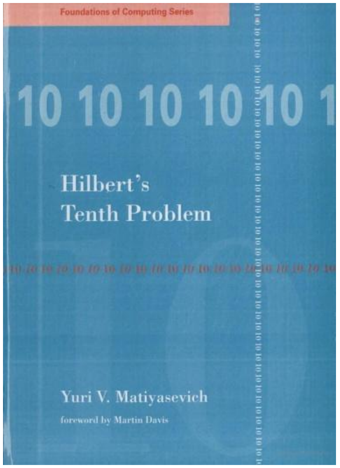


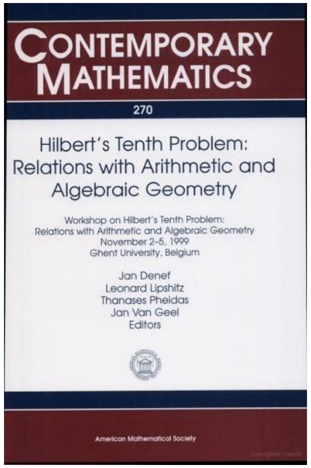
Corollary [Matiyasevich, 1970]: There is a fixed "universal" polynomial P such that for any Turing-enumerable set S there exists an integer n_0 such that

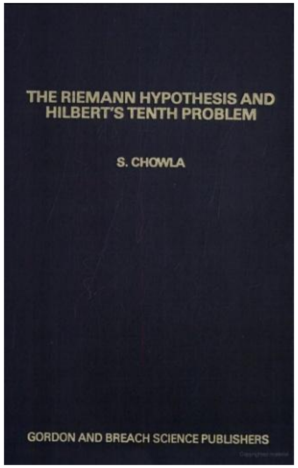
$$S = \{w \mid \exists\, x_1, x_2, \ldots, x_k \ni P(n_0, w, x_1, x_2, \ldots, x_k) = 0\}$$

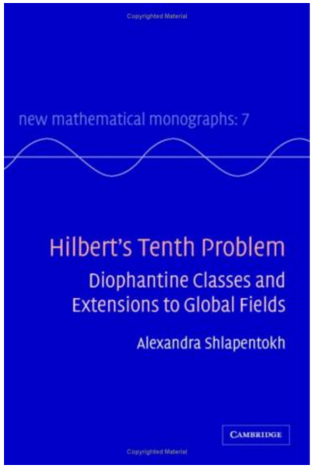
i.e., there is a fixed polynomial that can "output" any computable set, depending on one parameter.

This is an analogue of a universal Turing machine!









Hilbert's Tenth Problem

Q: What is the minimum Diophantine degree and dimension (i.e., number of variables) of a given Turing-enumerable set?

Theorem [Skolem]: degree 4 suffices.

Theorem [Matiyasevich]: dimension 9 suffices.

But there is a dramatic tradeoff between the degree and the number of variables.

$$
\begin{aligned}
&(k+2)(1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2\\
&-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[2n+p+q+z-e]^2\\
&-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2\\
&-[16r^2y^4(a^2-1)+1-u^2]^2-[n+l+v-y]^2-[(a^2-1)l^2+1-m^2]^2\\
&-[ai+k+1-l-i]^2-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1\\
&-(x+cu)^2]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2\\
&-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2\\
&-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2)
\end{aligned}
$$

This is analogous to finding small universal TMs (where there is a tradeoff between the alphabet size and the number of states).

From "Undecidable Diophantine Equations" by James P. Jones, Bulletin of the American Mathematical Society, vol 2, No 3, 1980, pp. 859-862.

Tradeoff between degree and the number of variables in universal polynomials:

Examples:

58 variables & degree 4 suffice
28 variables & degree 20 suffice
19 variables & degree 2668 suffice
14 variables & degree ~10^5 suffice
13 variables & degree ~10^{43} suffice
9 variables & degree ~10^{45} suffice

Corollary: 100 additions and/or multiplications suffice to "prove" any provable proposition.

Catch: using <u>very large</u> integers!

function appear. Next these are eliminated so that we obtain a system of purely polynomial equations.

THEOREM 3. *In order that $x \in W_{\langle z,u,y\rangle}$, it is necessary and sufficient that the following system of equations has a solution in positive integers.*

$$elg^2 + \alpha = (b - xy)q^2, \quad q = b^{5^{60}}, \quad \lambda + q^4 = 1 + \lambda b^5,$$

$$\theta + 2z = b^5, \quad l = u + t\theta, \quad e = y + m\theta, \quad n = q^{16},$$

$$r = [g + eq^3 + lq^5 + (2(e - z\lambda)(1 + xb^5 + g)^4 + \lambda b^5 + \lambda b^5 q^4)q^4][n^2 - n] + [q^3 - bl + l + \theta\lambda q^3 + (b^5 - 2)q^5][n^2 - 1],$$

$$p = 2ws^2r^2n^2, \quad p^2k^2 - k^2 + 1 = \tau^2, \quad 4(c - ksn^2)^2 + \eta = k^2,$$

$$k = r + 1 + hp - h, \quad a = (wn^2 + 1)rsn^2,$$

$$c = 2r + 1 + \varphi, \quad d = bw + ca - 2c + 4\alpha\gamma - 5\gamma, \quad d^2 = (a^2 - 1)c^2 + 1,$$

$$f^2 = (a^2 - 1)i^2c^4 + 1, \quad (d + of)^2 = ((a + f^2(d^2 - a))^2 - 1)(2r + 1 + jc)^2 + 1.$$

The equations of Theorem 3 have twenty eight unknowns, $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, w, \alpha, \gamma, \eta, \theta, \lambda, \tau, \varphi$. The degree is 5^{60}, however the equation $q = b^{5^{60}}$ can be replaced by certain others of low degree. In fact, by introducing some 30 additional unknowns and new equations one can reduce the degree of the system to 2. Then, by transposing terms to one side and summing squares one can construct a universal diophantine equation in 58 unknowns and degree 4.

Alternatively one may try instead to reduce the total number of unknowns, ν. In [6] Julia Robinson and Ju. Matijasevič showed that ν can be reduced universally to 13. More recently Matijasevič [5] has improved this to $\nu = 9$. The corresponding value of the degree, δ is however very large. The following table gives various simultaneous possibilities for δ and ν, sufficient for a universal equation.

THEOREM 4. *The following pairs are universal.*

$\nu = 58$,	$\delta = 4$	$\nu = 21$,	$\delta = 96$
$\nu = 38$,	$\delta = 8$	$\nu = 19$,	$\delta = 2668$
$\nu = 32$,	$\delta = 12$	$\nu = 14$,	$\delta = 2.0 \times 10^5$
$\nu = 29$,	$\delta = 16$	$\nu = 13$,	$\delta = 6.6 \times 10^{43}$
$\nu = 28$,	$\delta = 20$	$\nu = 12$,	$\delta = 1.3 \times 10^{44}$
$\nu = 26$,	$\delta = 24$	$\nu = 11$,	$\delta = 4.6 \times 10^{44}$
$\nu = 25$,	$\delta = 28$	$\nu = 10$,	$\delta = 8.6 \times 10^{44}$
$\nu = 24$,	$\delta = 36$	$\nu = 9$,	$\delta = 1.6 \times 10^{45}$

Hilbert's Tenth Problem

Q: Find an algorithm that determines whether a given Diophantine (i.e., multi-variable polynomial) equation has any ~~integer~~ rational solutions.

A: Still open!

CLAY MATHEMATICS INSTITUTE

March 15–16, 2007

One Bow Street, Cambridge, Massachusetts

Conference on Hilbert's Tenth Problem

Thursday, March 15

9:00	Coffee
9:15 - 9:25	Constance Reid, *Genesis of the Hilbert Problems*
9:25 - 10:00	George Csicsery, *Film clip on life and work of Julia Robinson, discussion*
10:15 - 11:15	Bjorn Poonen, *Why number theory is hard*
11:30 - 12:30	Yuri Matiyasevich, *My collaboration with Julia Robinson*
	Break for lunch
2:30-3:30	Martin Davis, *My collaboration with Hilary Putnam*
3:45-4:45	Maxim Vsemirnov, *TBA*
7:30	**Museum of Science • Film Screening** Scenes from *Julia Robinson and Hilbert's Tenth Problem*, a documentary by George Csicsery, will be screened in Cahner's Theater (Blue Wing, Level 2, Museum of Science), and followed by a panel discussion with filmmaker George Csicsery, mathematician Yuri Matiyasevich, and biographer Constance Reid. This event is free and open to the public.

Friday, March 16

8:30	Coffee
9:00–10:00	Yuri Matiyasevich, *Hilbert's Tenth Problem: What was done and what is to be done*
10:15–11:15	Bjorn Poonen, *Thoughts about the analogue for rational numbers*
11:30–12:30	Alexandra Shlapentokh, *Diophantine generation, horizontal and vertical problems, and the weak vertical method*
	Break for lunch
2:00–3:00	Yuri Matiyasevich, *Computation paradigms in the light of Hilbert's tenth problem*
3:15–4:15	Gunther Cornelisson, *Hard number-theoretical problems and elliptic curves*
4:30–5:30	Kirsten Eisentrager, *Hilbert's Tenth Problem for algebraic function fields*

Hilbert's 10th Problem (1900): is there an algorithm for deciding whether a polynomial equation with integer coefficients has an integer solution?

$$x^2 - (a^2 - 1)y^2 = 1$$

Photo credits (top to bottom): Julia Robinson, courtesy of Constance Reid; Yuri Matiyasevich, photo by George Csicsery; David Hilbert, courtesy AK Peters, Ltd.

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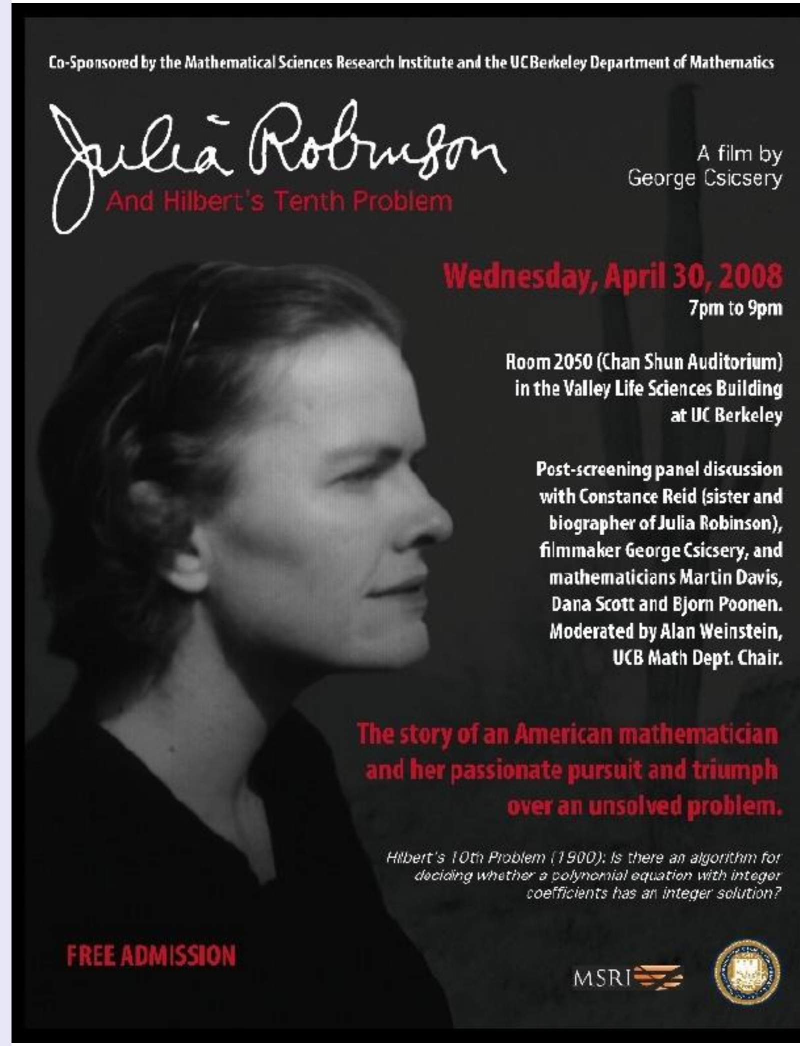
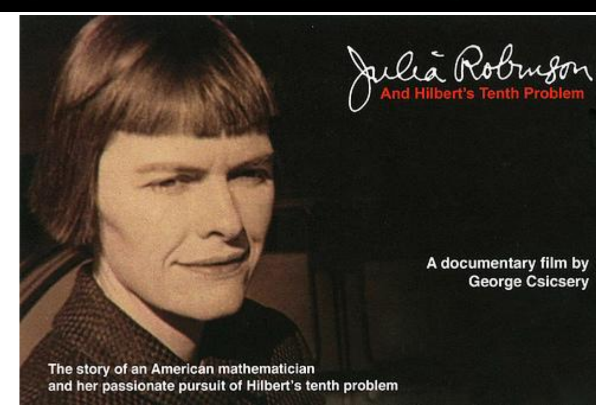
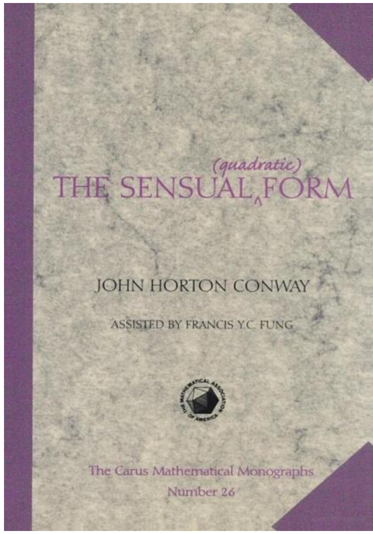




Hilbert's Problems

Problem 11: Solving quadratic forms with algebraic numerical coefficients.

Status: Partially solved by Hasse (1923).

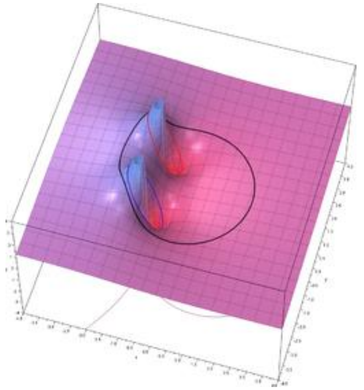


Problem 12: Extend the Kronecker–Weber theorem on abelian extensions of the rational numbers to any base number field.

Status: Still unsolved.

Problem 13: Solve all 7-th degree equations using functions of two parameters.

Status: Partially solved by Kolmogorov (1956), Arnold (1957), and Shimura (1976).



Problem 14: Proof of the finiteness of certain complete systems of functions.

Status: Counter-examples found by Nagata (1959).

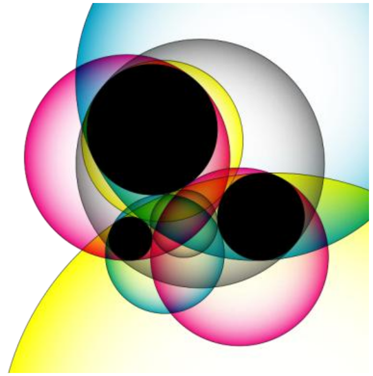
Hilbert’s Problems

Problem 15: Find a rigorous foundation for Schubert's enumerative calculus.

Status: Partially resolved.

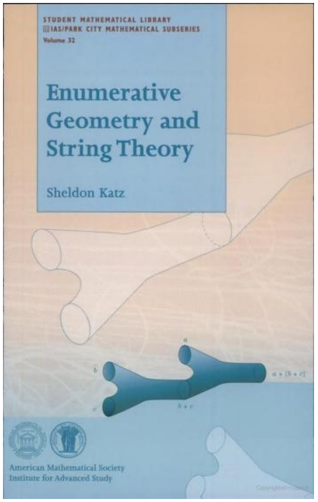
Problem 16: Topology of algebraic curves and surfaces.

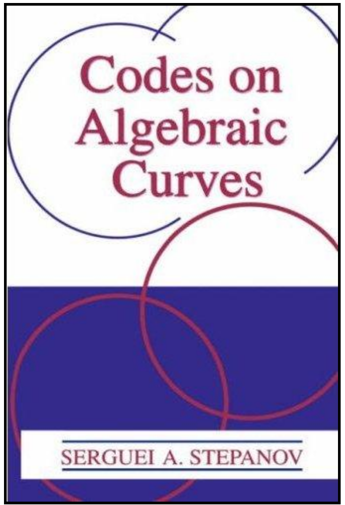
Status: Open-ended: some results, but unresolved.

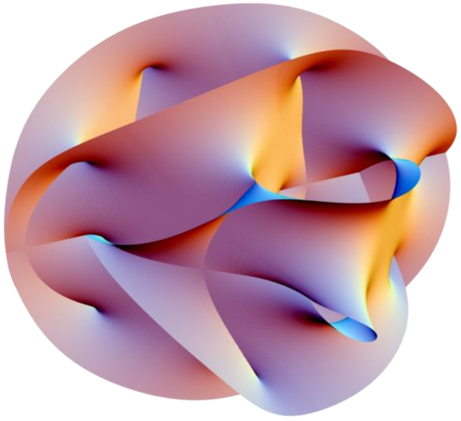


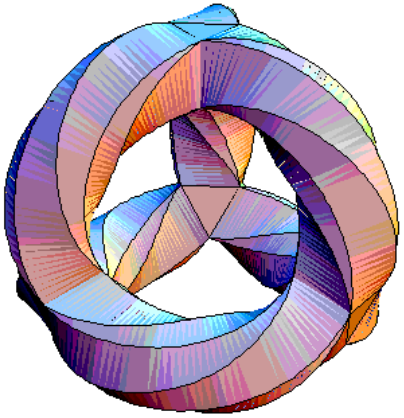
Problem 17: Expression of definite rational function as quotient of sums of squares

Status: Resolved in the affirmative by Artin (1927) and Delzel (1984).





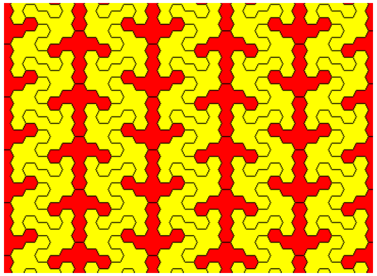




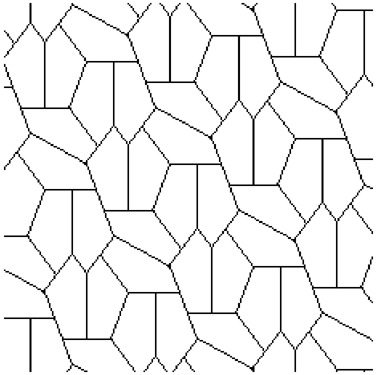
Hilbert's Problems

Problem 18: Is there a non-regular, space-filling polyhedron? What is the densest sphere packing?

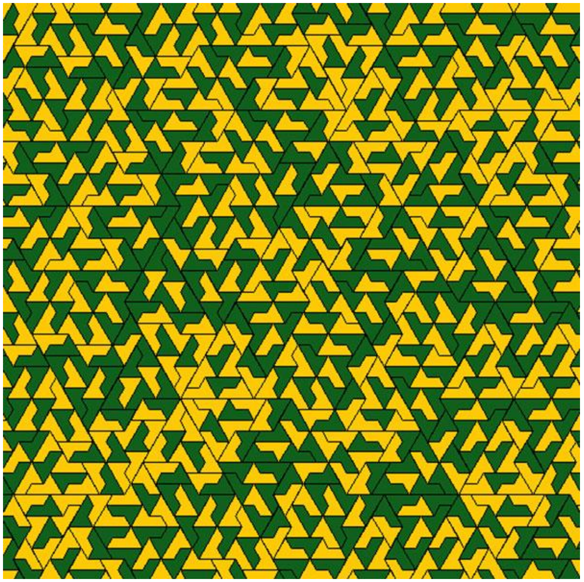
Status: Anisohedral tilings were found in 3D by Reinhardt (1928), and for 2D by Heesch (1935).

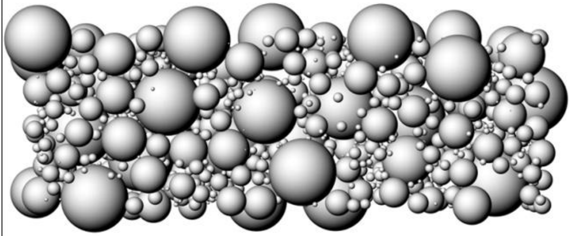


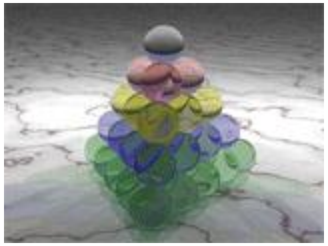
Sphere packing in 3D (Kepler's problem, 1611) was solved by Toth (1953) and Hale (1998). Regular sphere packing in 24 dimensions was solved by Cohn and Kumar (2004), where the "kissing number" is 196,560.

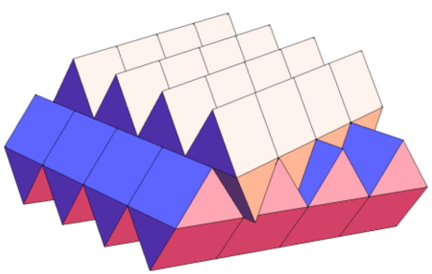


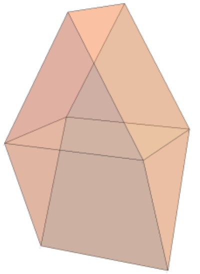
Many related open problems remain, including non-regular, non-uniform, and ellipsoid packings.









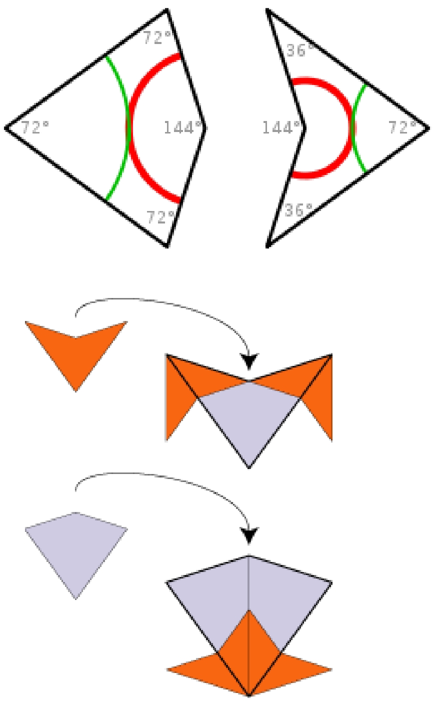


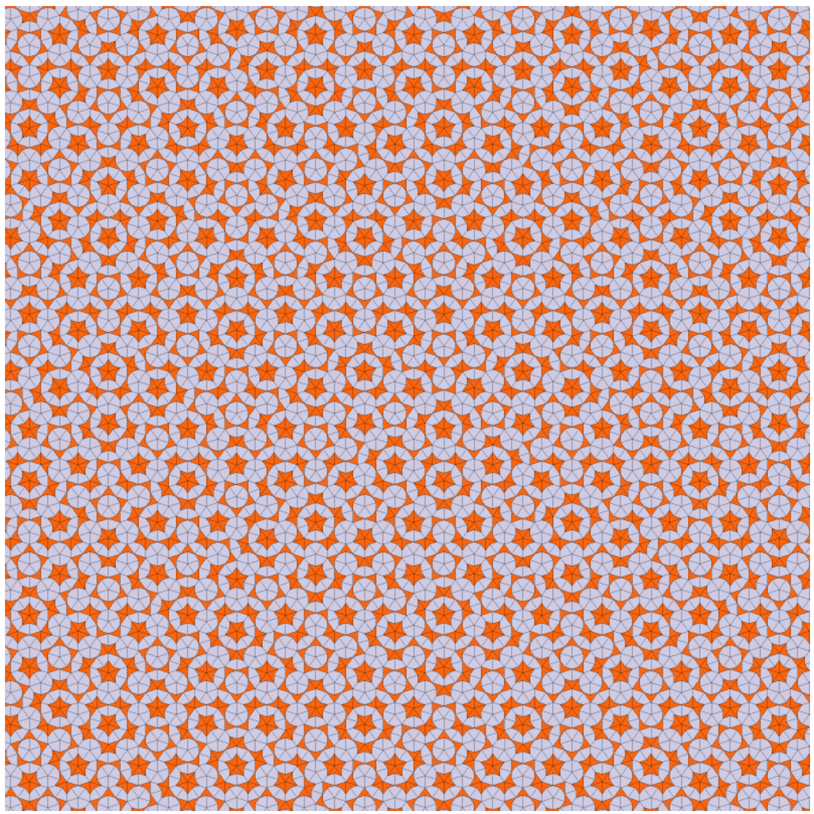
Aperiodic Tilings

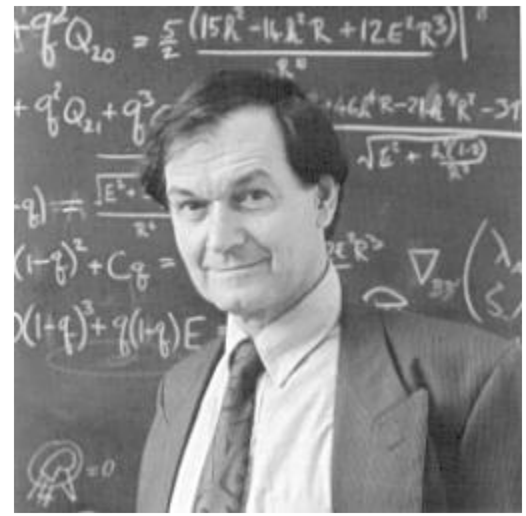
Goal: tile the entire plane without overlaps, non-periodically

- Non-periodic tiling is not equal to a translation of itself
- Aperiodic tile set admits only non-periodic tilings

“Kites and Darts” 2-tile aperiodic set, Roger Penrose, 1974





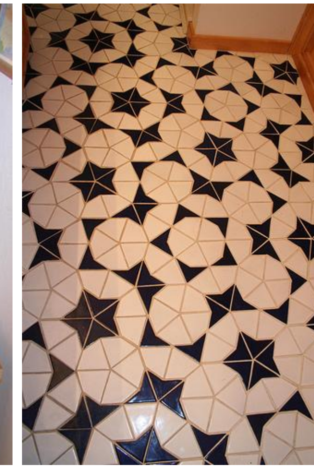


Open question:

$\exists$ a single-tile 2D aperiodic tiling?

Aperiodic Tilings

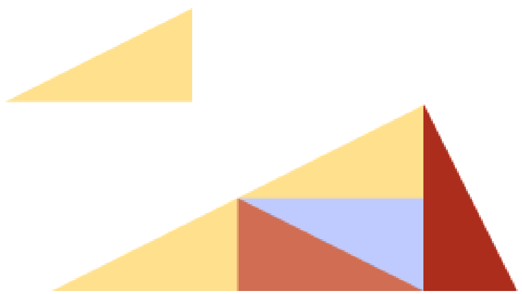
Penrose tilings in architecture and design:

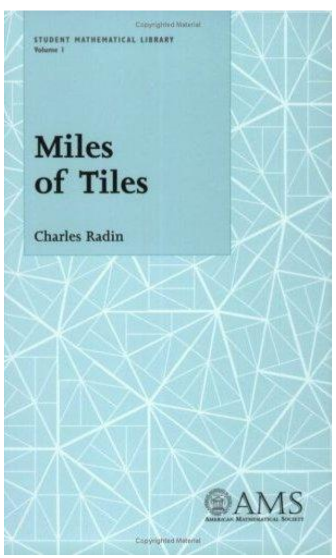


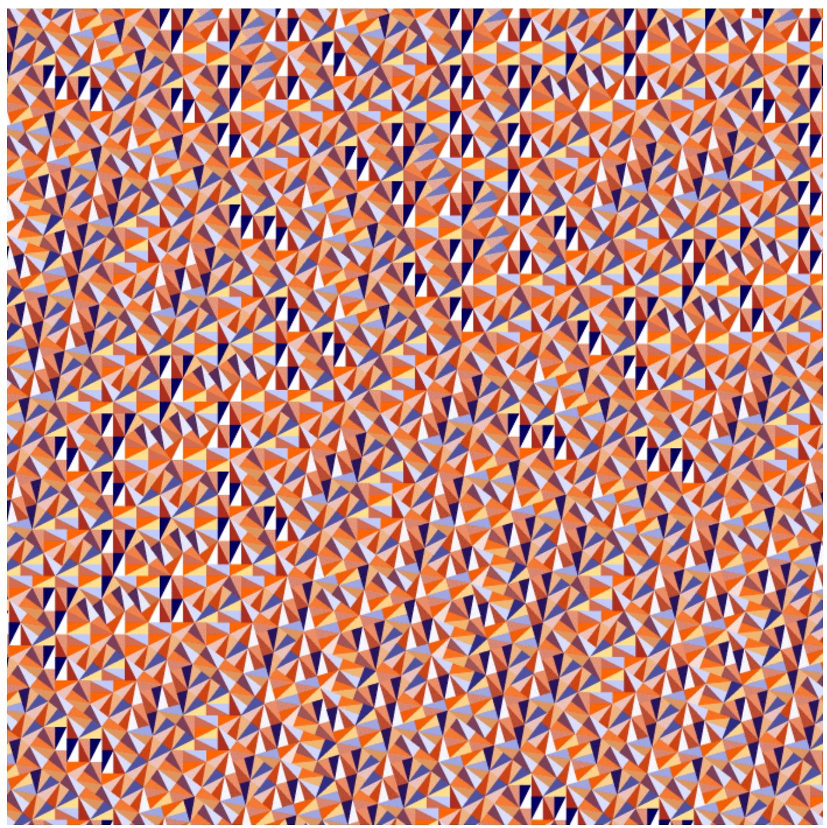
Aperiodic Tilings

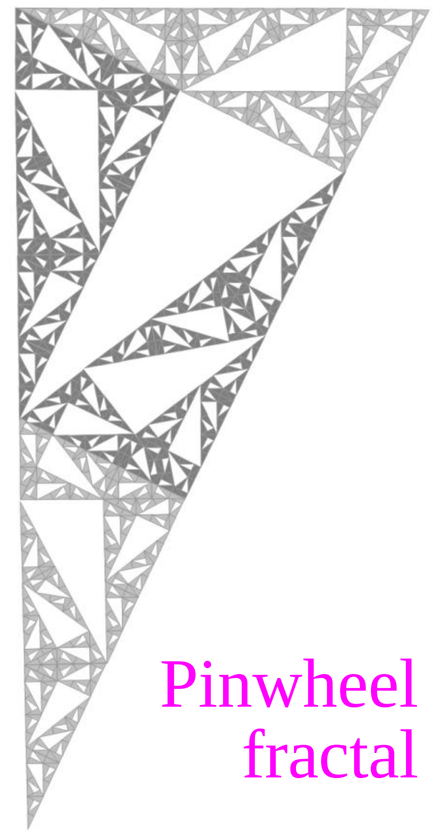
"Pinwheel tiling", John Conway and Charles Radin, 1992

- Tiles occur in infinitely many orientations, with uniform distribution!
- Despite irrational edge lengths and incommensurable angles, all vertices of tiles have rational coordinates!









Pinwheel fractal

Aperiodic Tilings

"Pinwheel tiling", John Conway and Charles Radin, 1992

SCIENCE

Bathroom tiling to drive you mad

Ian Stewart

AN AMERICAN mathematician has come up with what is probably the strangest way ever of covering a floor or wall with tiles. The set of tiles which has been devised by Charles Radin of the University of Texas at Austin can only be assembled in a highly complex way (*Annals of Mathematics*, vol 109, p 661).

The usual way of assembling tiles is in a periodic pattern, one that starts with a basic unit, which is repeated at regularly spaced intervals. However, more complex patterns of tiling are perfectly possible and the subject of aperiodic tilings was created by the philosopher Hao Wang in 1961. Wang was studying the existence or otherwise of certain "decision procedures" in mathematical logic—ways of working out in advance whether certain problems have solutions—when he came to the surprising conclusion that the problem could be reformulated in terms of tiles.

Choose a finite collection of shapes and call them prototiles. A tiling is then a way to assemble perfect copies of those prototiles so that they cover the entire infinite plane without any gaps or overlaps. Wang discovered that he could design prototiles that corresponded to various logical statements, in such a way that the rules for fitting prototiles together corresponded exactly to the rules of logical deduction. So, in effect, a tiling pattern corresponded to a logical proof. This new viewpoint led Wang to ask whether there existed a set of prototiles that could tile the plane, but could not tile it periodically.

Tiling a plane aperiodically turns out to be easy. It can be done with a single domino-shaped prototile. First, however, it is necessary to tile the plane with squares. Then each square is divided into two dominos by splitting it in half in either the vertical or horizontal direction. If the pattern of verticals and horizontals is aperiodic, so too is the tiling: the easiest method is to vary the directions randomly. However, dominos can also tile the plane periodically—for example, by making all splits point the same way.

Wang wanted something much more subtle: a set of prototiles that produced only aperiodic tilings. Such a set of tiles was found in 1966 by his student Robert Berger. The best known of such sets are the Penrose tilings, introduced by Roger Penrose of the University of Oxford in 1977; these produce tilings with fivefold "almost" symmetries.

Radin notes that: "All published examples . . . have the feature that in every tiling each prototile only appears in finitely many orientations." For instance, dominos can be laid down horizontally or vertically but not oriented at any other angle; and Penrose tiles rotate only through multiples of an angle of 36°. This means that if the set of prototiles is expanded so that it includes a copy of each prototile in each orientation, then the new prototiles can tile the whole plane without being rotated. Only translations of these "oriented prototiles" are then needed.

Radin's new discovery is a set of prototiles that are forced to appear in an infinite number of orientations. Because periodic tilings involve only a finite number of directions—the ones in the basic repeating unit—Radin's tilings are necessarily aperiodic.

His starting point is an idea thought up by John Horton Conway of Princeton University in New Jersey. Begin with a "half-domino" prototile, a right triangle of sides 1 and 2 units (whose hypotenuse is 5 units). This can be surrounded by four copies of itself in order to create a triangle of the same shape, but larger and rotated through an angle (see Figure). The process can be thought of as defining a "level-l" tiling of part of the plane with five triangular tiles. The construction can now be repeated, surrounding the level-1 set of five tiles with four copies of those sets to make an even larger and further rotated triangle that is composed of 25 of the original prototiles: this is known as the level-2 tiling.

Continuing this "expansion" process indefinitely from each level to the next leads to a strange, random-looking tiling of the infinite plane by half-dominos (see Figure), called the Conway tiling. Because the angle of rotation at each stage does not exactly divide into an integer number of full turns, the half-domino appears in an infinite number of different orientations throughout the plane.

However, this particular prototile can also tile the plane periodically. This can be done if two half-dominos are stuck together to make a domino and the plane is tiled periodically with those. To eliminate these periodic possibilities, Radin modifies the construction so that certain features of the Conway tiling, in particular its hierarchical structure into levels, cannot be avoided.

The essential idea is an old one: the edges of prototiles can be "labelled" with marks or symbols, with the extra rule that adjacent tiles must have matching labels along their common edges. This produces a larger set of labelled prototiles with more restrictive tiling rules. The point is that the labels can be realised by making notches in the edges of one tile and adding protruding lugs to match them in the adjacent tile. By using a different shaped notch/lug pair for each symbol used as a label, we can convert labelled prototiles into ordinary ones of more complicated shapes.

It is, of course, easier to think about simple shapes that have labelled edges, and this is the way in which Radin proceeds. His prototiles are labelled half-dominos, and he invents a complicated range of different labels whose matching rules cleverly force the appearance of the same structure as the Conway tiling.

It is astonishing that such a simple shape as half a domino can have such curious implications, and it shows that even in today's complex world mathematics can still advance by looking at a simple idea in a new way. □

World's most complex tiling? Surround a triangular "half-domino" by four more. Then simply repeat . . .

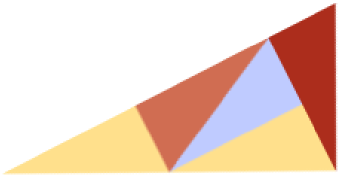
Federation Square
Melbourne, Australia

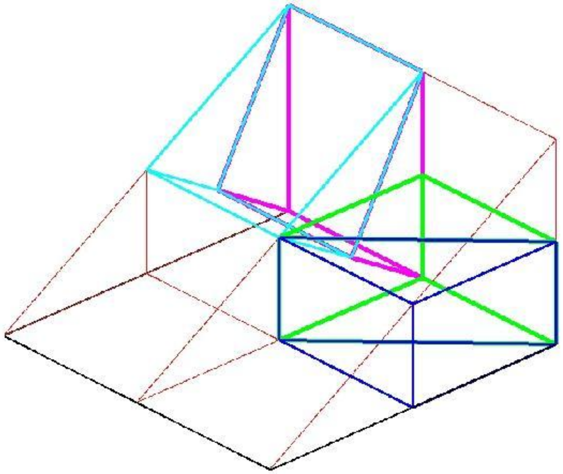
3D Aperiodic Tilings

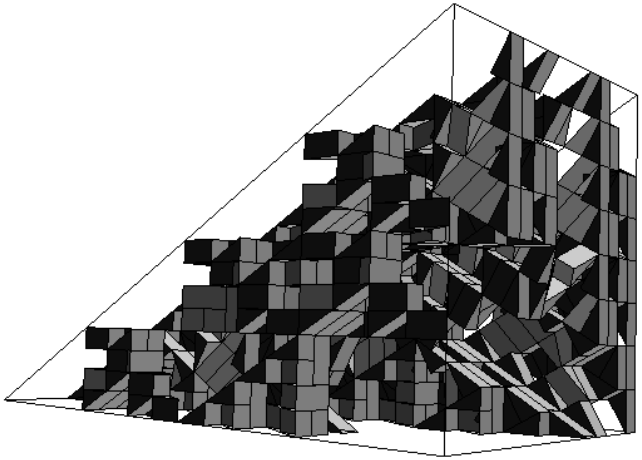
Goal: tile all of 3D space non-periodically

"Quaquaversal" non-periodic tiling of 3D space,
John Conway and Charles Radin, 1998

- Generalization of 2D Pinwheel tiling







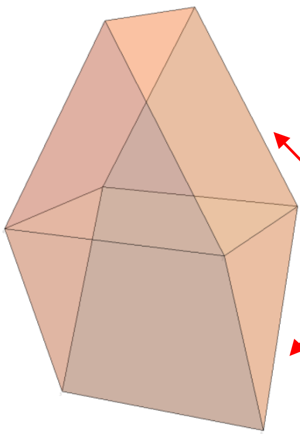
Q: ∃ a single-tile aperiodic 3D tiling?

(i.e., that does not admit any periodic tiling?)

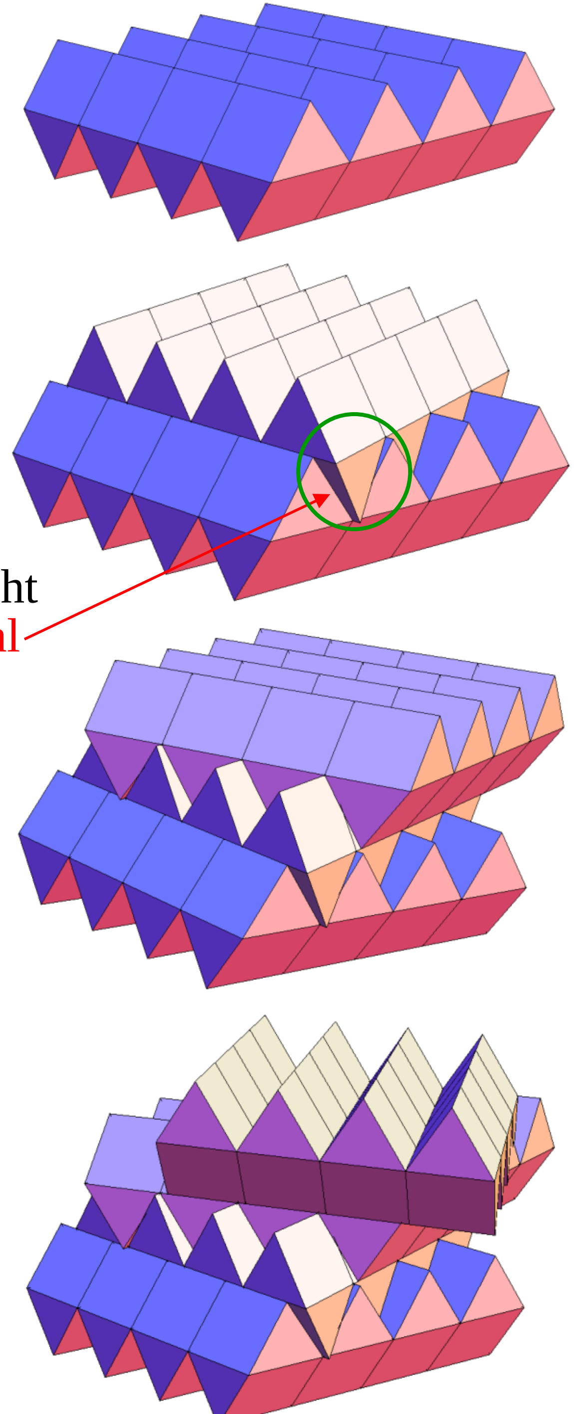
A: Yes! (yet this is still open for 2D)

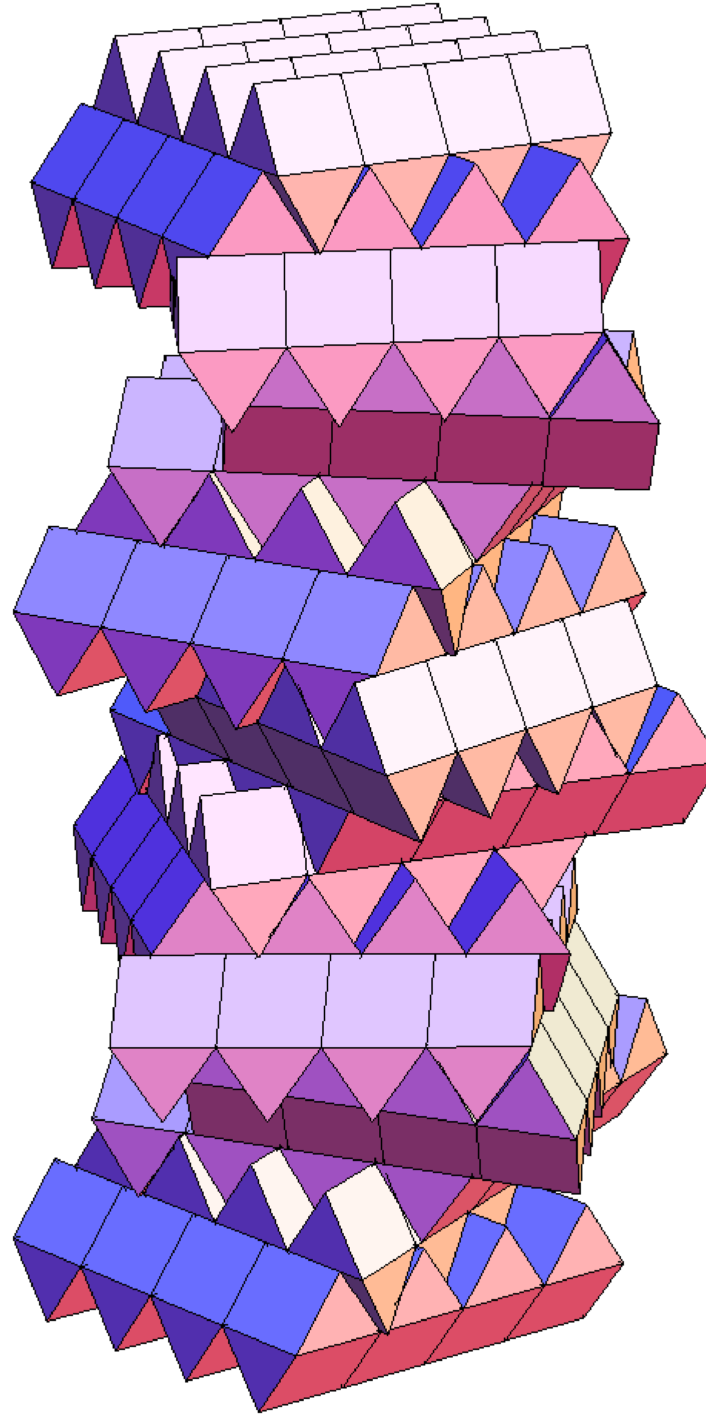
Aperiodic 3D Tiling

The Schmitt-Conway "biprism" tiles 3D space aperiodically using 1 convex tile!



Note slight irrational skew!





This is more than Hilbert asked for, since the biprism tiling is also anisohedral, and with an infinite number of tile orientations!

Undecidability of Tiling Problem

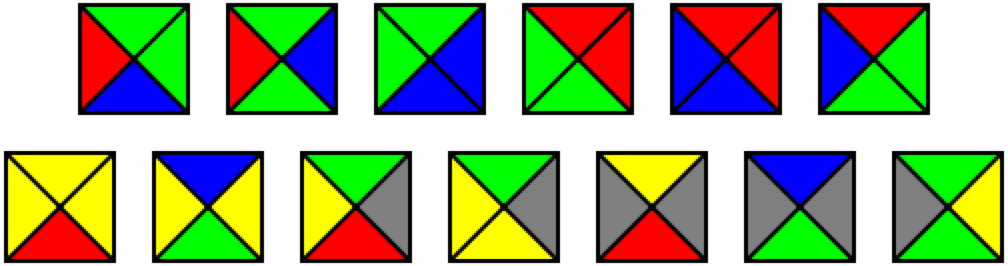
Q [Wang, 1961]: Is there an algorithm for determining whether a given set of tiles can tile the entire plane? (Tiles can not be rotated)

Wang gave a decision algorithm for periodic tilings (and falsely assumed that non-periodic tilings do not exist).

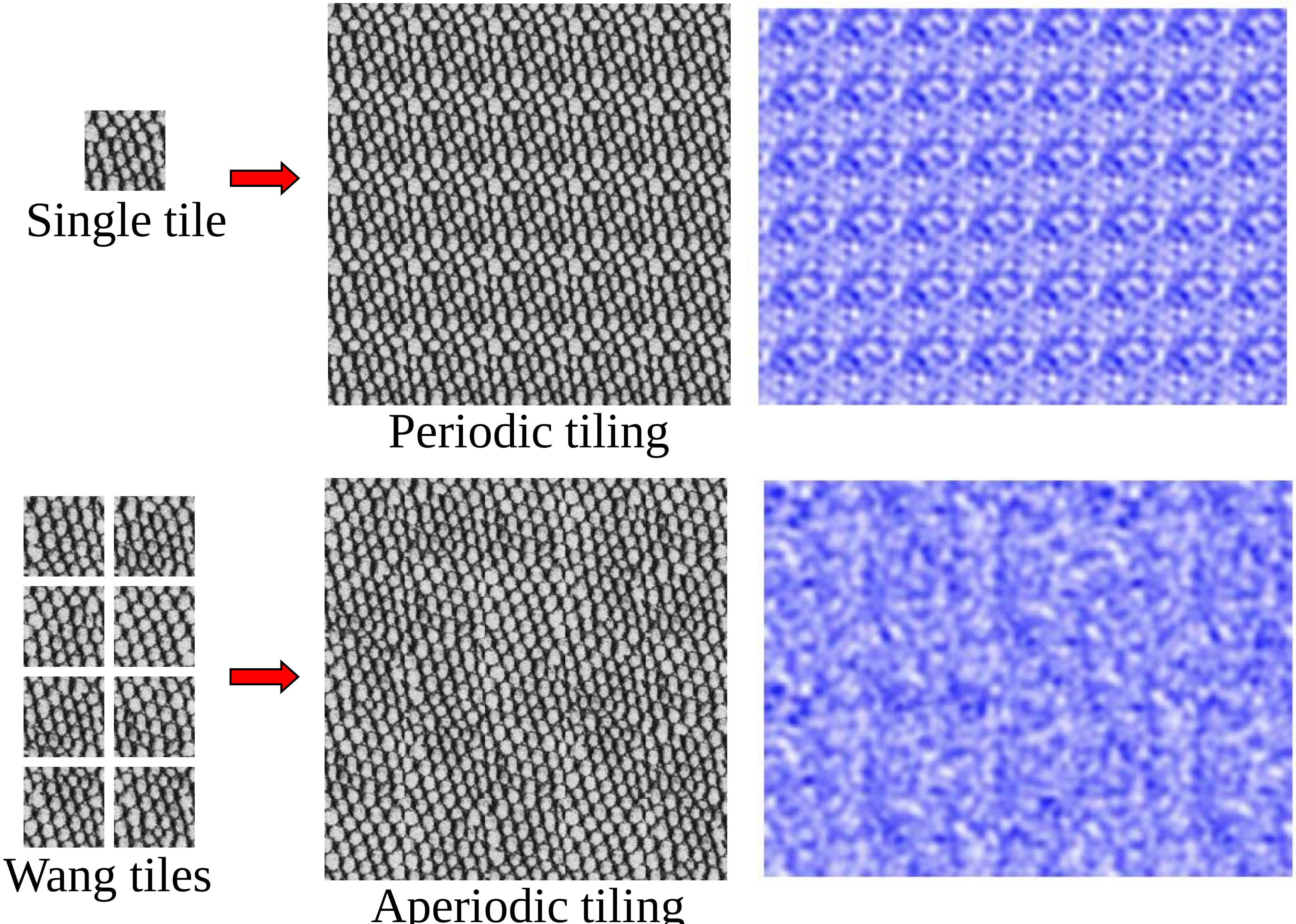
Theorem [Berger, 1966]: Tiling is undecidable.
Proof idea: A tiling can "simulate" an arbitrary Turing computation.

Berger discovered a set of 20,426 Wang tiles that can tile the plane only aperiodically, and conjectured that smaller sets exist.

Theorem [Culik, 1996]: The following 13 tiles is an aperiodic tiling set.



Aperiodic Tiling for Texure Generation



SIGGRAPH 2006

Recursive Wang Tiles for Real-Time Blue Noise

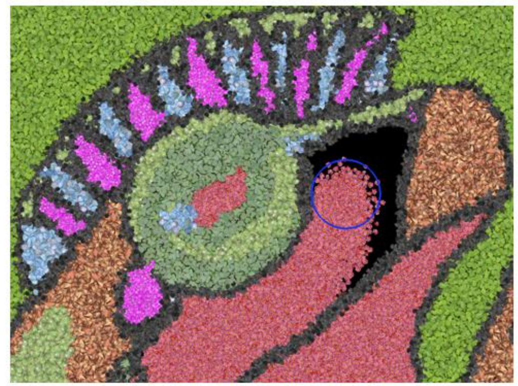
Johannes Kopf
University of Konstanz

Daniel Cohen-Or
Tel Aviv University

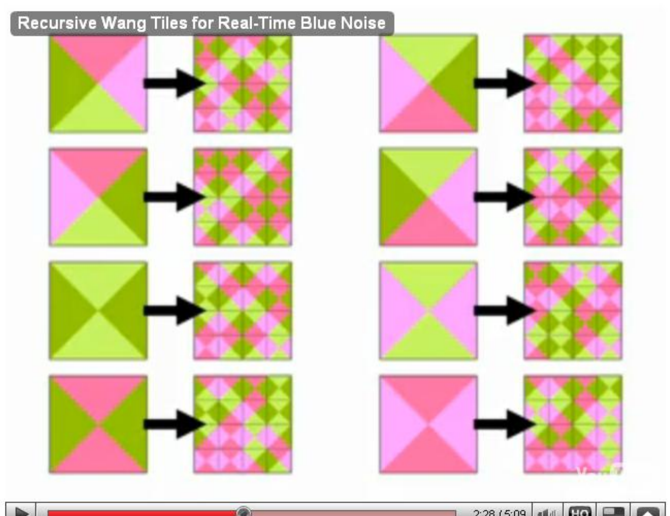
Oliver Deussen
University of Konstanz

Dani Lischinski
The Hebrew University

Zooming into a stippled non-photorealistic rendering. Each image shows a subset of the same implicitly infinite point set: while zooming in, more points are shown to maintain the apparent density. Only the local visible area of the point set was evaluated for each image.



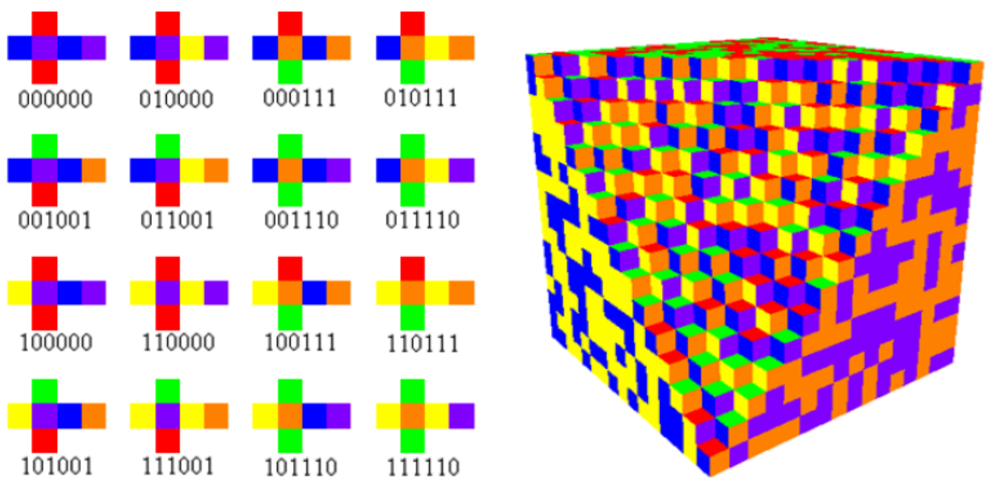
Interactive texture painting application. The user controls the placement of textons by directly painting individual density maps for different texton classes using various brushes. The high speed of our technique allows computing the instance positions on-the-fly from the density maps only where needed at any given time.



3D "Wang Cubes"

Generalizations to higher dimensions: "Wang cubes"

16 Wang cubes and a partial aperiodic 3D tiling:



Applications in graphics:

- Texture generation
- Volume rendering
- Video synthesis
- Geometry placement
- Self assembly

Wang Cubes for Fast Geometry Placement & Video Synthesis

Peter G. Sibley†, Philip Mongomery, G. Elisabeta Marai
Brown University

1. Abstract

We present an extension of Cohen's Wang Tiles to three dimensions: Wang Cubes. Cubes are filled with video or Poisson distributed points to perform realtime video synthesis or geometry placement. Video synthesis from a sample is useful for generating dynamic backgrounds for games or special effects but costly in terms of storage and runtime. Randomly positioning non-overlapping 3D geometry is useful for simulations and games but also costly. We propose Wang Cubes where we only store 32 cubes and generate, at runtime, large amounts of synthesized video, or Poisson distributed geometry

2. Methods

Cohen et al. introduced a fast and simple stochastic algorithm to generate an aperiodic tiling of the plane with as few as eight Wang Tiles (oriented squares with color associated edges). Cohen et al. used these tilings for texture synthesis and 2D geometry placement.

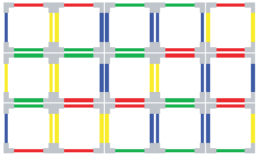


Figure 1: A valid tiling of Wang Tiles , from Cohen et al.

We extend these applications to the 3D case, where cubes with colored faces replace tiles. 32 cubes are sufficient to tile space. The extended tiling algorithm iterates through the space, placing a cube at each point (Figure 2). The 32 cubes contain either Poisson ball distributed points or video data and are tiled at runtime to generate large stretches of 3D geometry or video sequences.

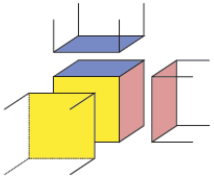


Figure 2: Aligning face colors of Wang Cubes

To place geometry data, we use dart-throwing to fill each cube with Poisson distributed points. Several iterations of Lloyd's relaxation are applied to prevent points near boundaries from violating the minimum distance constraint in tiling.

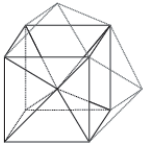


Figure 3: Assembling six octahedra of video to form one cube.

To fill the cubes with video data, we cut six octahedra from the video stream and stitch them together through the graph cut method of Kwatra et al. Then, the result is trimmed into a cube (Figure 3). Each original octahedron is associated with a face color. A xy plane in this cube corresponds to a frame of video and the z axis corresponds to time (Figure 4).

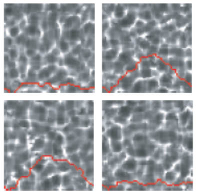


Figure 4: Several slices of one cube showing the seam of the bottom tetrahedron.

†{pgs,pgm,gem}@cs.brown.edu

3. Geometry Results

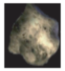


Figure 6: Single Asteroid.

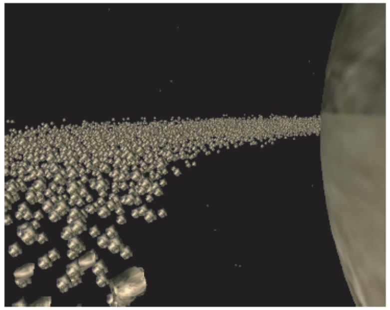


Figure 7: Saturn Asteroid belt, 5959 asteroid instances placed using tiling of 3972 cubes each with 15 Poisson distributed points. Note, this took only seconds to generate. Filling the same region with dart throwing is simply infeasible.

As a geometry placement application, we modeled the asteroid belt of Saturn with 5958 asteroids (Figure 6) constructed from 3972 tiled cubes with 15 points in each cube. The asteroids are placed according to a Poisson distribution in this large area (Figures 7 and 8). It only took 15 minutes to precompute the cubes, and under 20 seconds to tile them. Note that filling the same region with dart throwing is simply infeasible. Teapot geometry and sheep billboard distributions are shown in Figures 9 and 11. These tests were performed on an AMD Athlon XP 1800 with 512MB of memory.

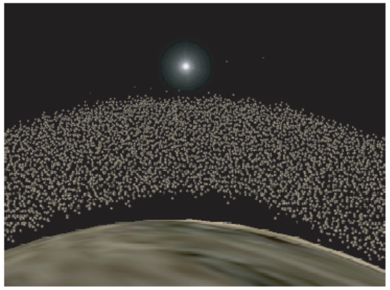


Figure 8: An overhead of the Saturn Asteroid belt.

Figure 9: An infinite teapot field. Note the teapot instances are neither colliding nor have a regular position pattern.

4. Video Results

We constructed a cube set (64*64*64 voxels per cube) from a video of simulated shallow pool caustics. Three vertical slices through two cubes tiled horizontally are shown in Figure 10. Note how the vertical seam in the middle of each frame is invisible. An infinite caustics pool (both in space and time) could be generated in this manner.

In order to keep our computation feasible, we constrained the cuts to lie near the intersecting triangles of the octahedra. We have noticed temporal artifacts in the videos, a growing and shrinking square-discontinuity. We believe these are caused by constrained cuts and small cube sizes.

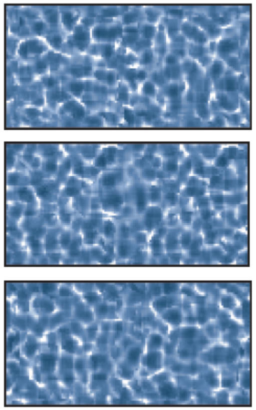


Figure 10: Vertical slices from two tiled Wang cubes. Note that the vertical middle seam of each frame is invisible

5. Discussion and Future Work

For video synthesis, we restricted the space searched for a min-cut surface, sacrificing quality of the cut for faster execution. Also, because of computational constraints we could only use quite small cube sizes (64*64*64). Implementing known randomized max-flow algorithms to approximate the cut could yield much lower preprocessing, which would allow for less constrained cuts and eliminate temporal artifacts. Incorporating newer texture synthesis techniques could produce better quality cubes.

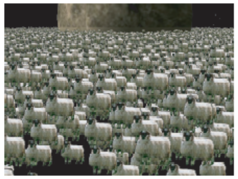


Figure 11: A sheep belt instead of an asteroid belt.

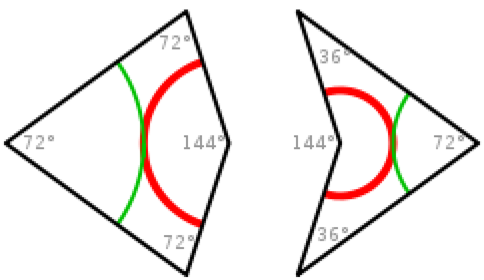
References

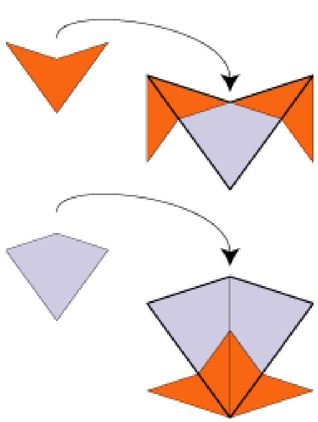
Cohen, M.F., Shade, J.,Hiller,S., and Deussen,O. 2003.
Wang tiles for image and texture generation. *ACM Trans. Graph.* 22,3,287-294.
Kwatra,V., Schödl,A.,Essa,I. Turk,G., and Bobick,A. 2003.
Graphcut textures: Image and video synthesis using graph cuts. *ACM Trans. Graph. 22,3,277-286.*

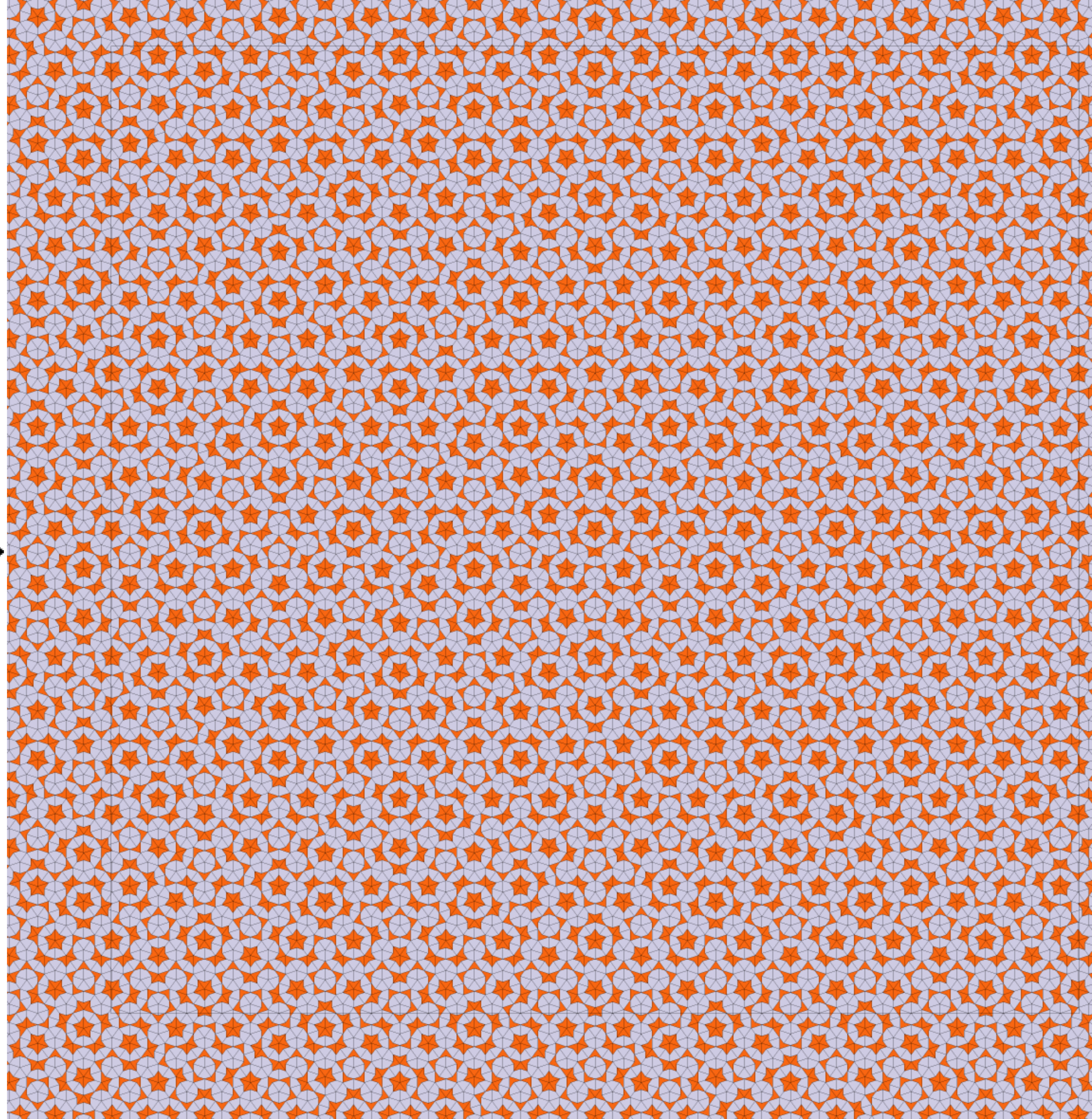
SIGGRAPH2004

Aperiodic Tilings

"Kites and Darts"
Roger Penrose, 1974

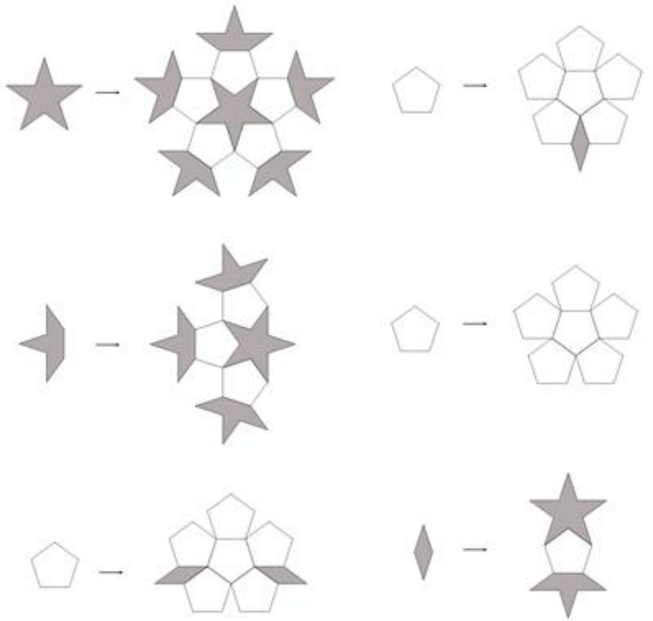


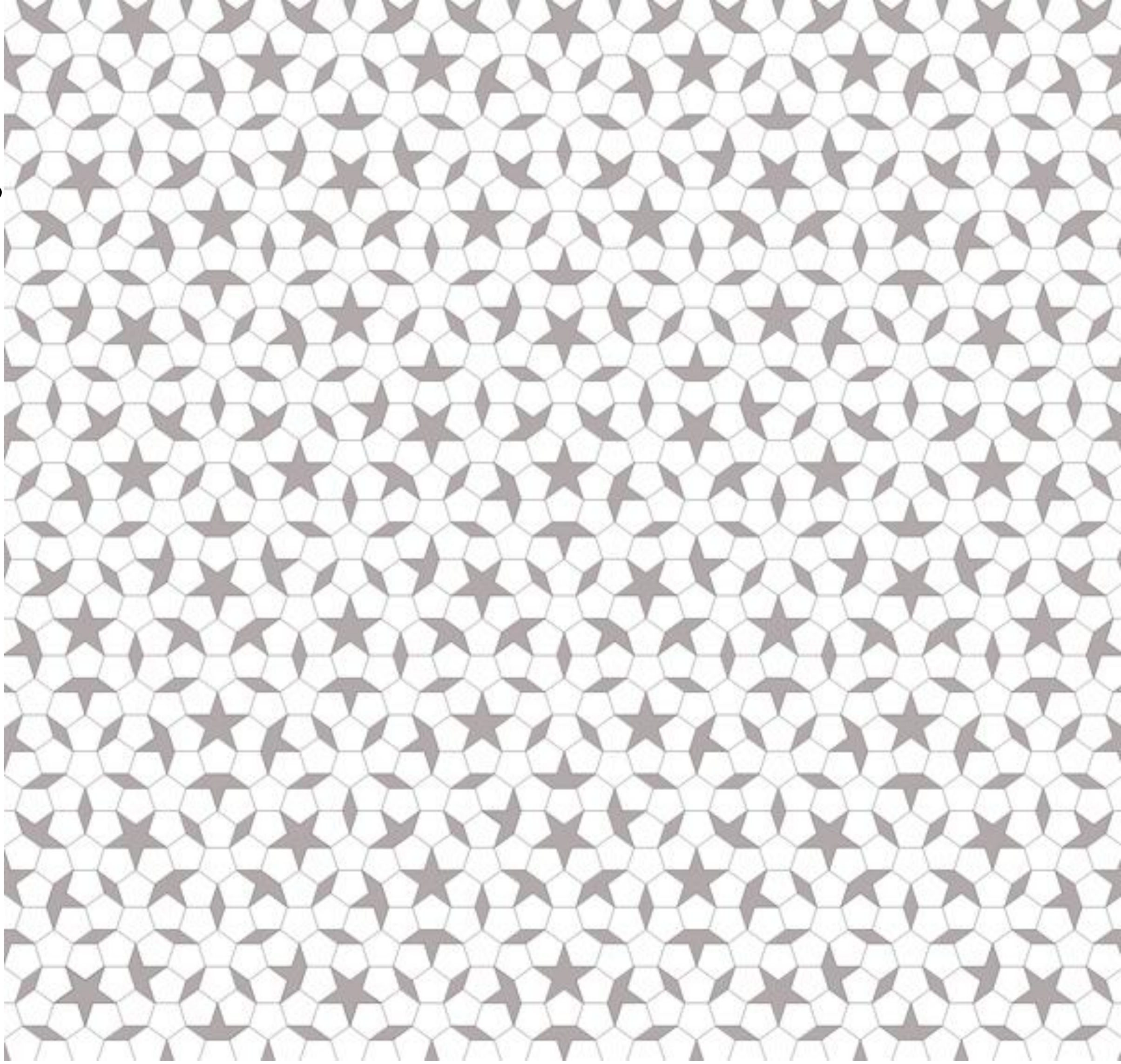




Aperiodic Tilings

“Pentagon, Boat, and Star”
Roger Penrose, 1974

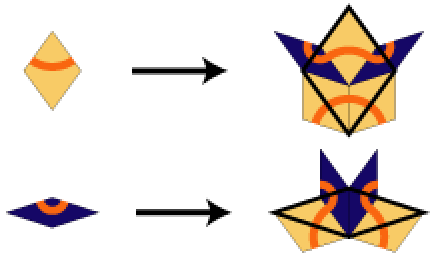


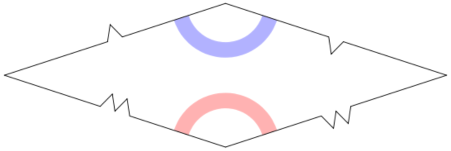


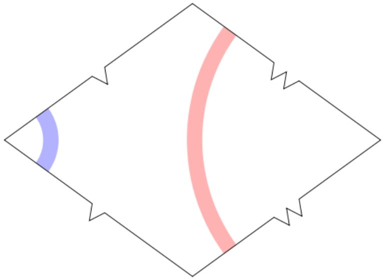
Aperiodic Tilings

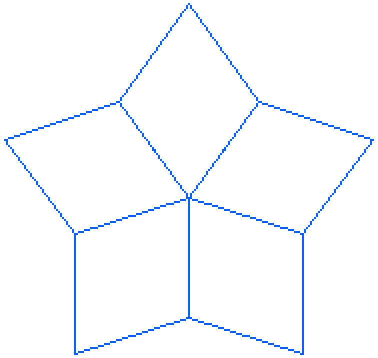
"Penrose
Rhombuses"

Roger Penrose, 1974





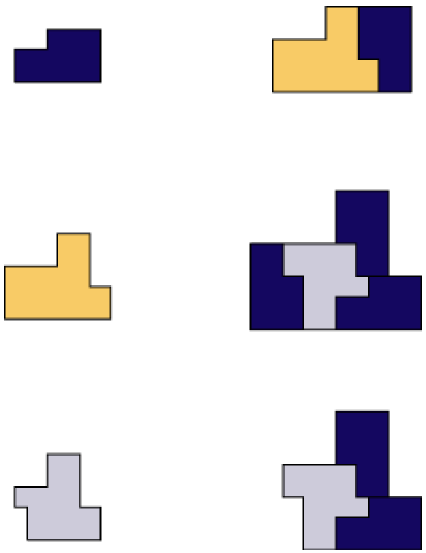


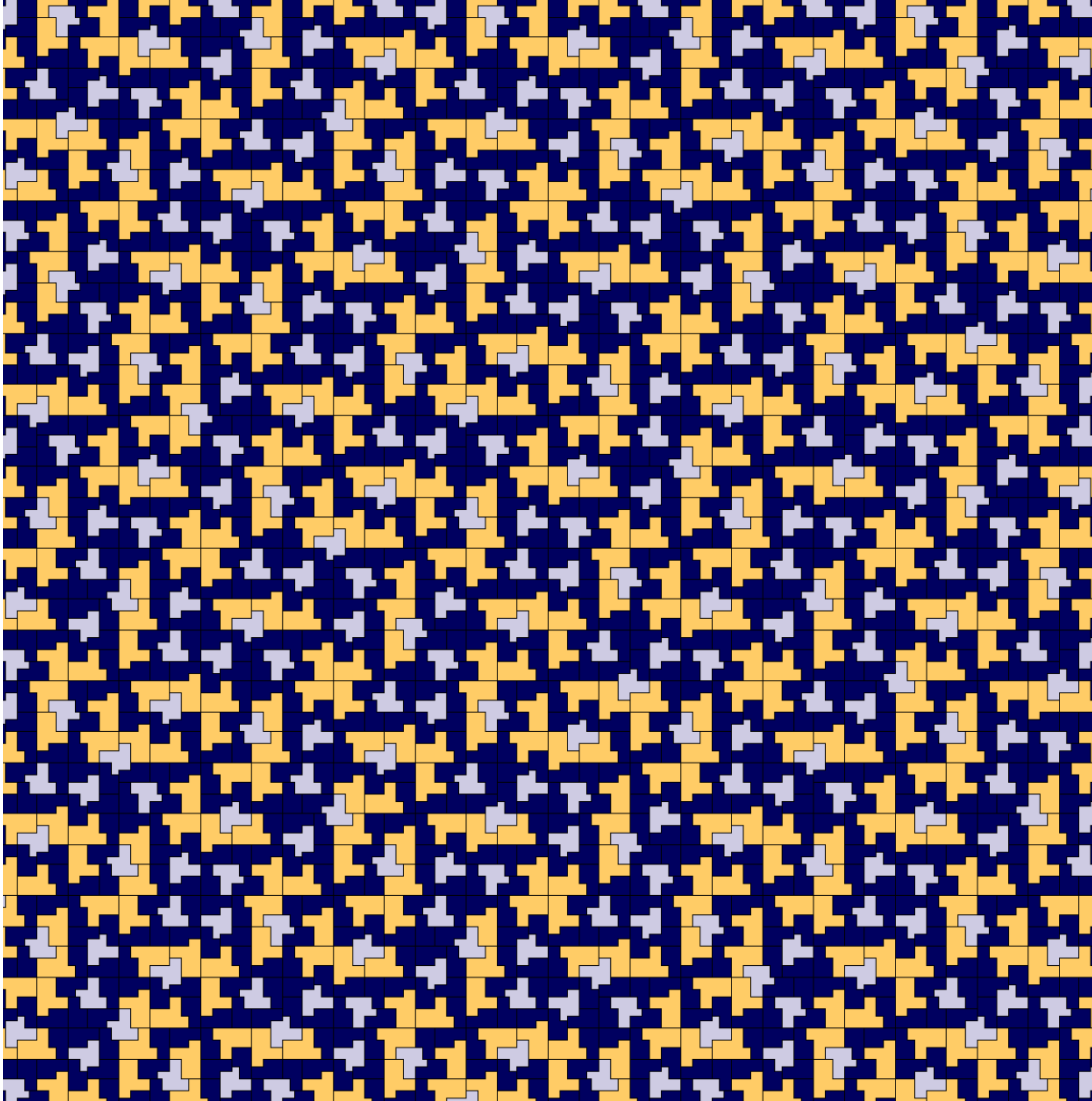


Aperiodic Tilings

“Ammann A3”

Robert Ammann,
1977

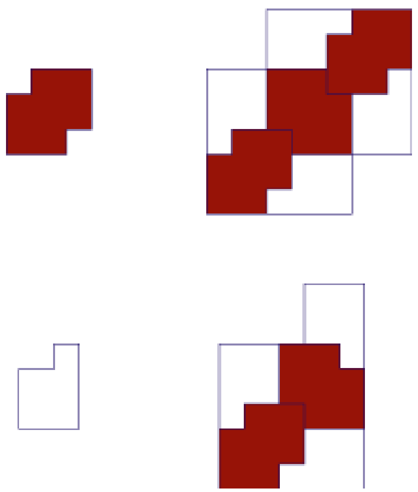


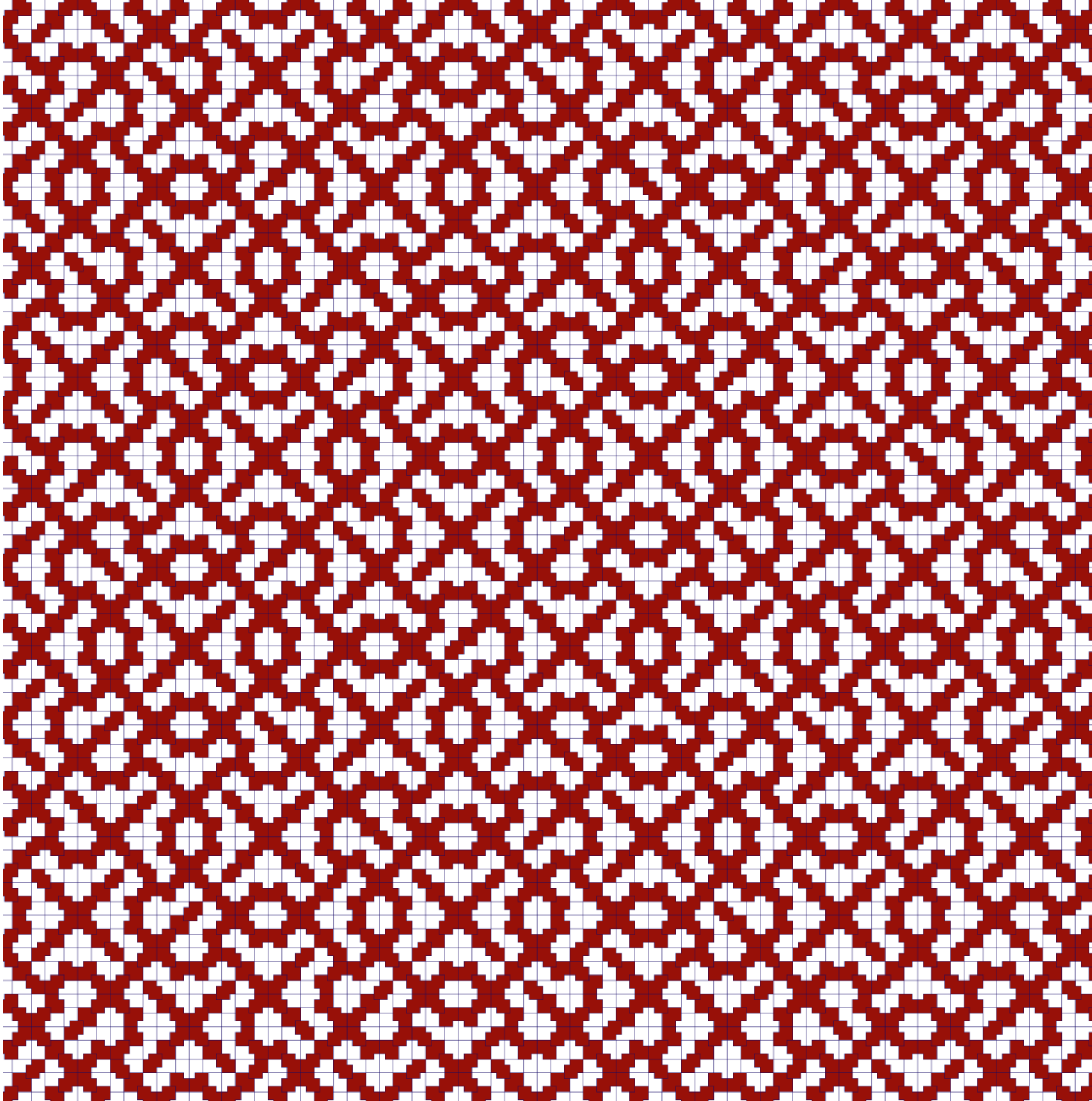


Aperiodic Tilings

"Ammann A4"

Robert Ammann,
1977

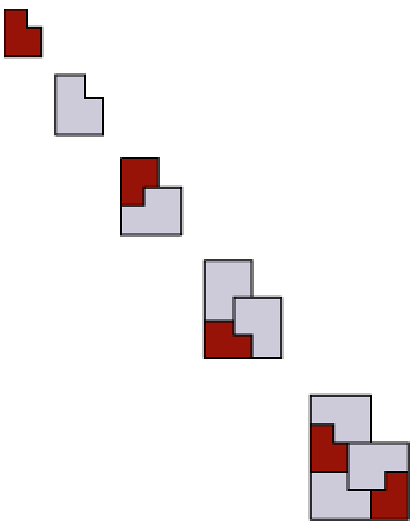


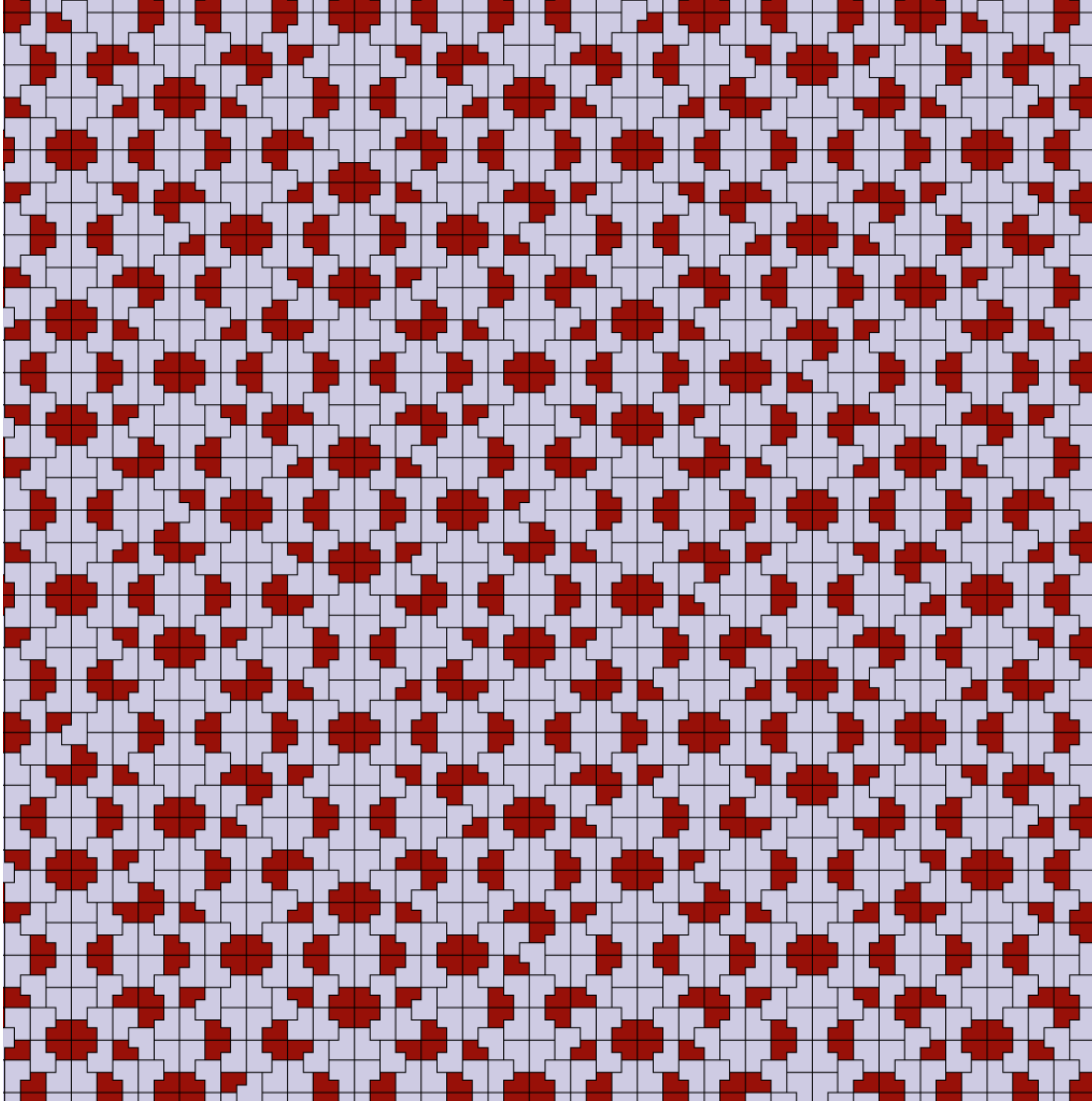


Aperiodic Tilings

"Ammann Chair"

Robert Ammann, 1977

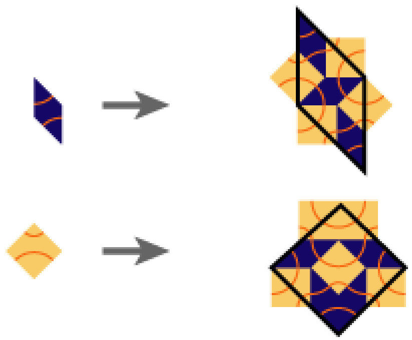


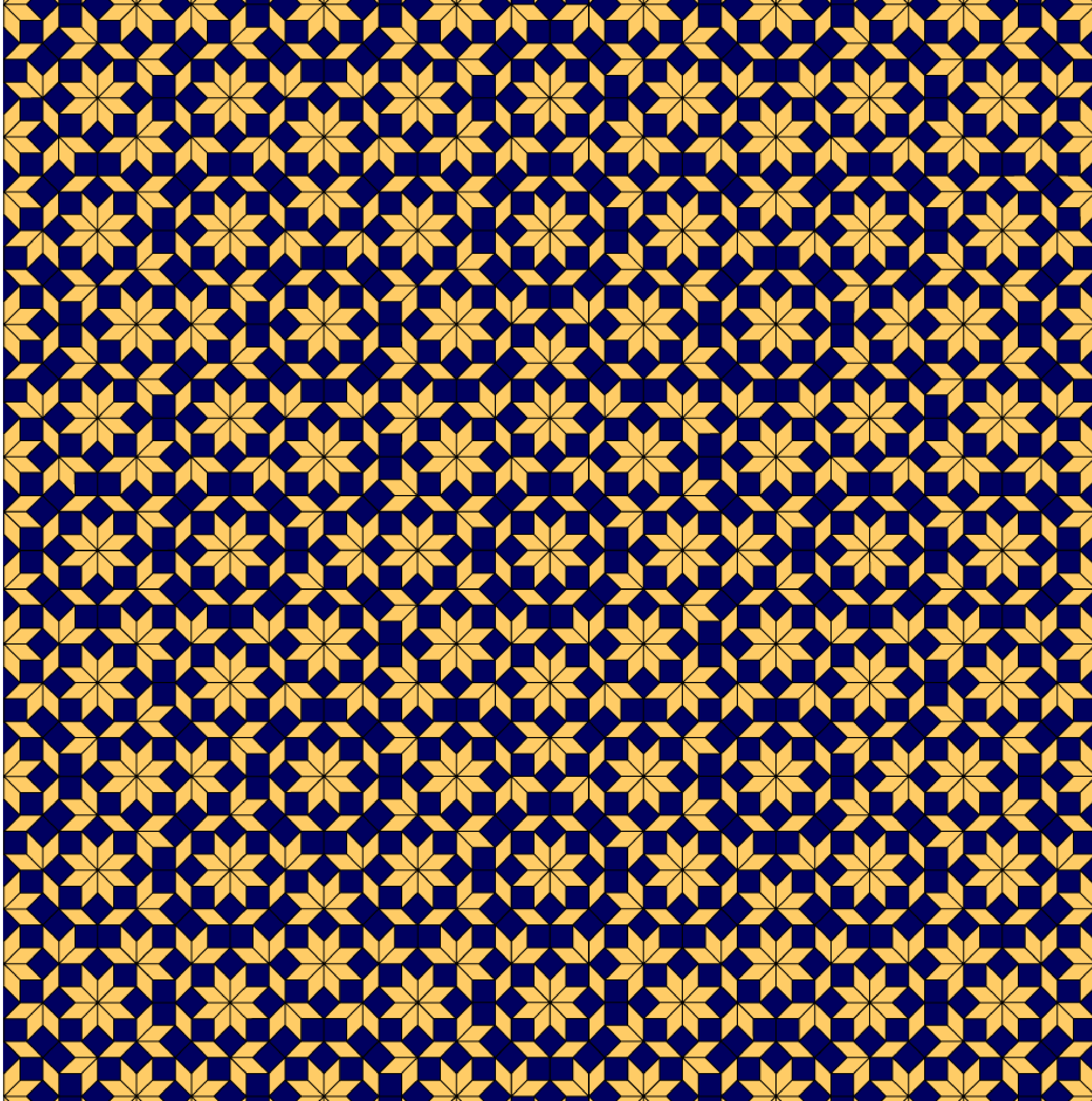


Aperiodic Tilings

“Ammann
Beekner”

Robert Ammann,
1977

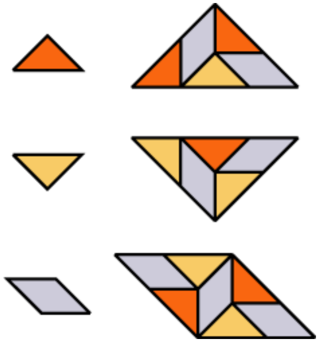


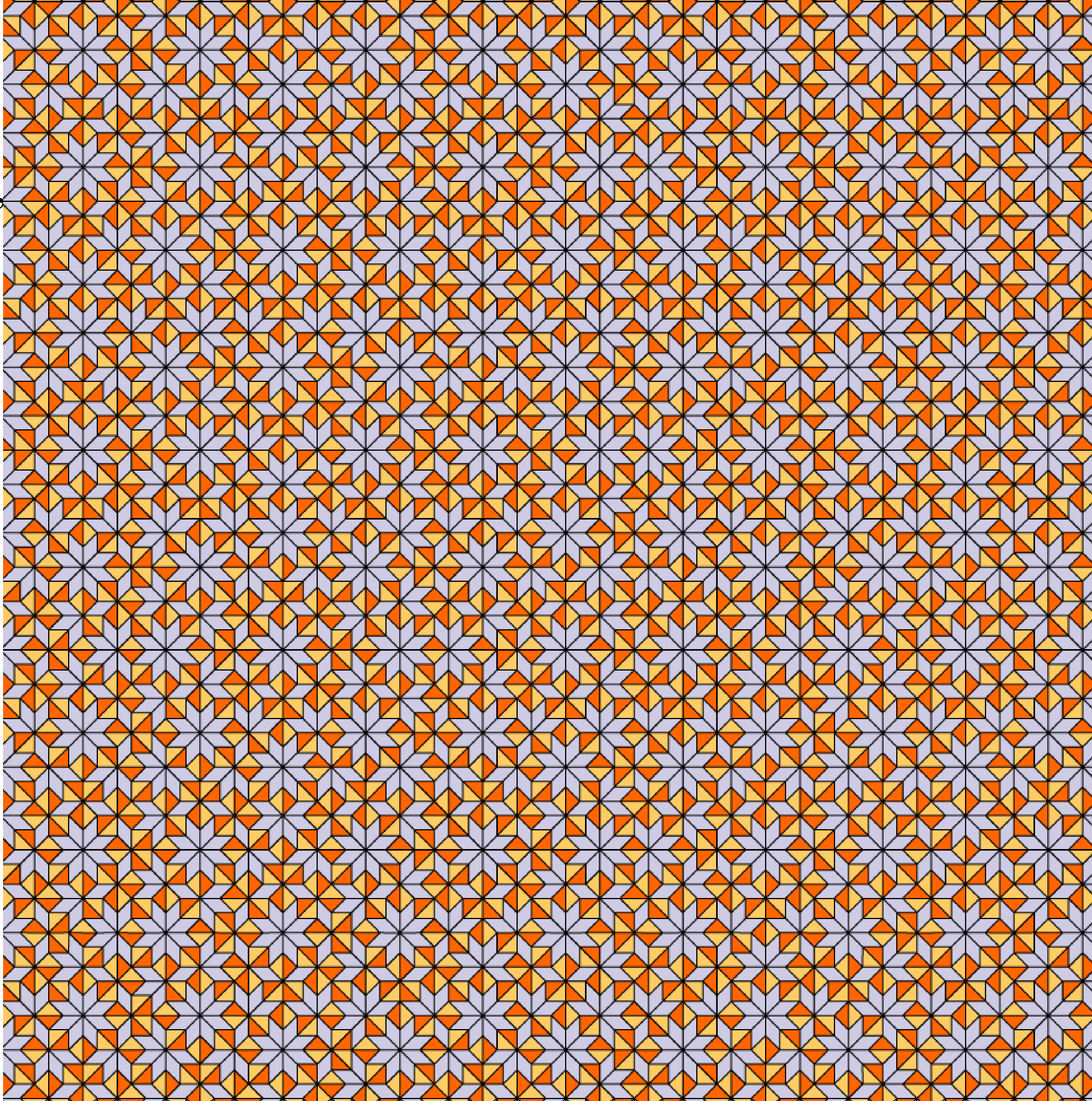


Aperiodic Tilings

"Ammann Beekner Rhomb triangle"

Robert Ammann, 1977

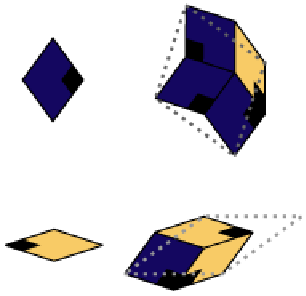


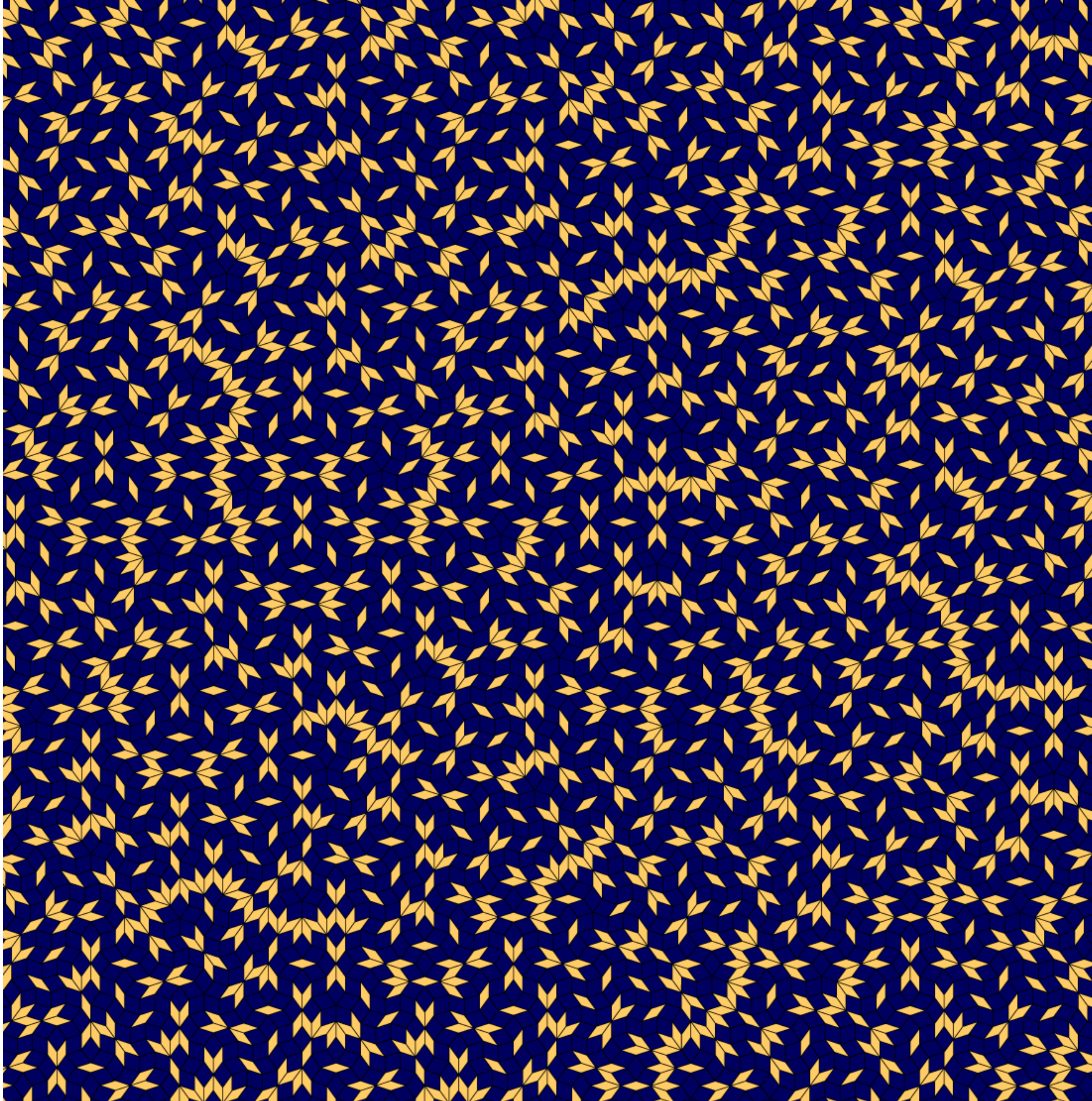


Aperiodic Tilings

“Binary”

F. Lançon, 1988

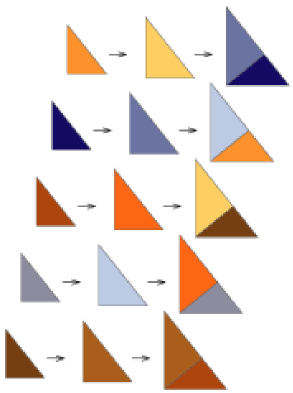


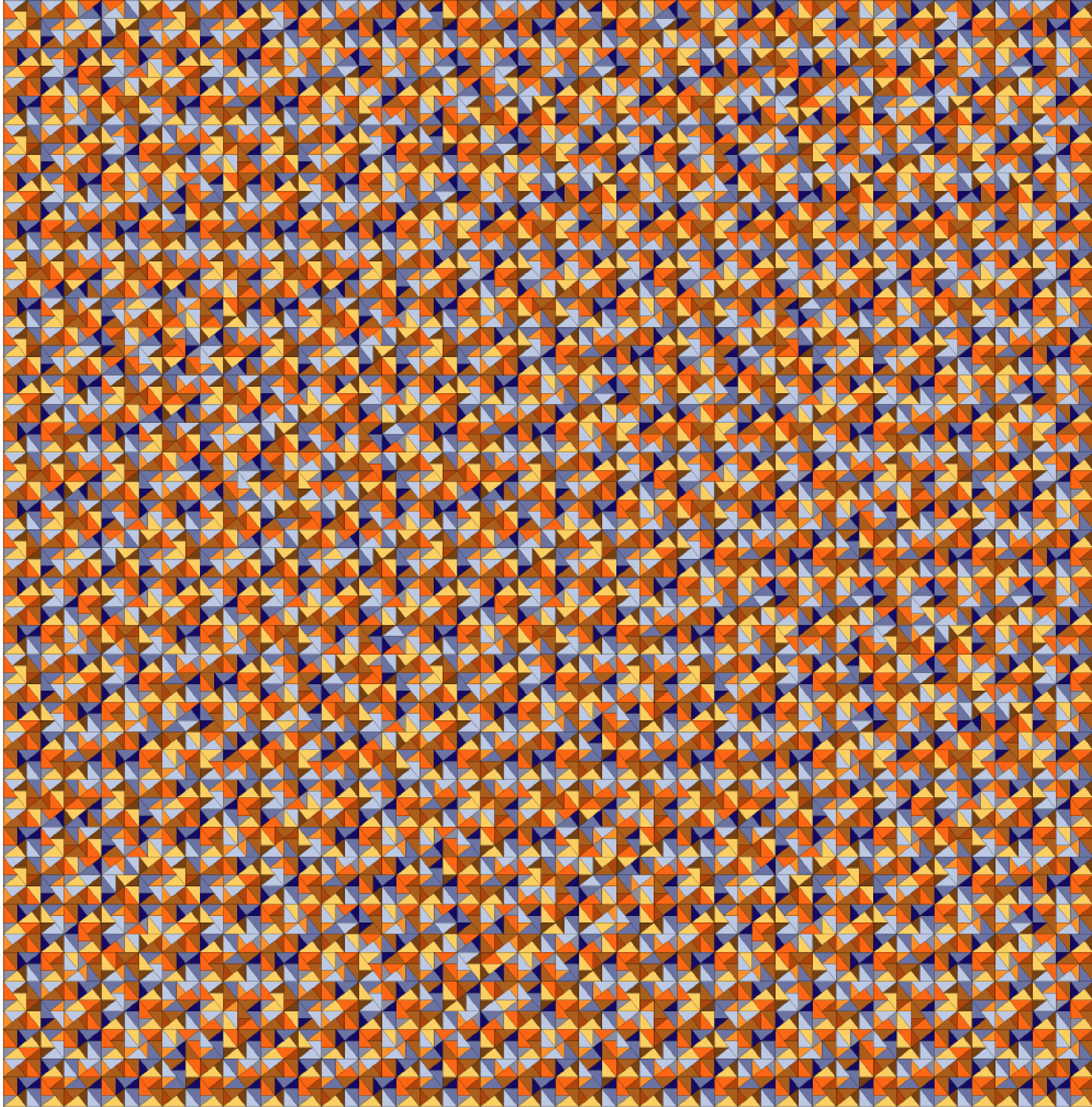


Aperiodic Tilings

“Colored Golden Triangle”

Ludwig Danzer and G. van Ophuysen

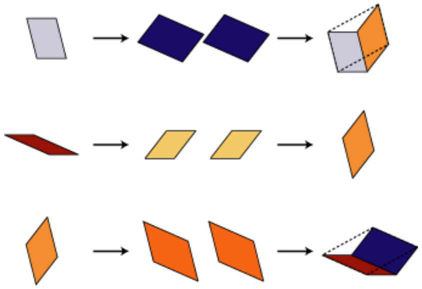


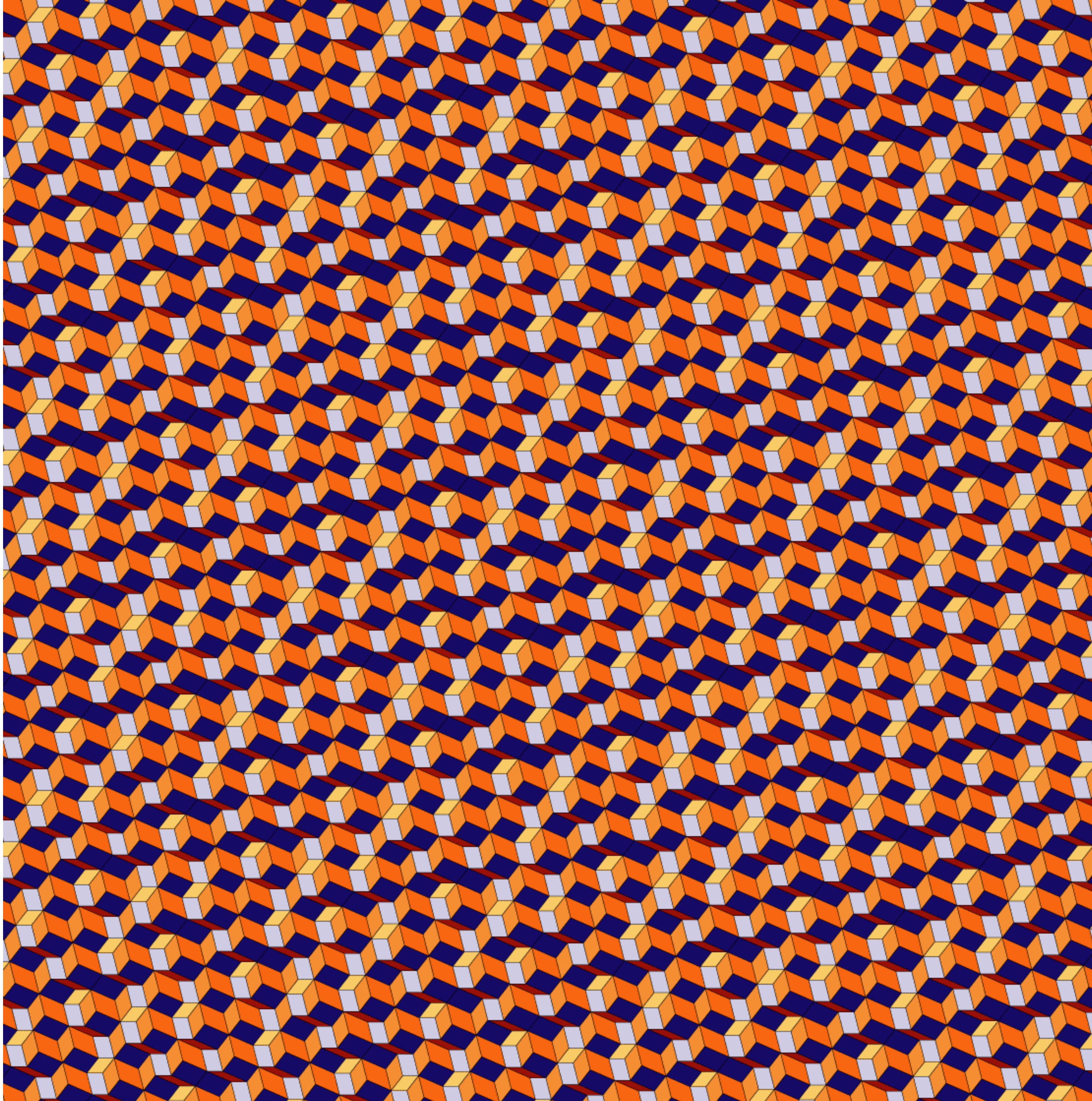


Aperiodic Tilings

“Conch”

G. Rauzy, 1982

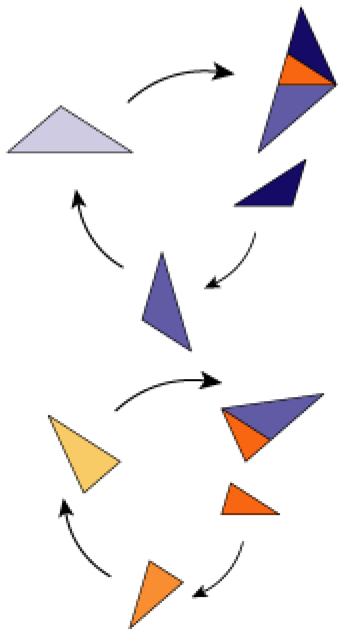


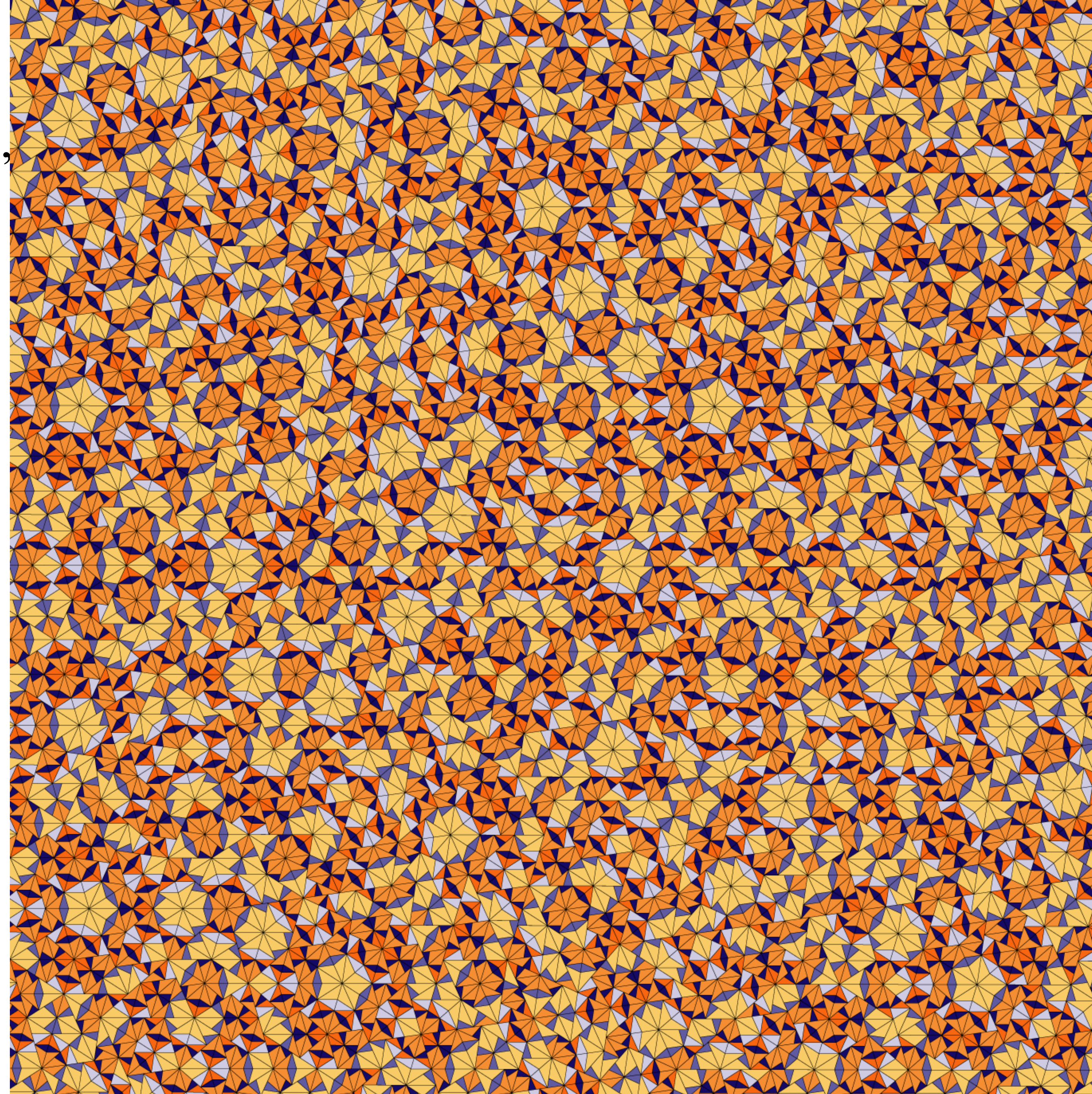


Aperiodic Tilings

"Cubic Pinwheel"

E. Harriss

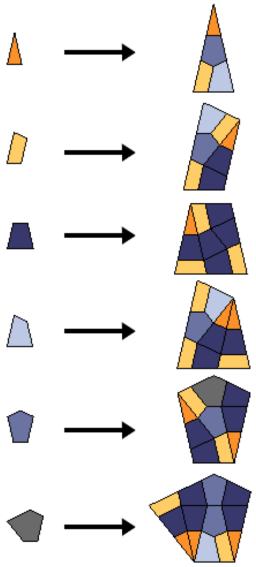


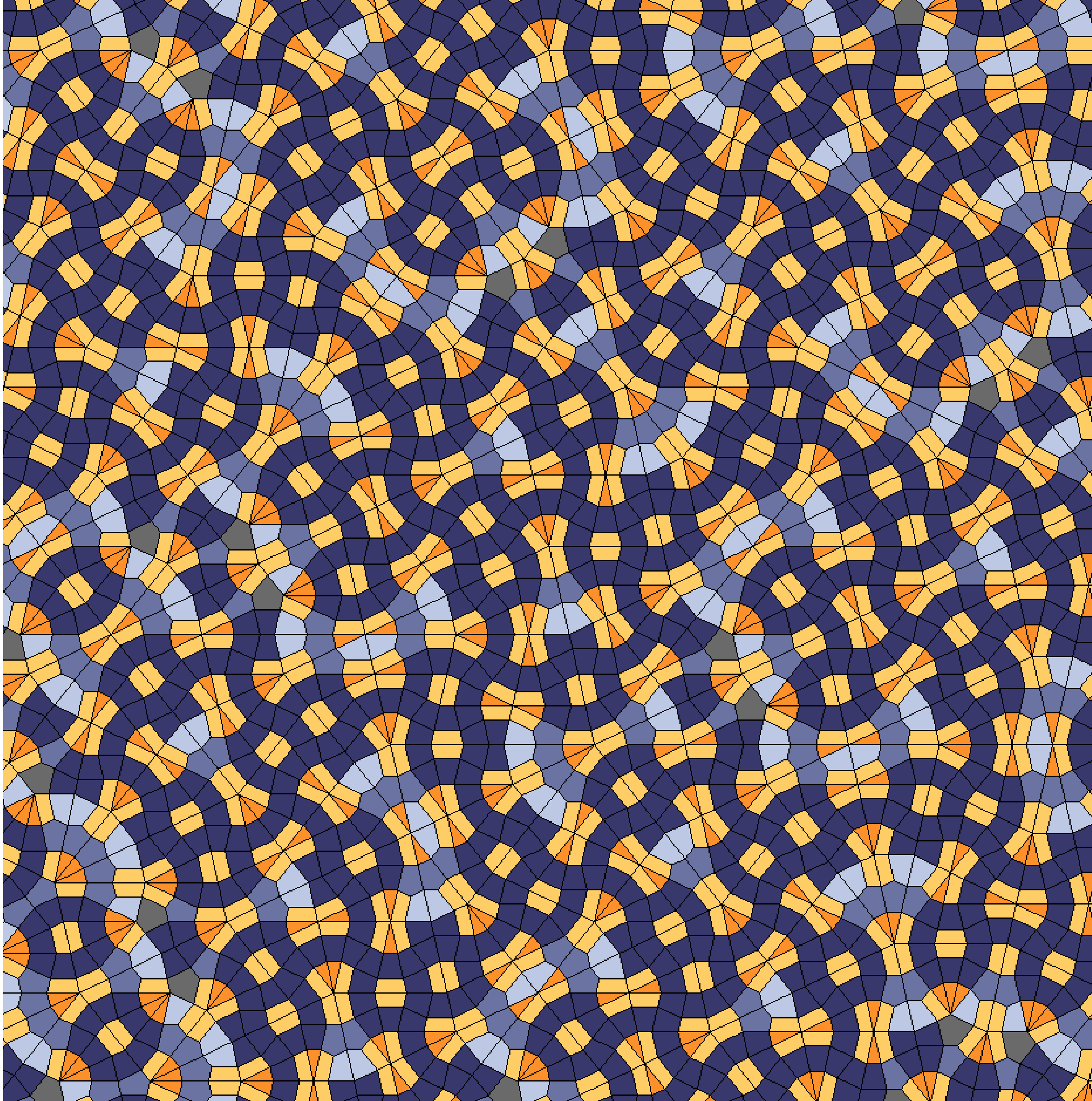


Aperiodic Tilings

"Cyclotomic rhombs 7-fold"

Ludwig Danzer and D. Frettlöh

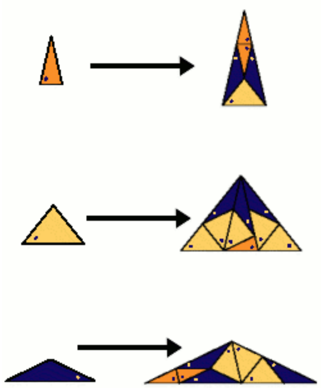


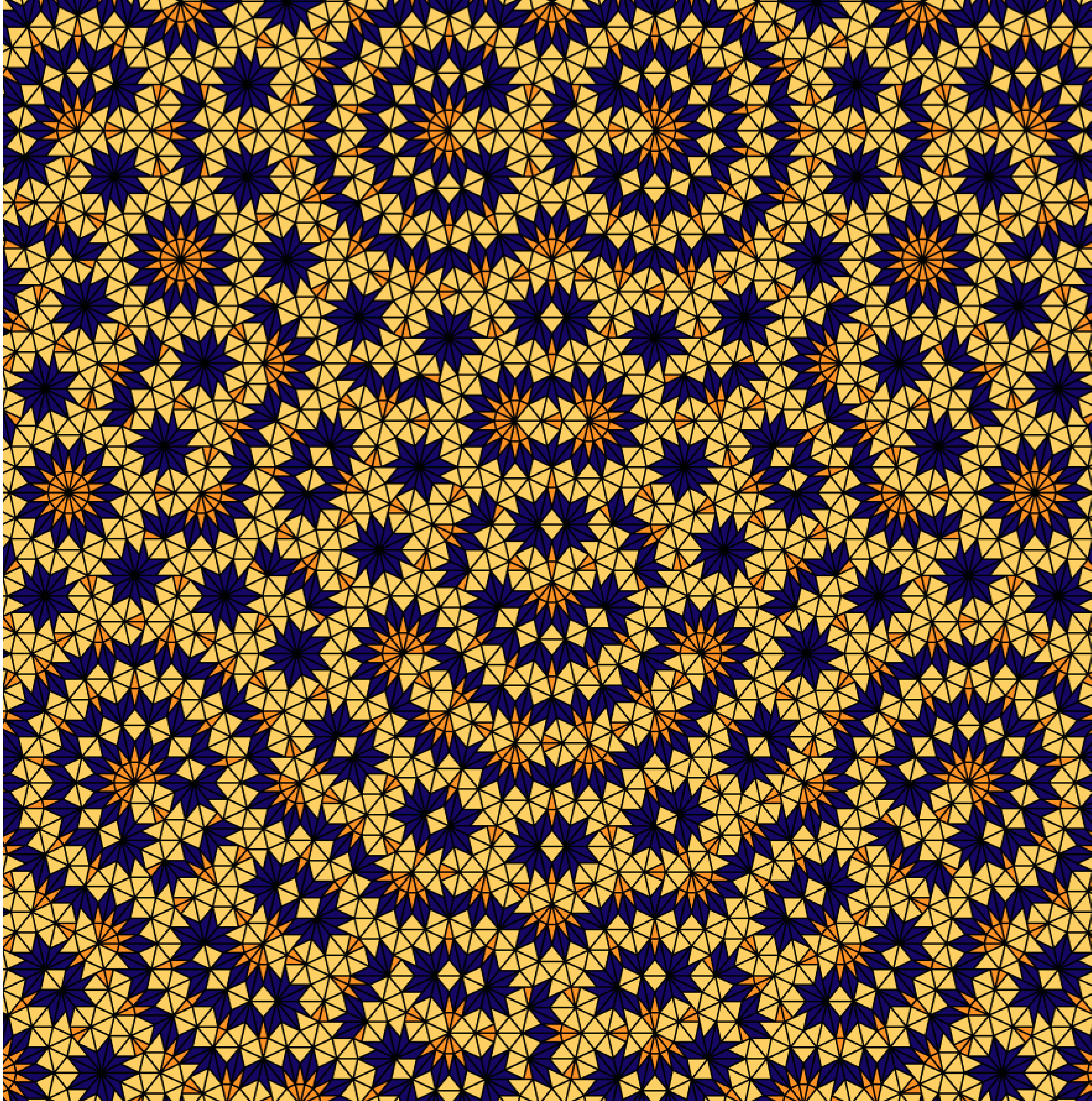


Aperiodic Tilings

“Danzer 7-fold”

K.-P. Nischke and Ludwig Danzer, 1996

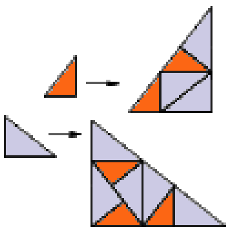




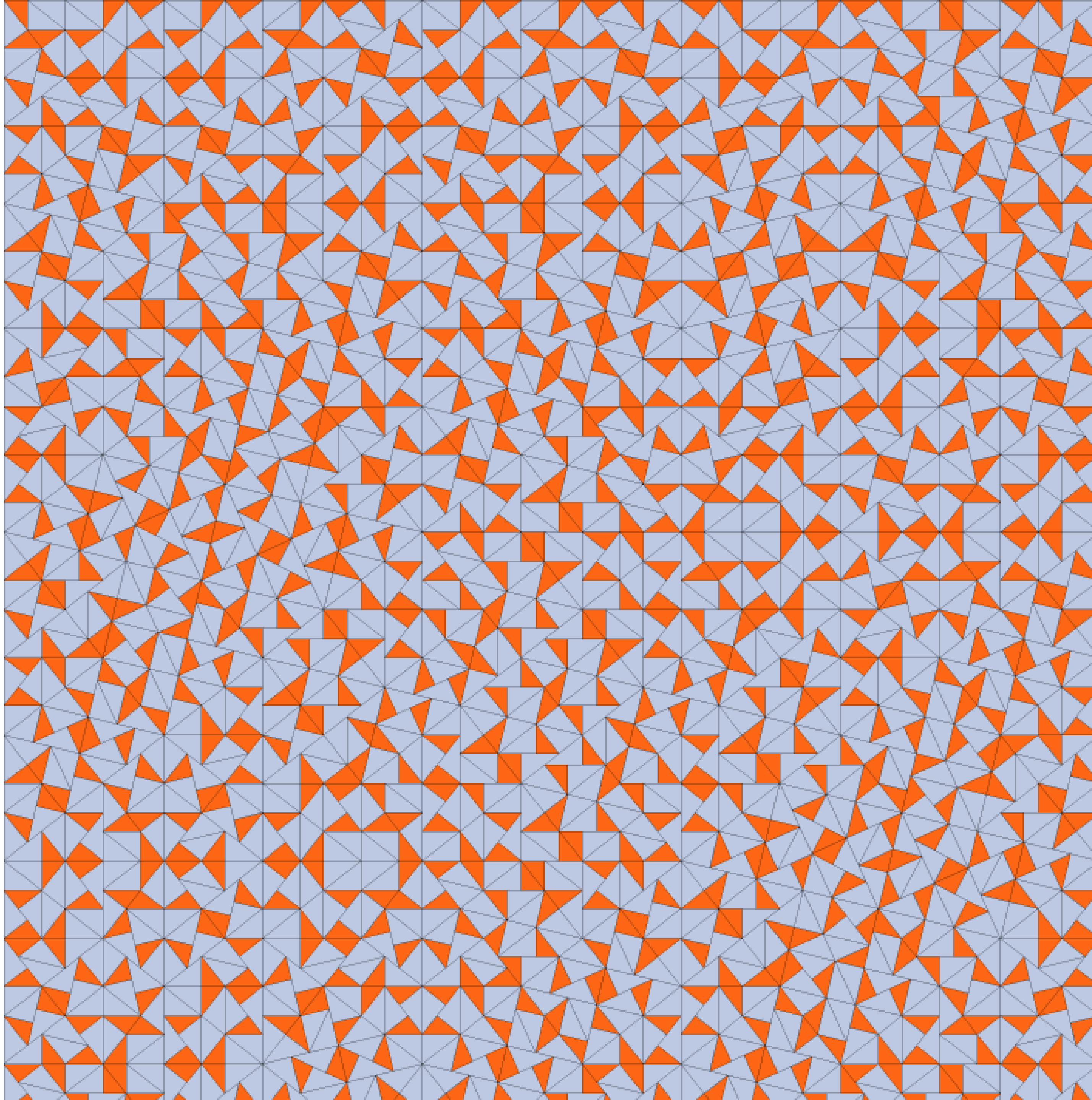
Aperiodic Tilings

“Golden Pinwheel”

D. Frettlöh



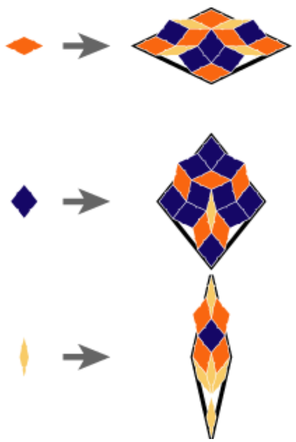
Tiles occur in infinitely many orientations!

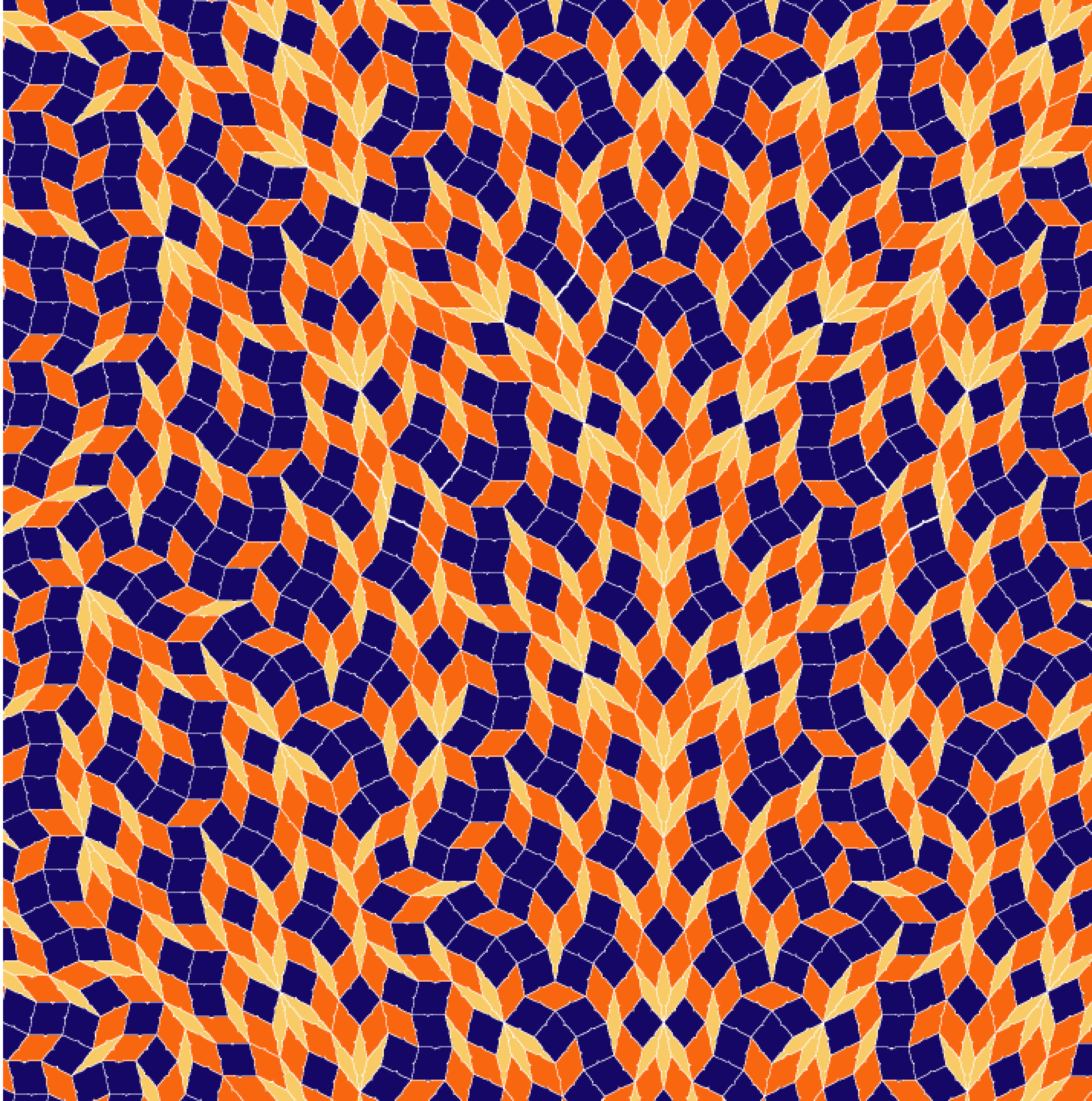


Aperiodic Tilings

"Goodman-Strauss 7-fold rhomb"

C. Goodman-Strauss

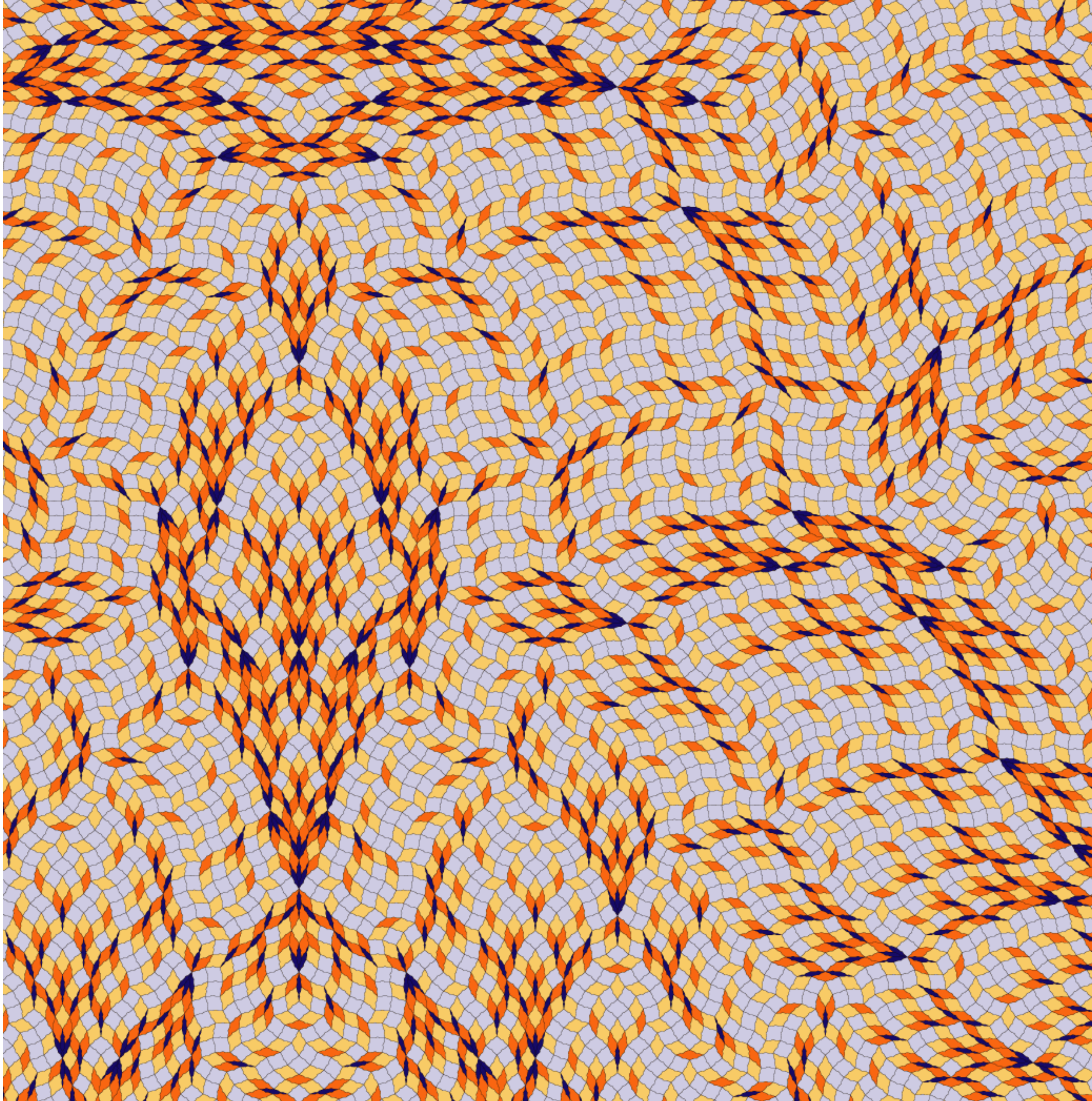




Aperiodic Tilings

"Harriss's 9-fold rhomb"

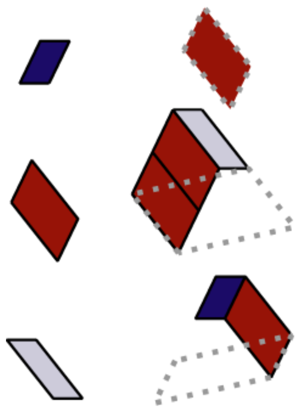
E. Harriss

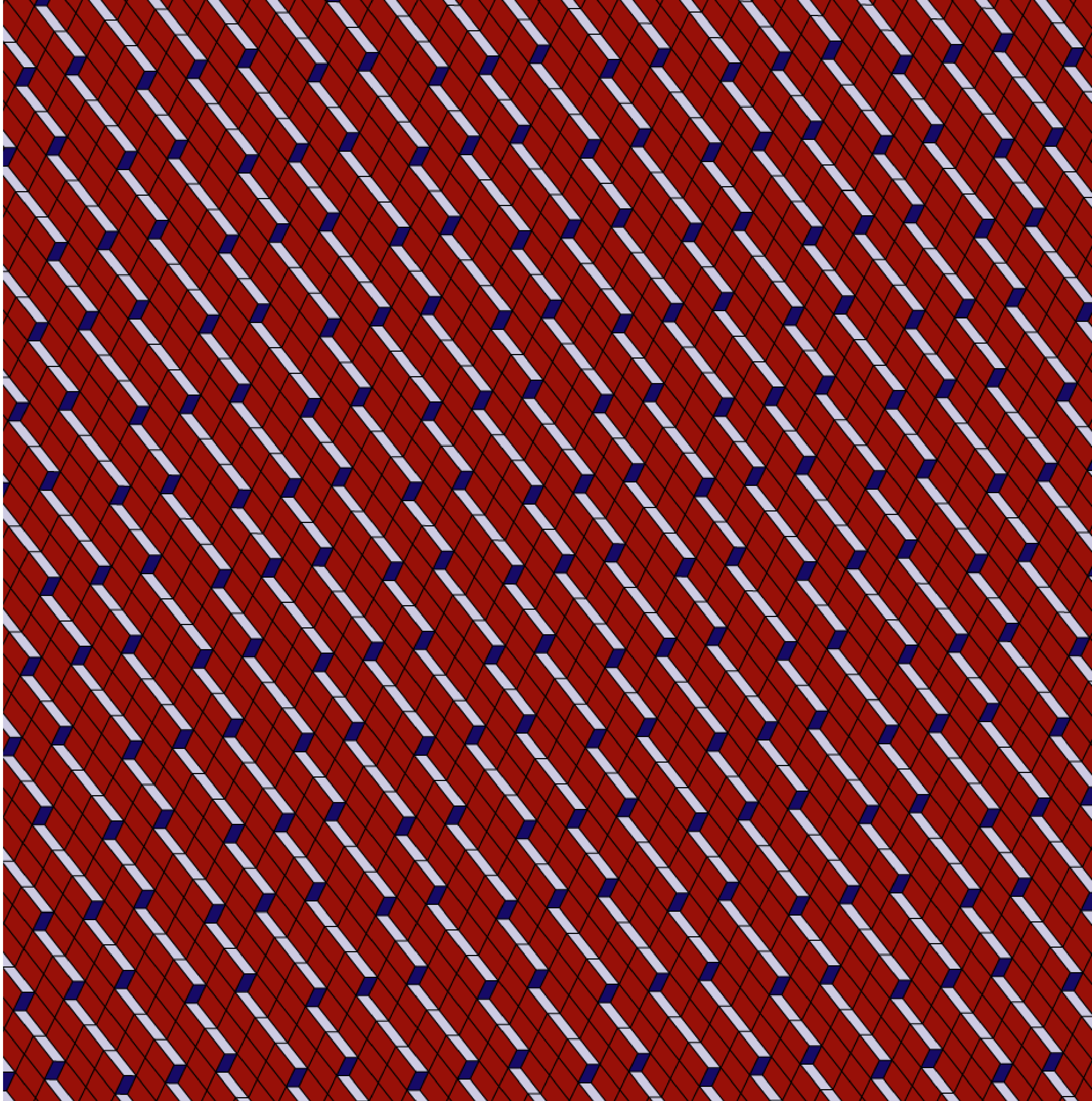


Aperiodic Tilings

“Kenyon (1,2,1) Polygon”

R. Kenyon

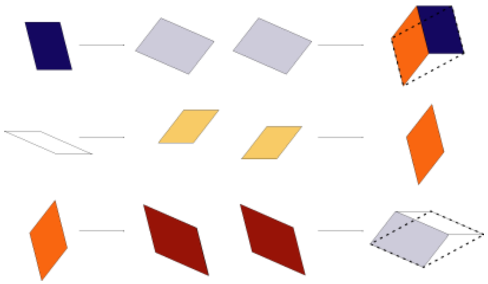


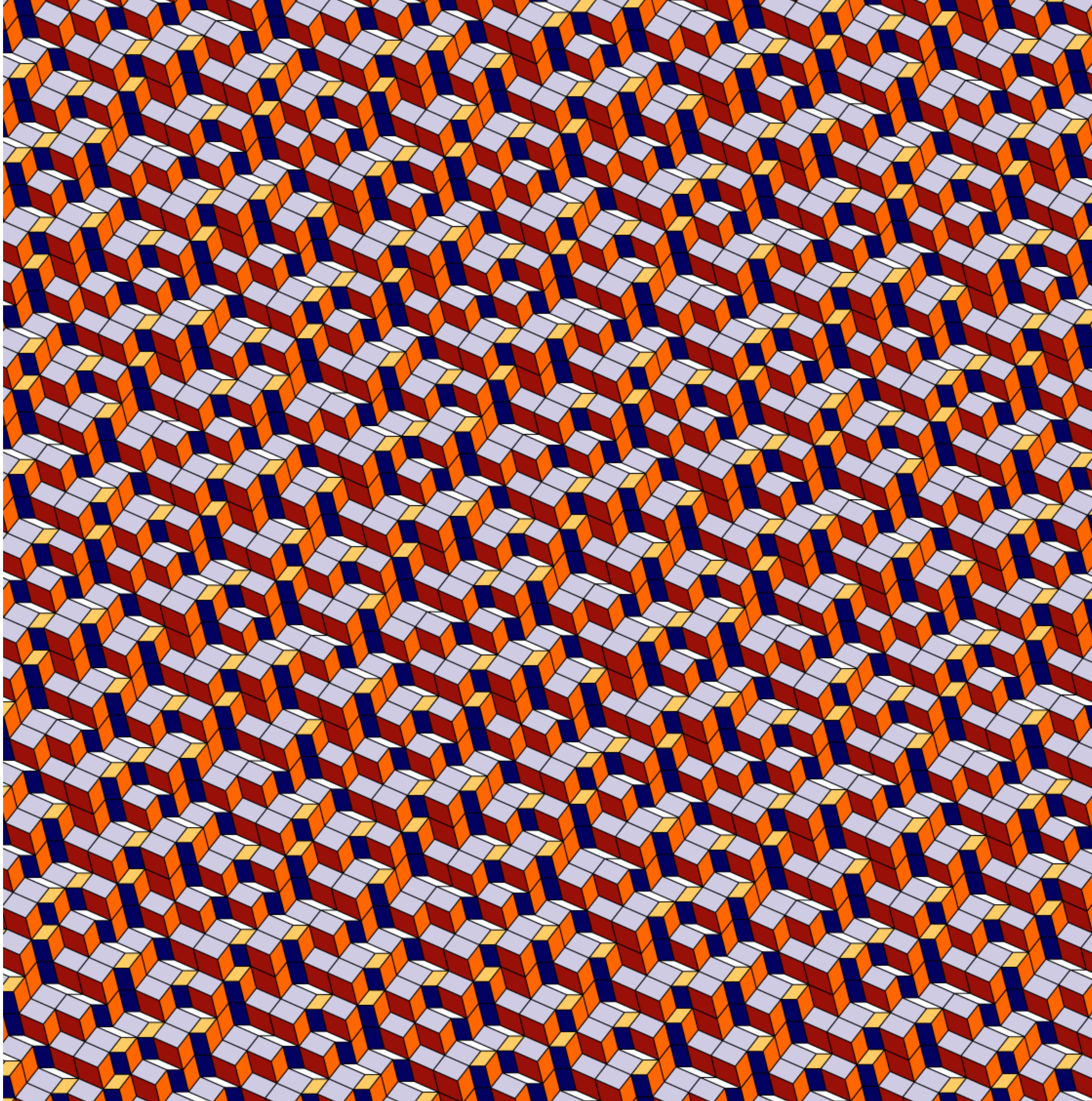


Aperiodic Tilings

"Kenyon 2
Polygonal"

R. Kenyon

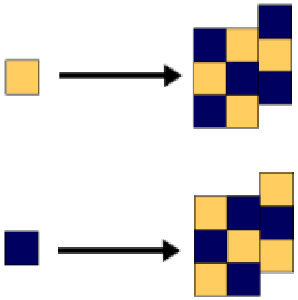


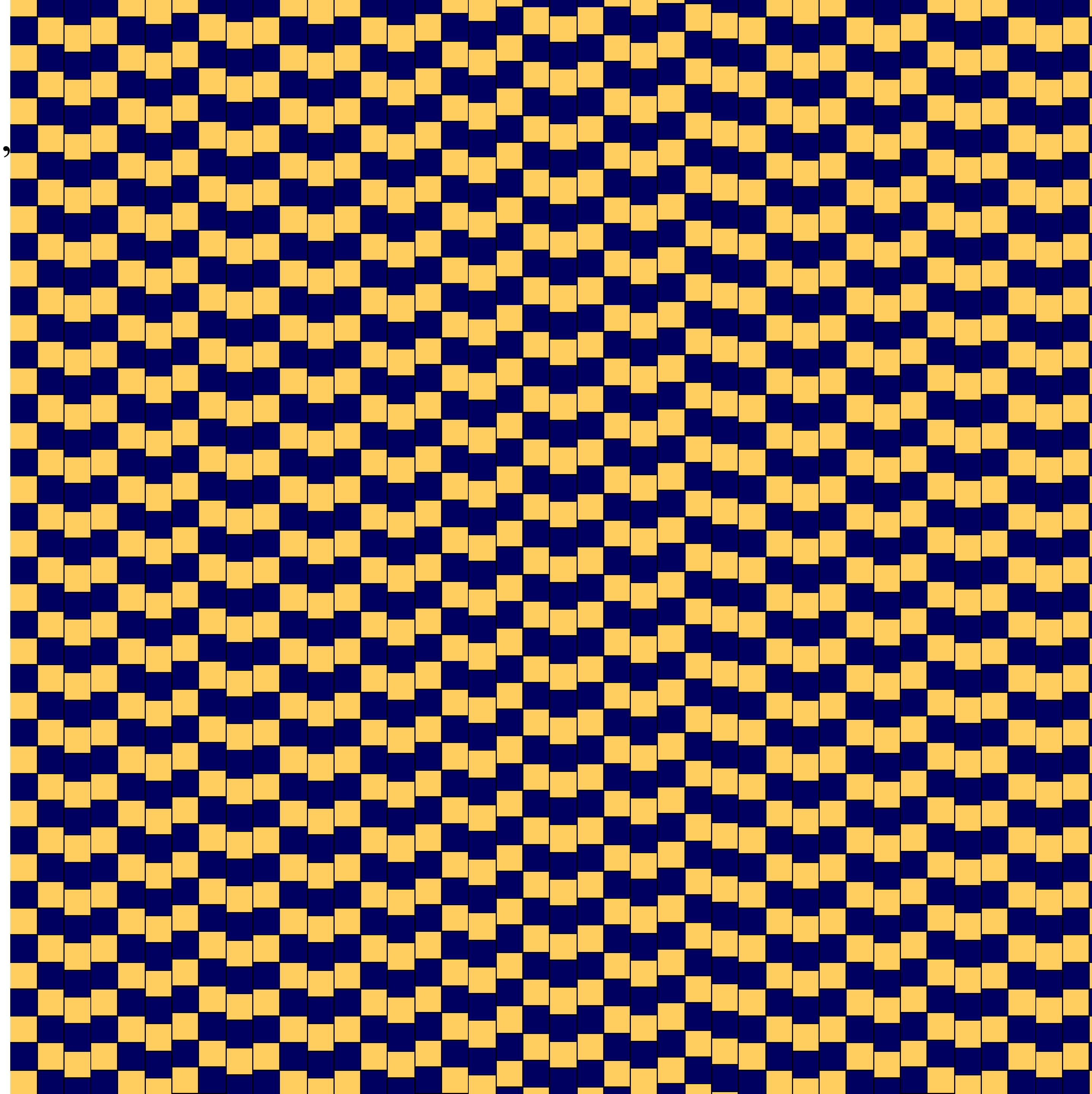


Aperiodic Tilings

“Kenyon non FLC”

R. Kenyon

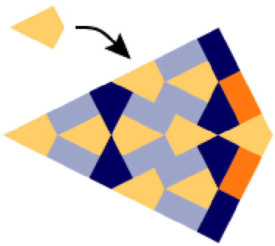


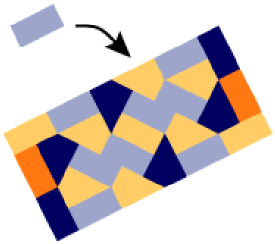


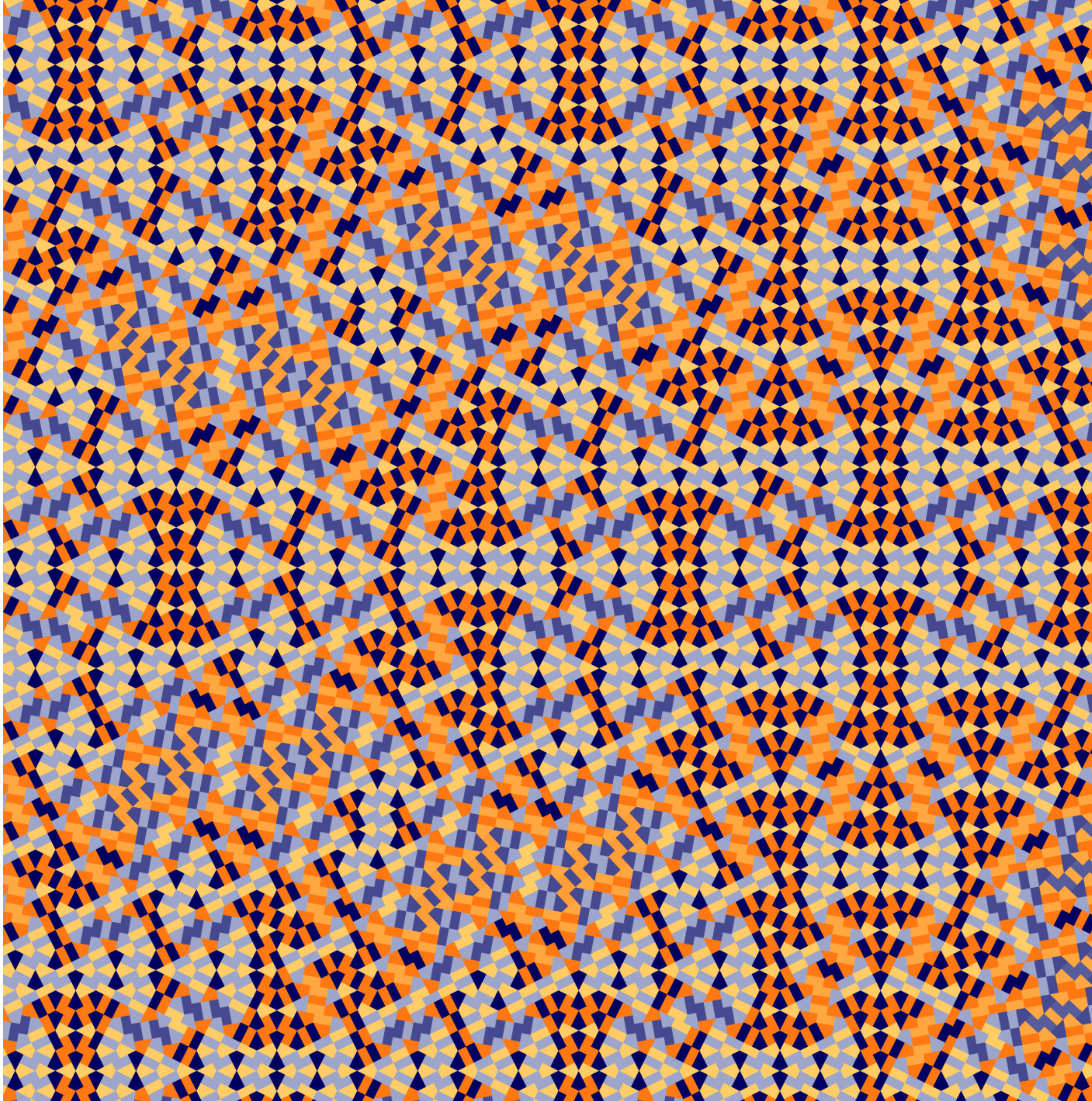
Aperiodic Tilings

"Kite-Domino"

D. Frettlöh and M. Baake, 1994



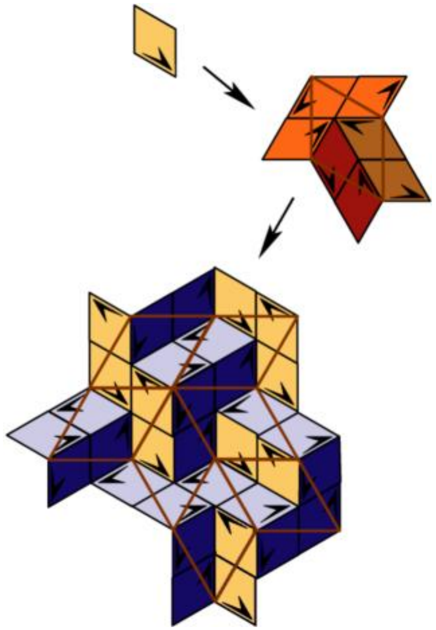


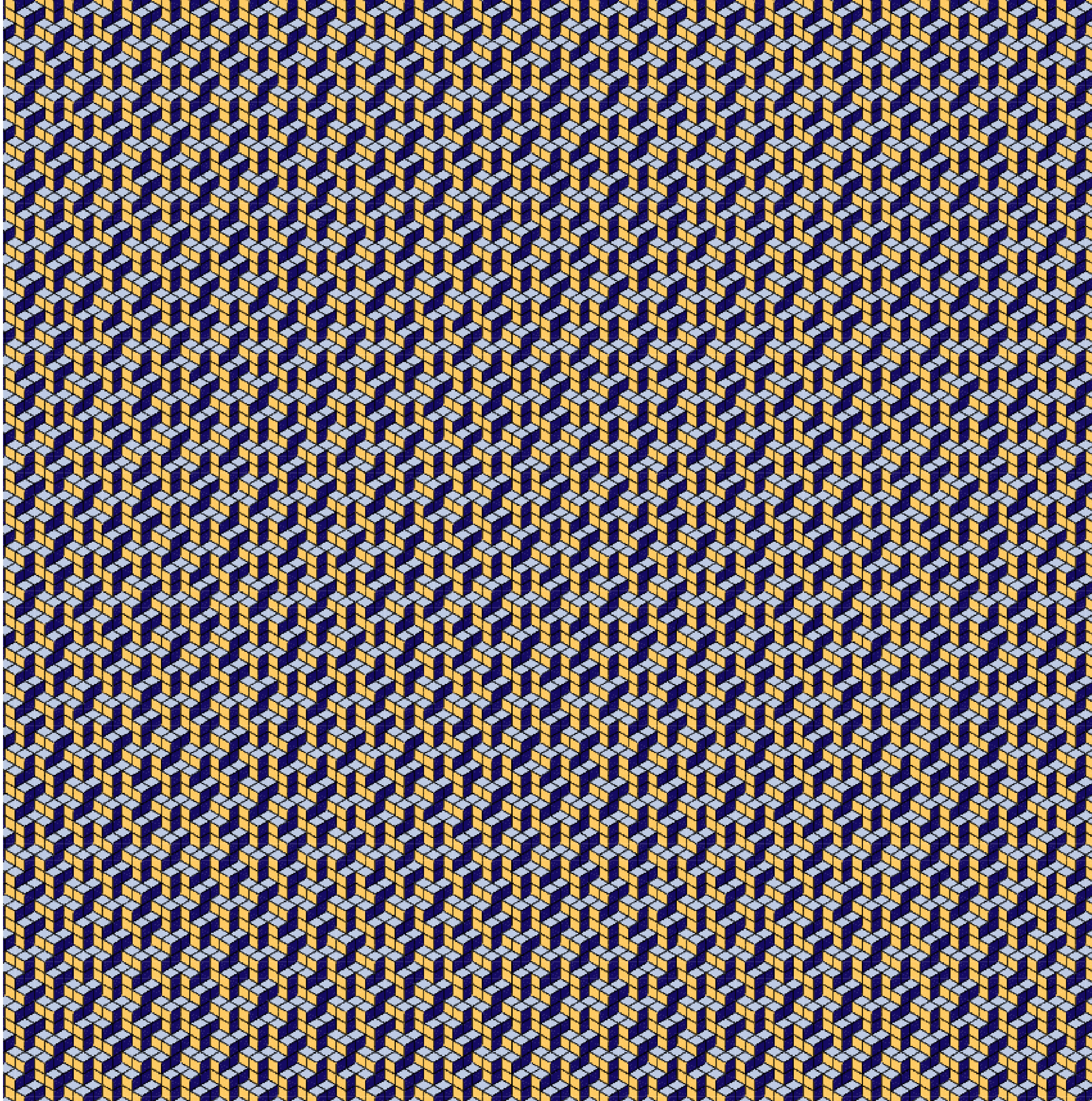


Aperiodic Tilings

"Lord"

E. Lord

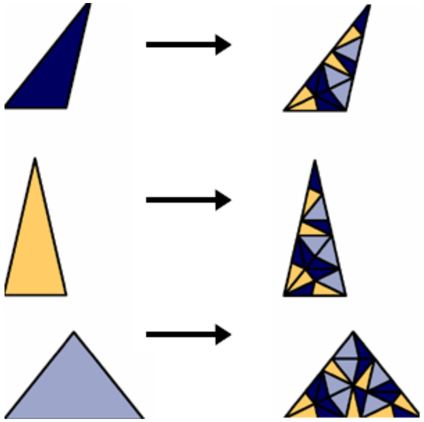


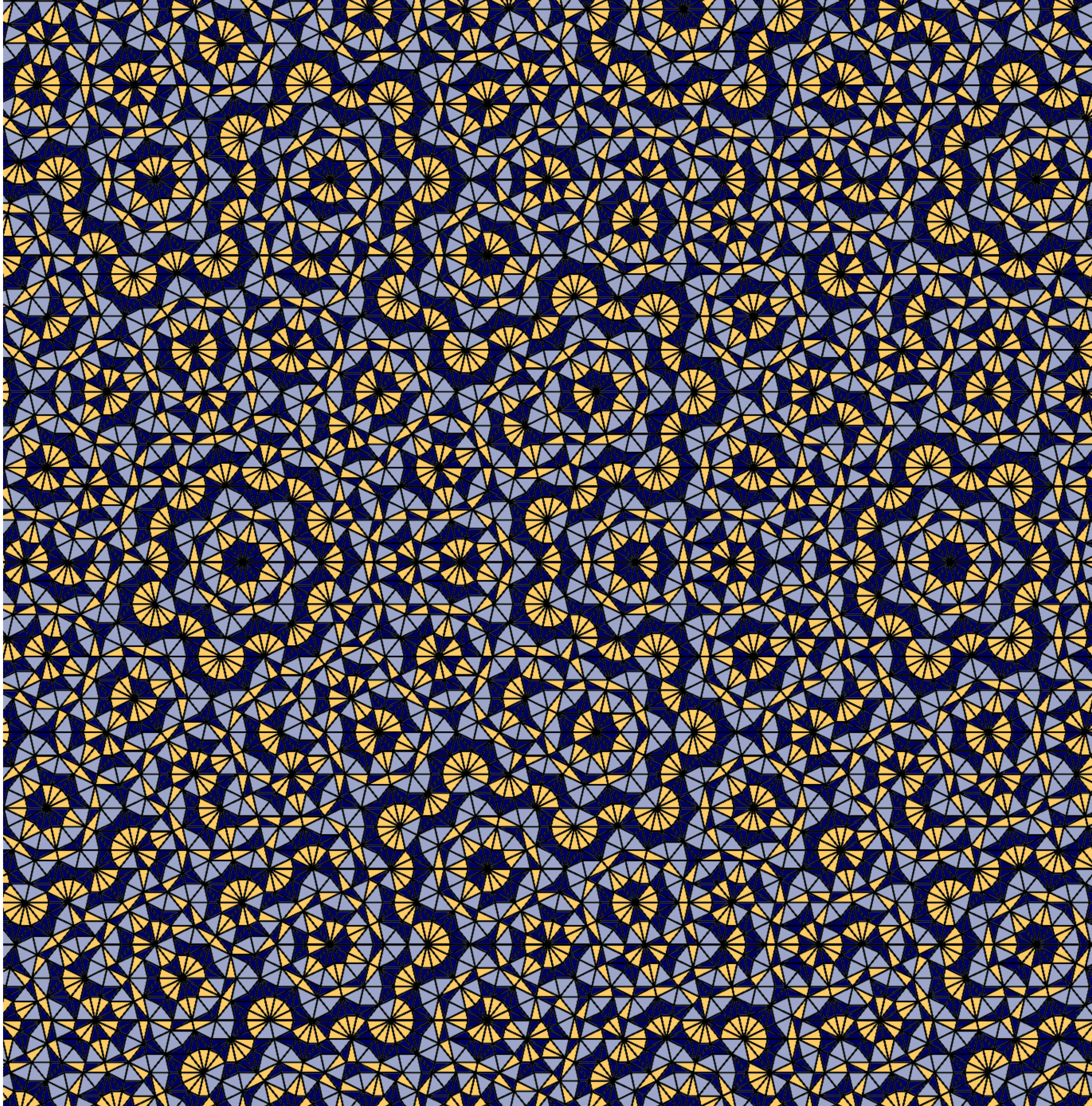


Aperiodic Tilings

“Maloney’s 7-fold”

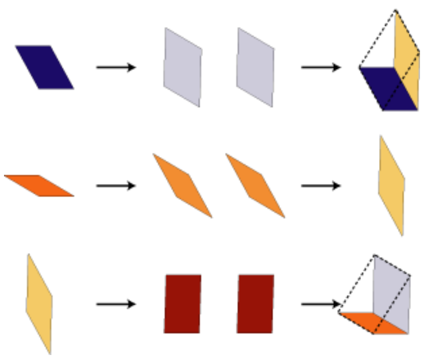
G. Maloney

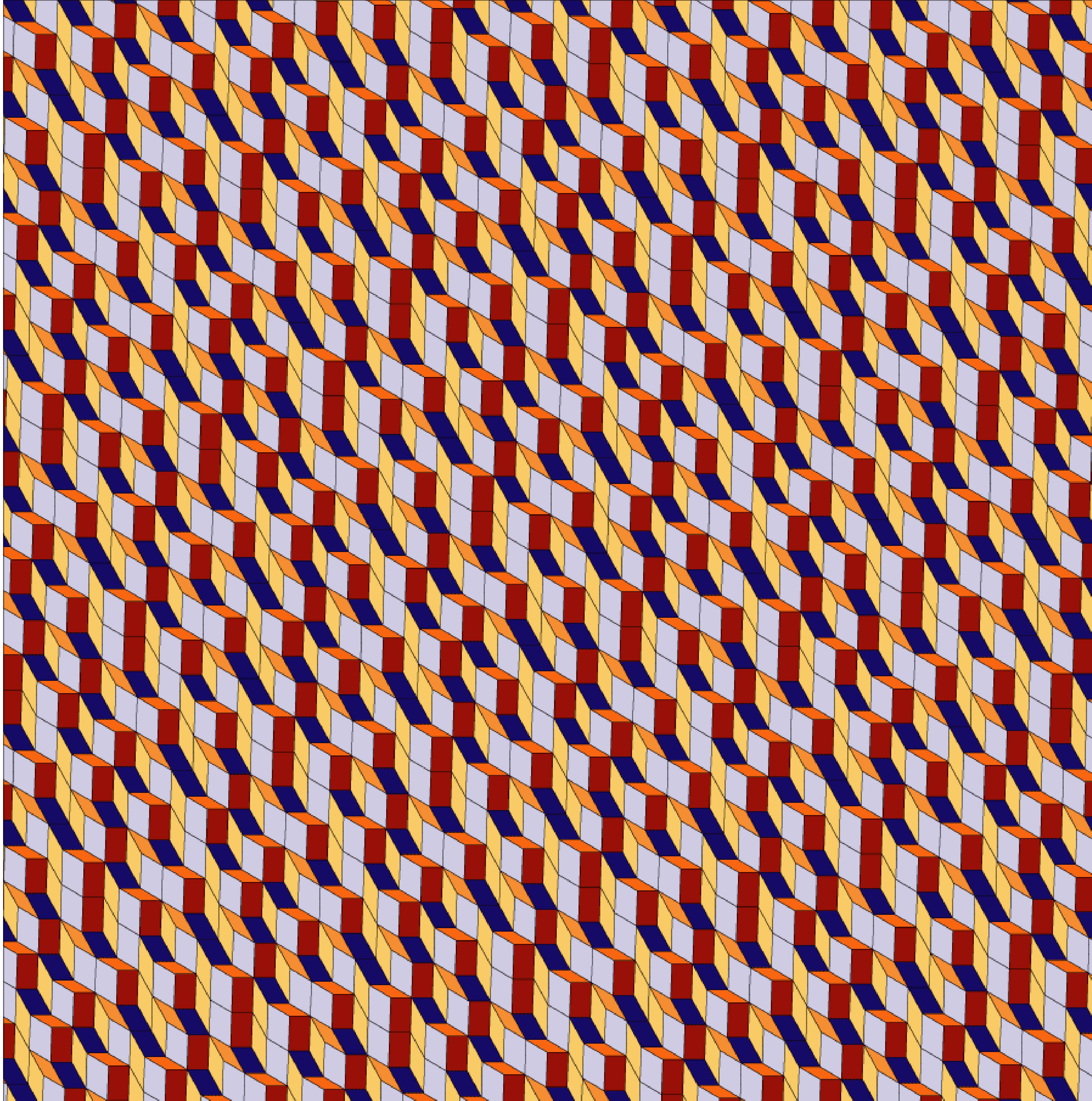




Aperiodic Tilings

"Nautilus"
P. Arnoux,
M. Furukado,
E. Harriss,
and S. Ito





Aperiodic Tilings

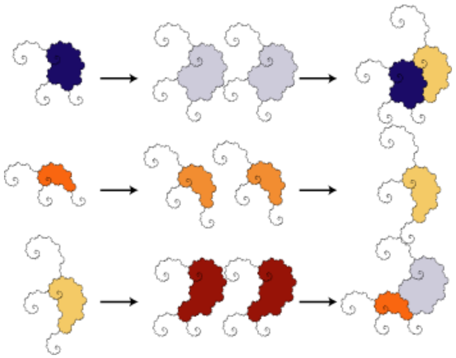
"Nautilus (volume hierarchic"

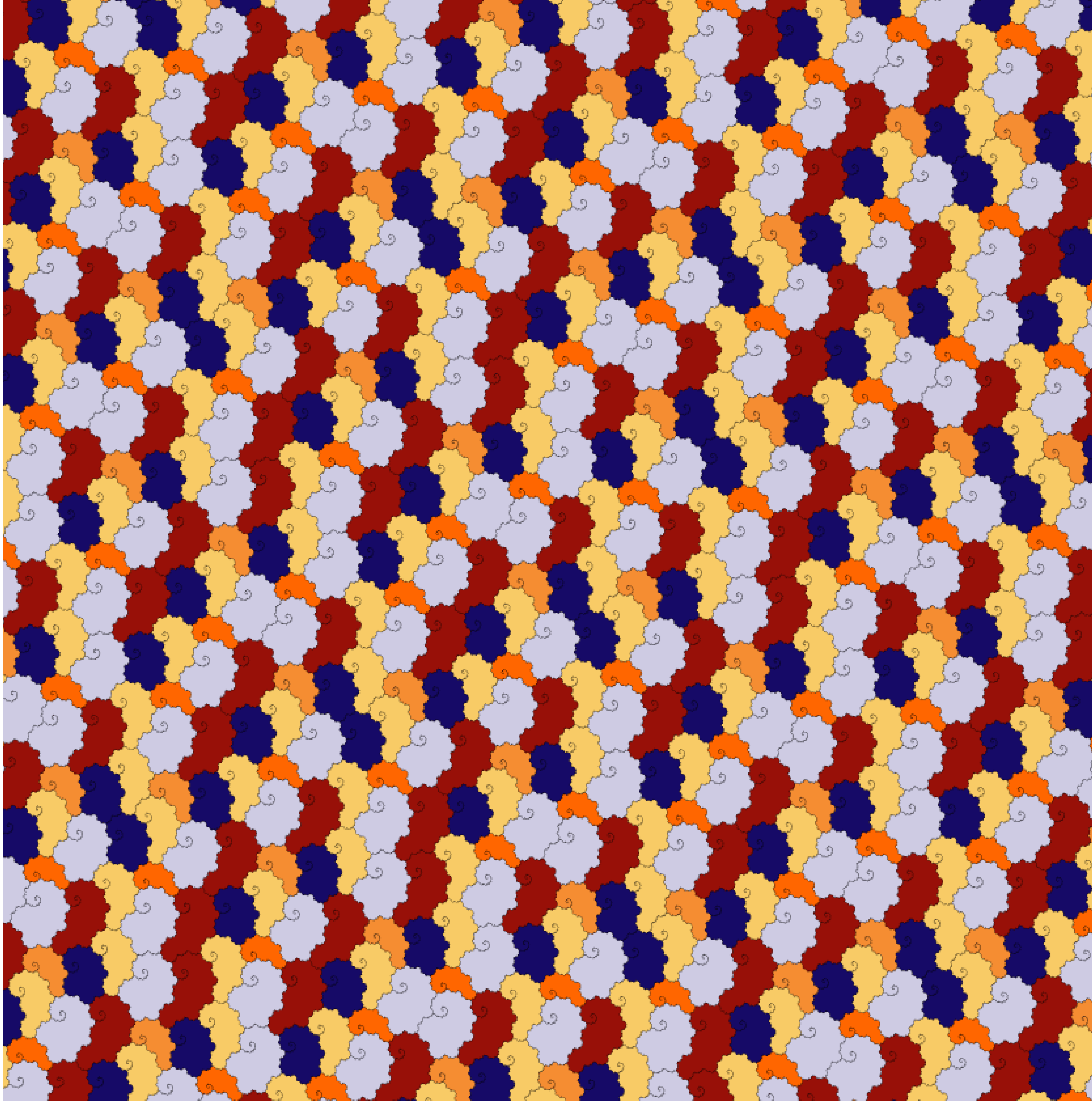
P. Arnoux,

M. Furukado,

E. Harriss,

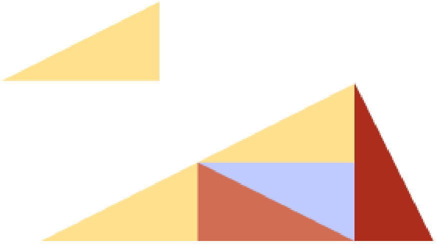
and S. Ito





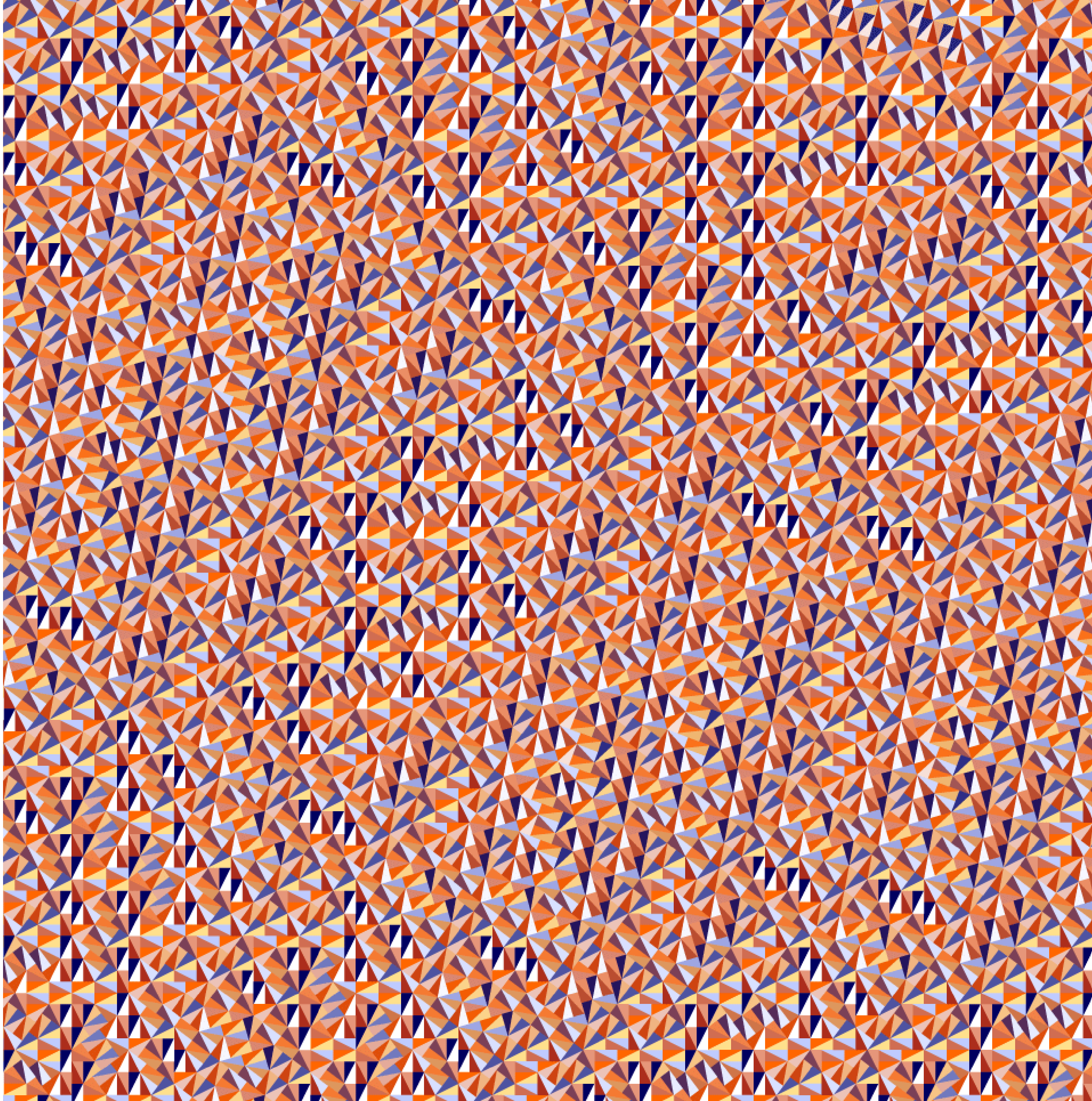
Aperiodic Tilings

"Pinwheel"
John Conway
and C. Radin



Tiles occur in infinitely many orientations!

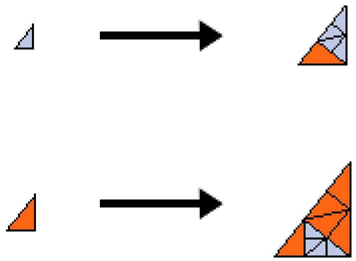
Despite irrational edge lengths and incommensurable angles, all vertices of tiles have rational coordinates!

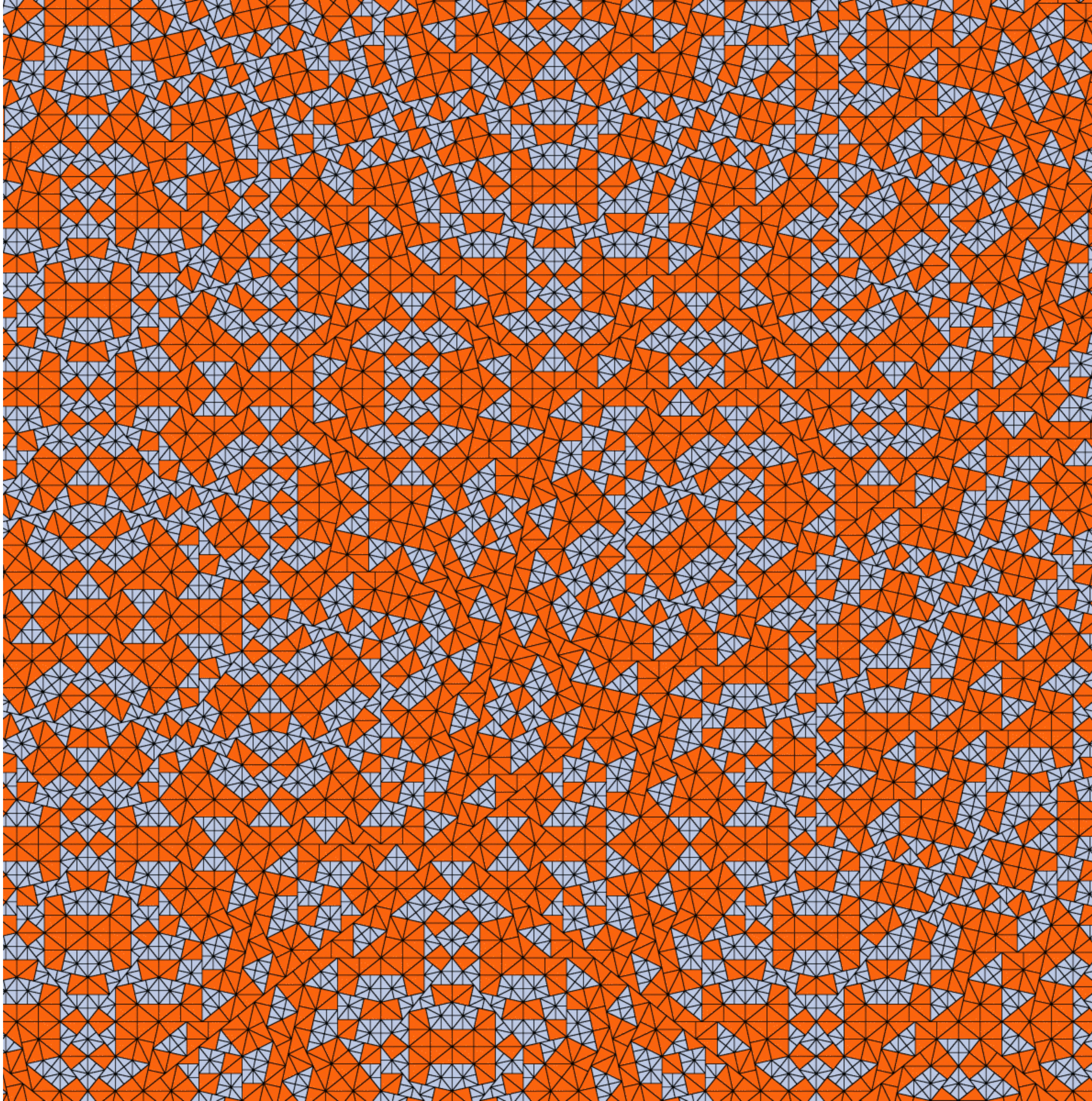


Aperiodic Tilings

“Pinwheel-3-1”

L. Sadun, 1998

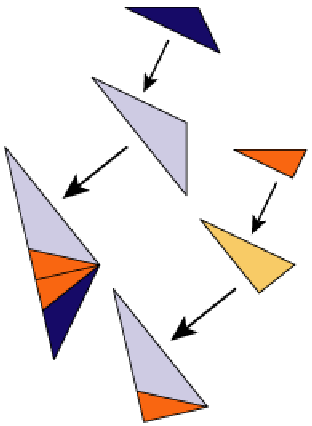




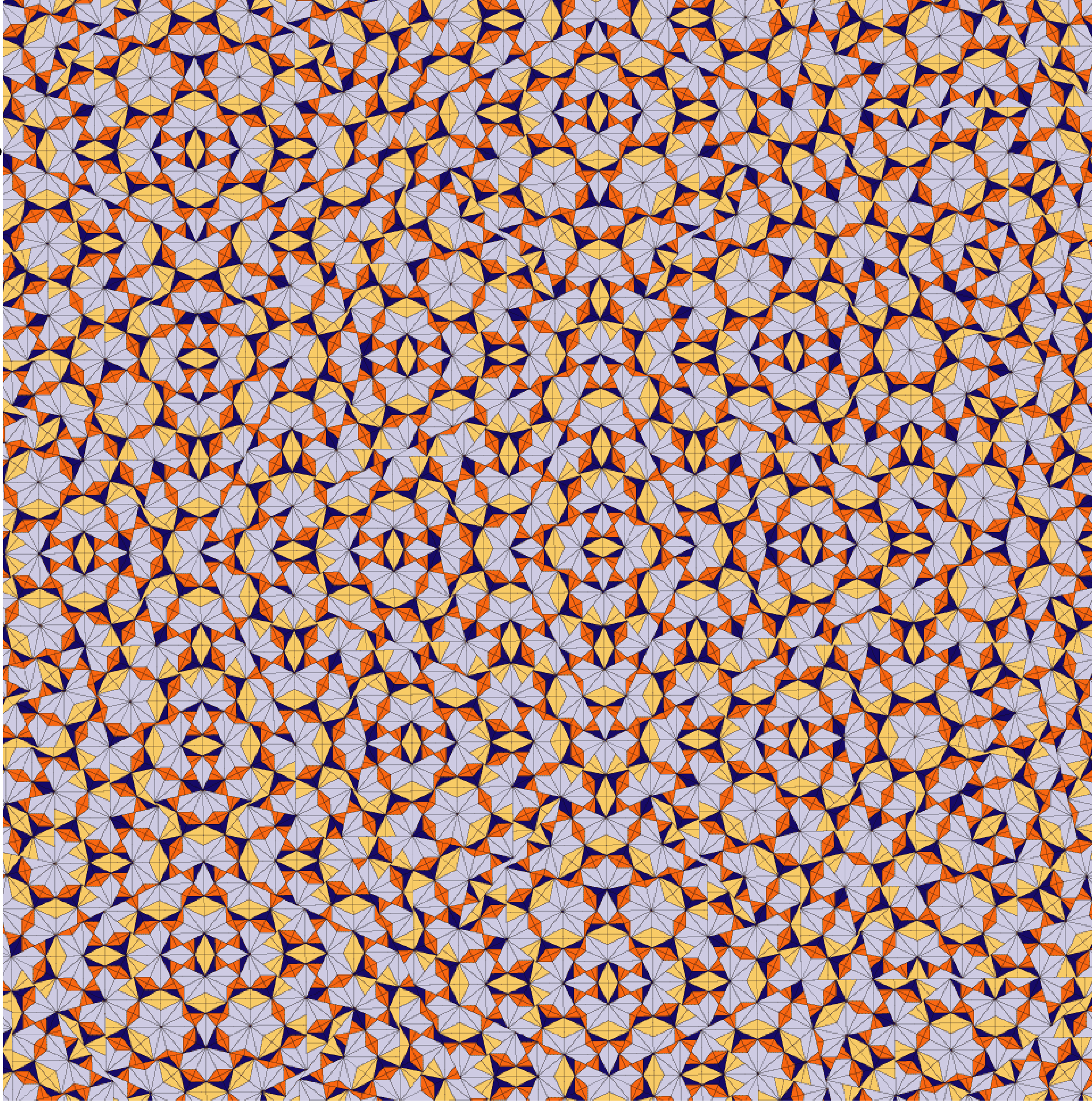
Aperiodic Tilings

“Quartic Pinwheel”

L. Sadun, 1998

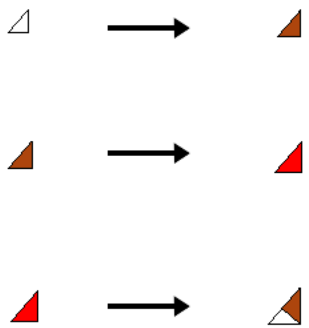


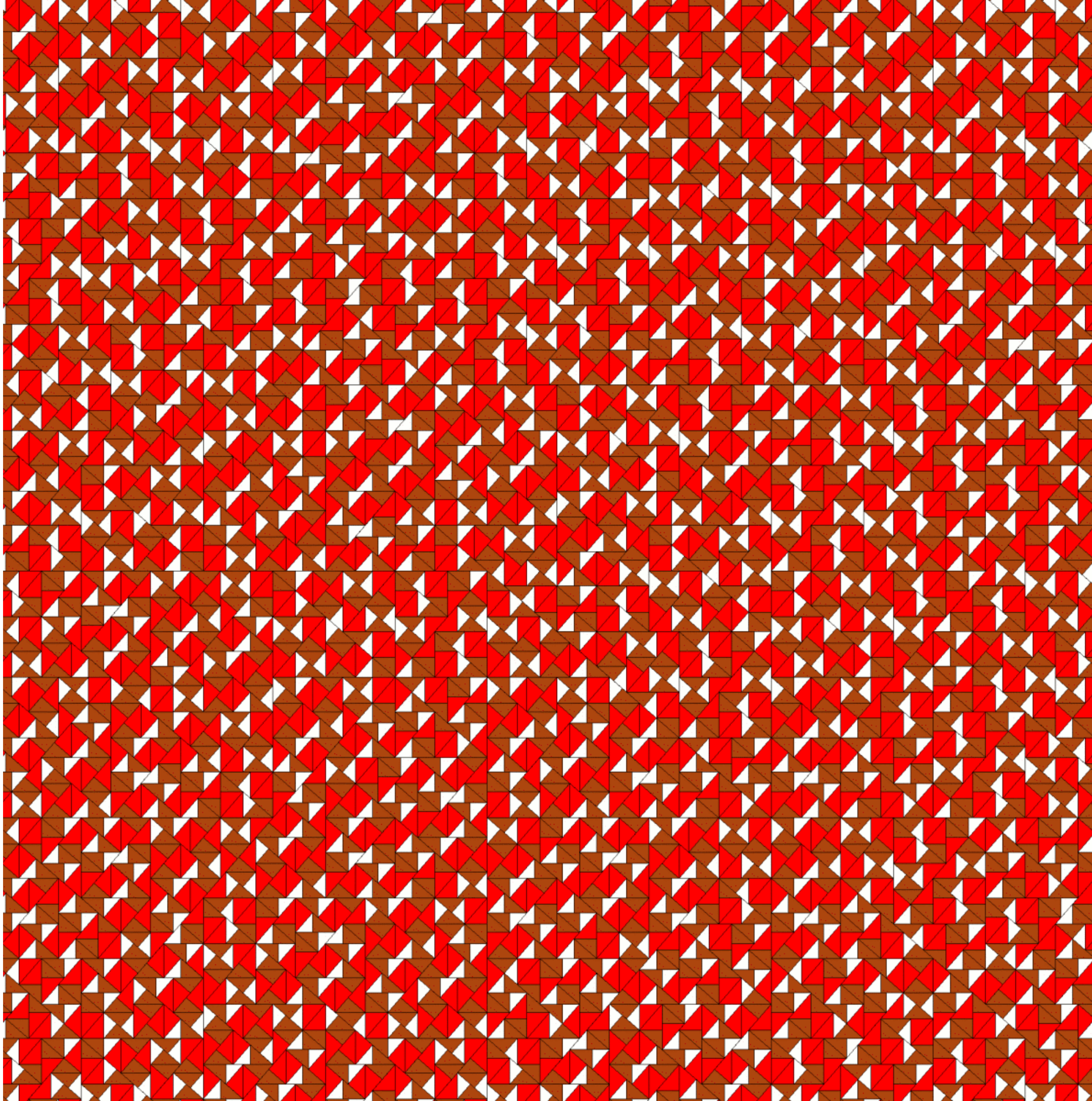
Tiles occur in infinitely many orientations!



Aperiodic Tilings

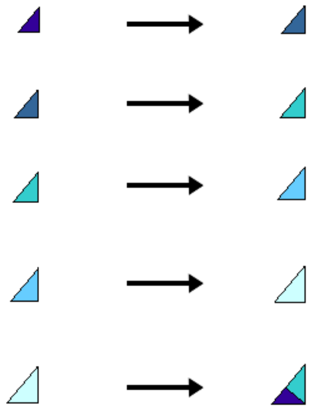
"Pythagoras-3-1"
J. Pieniak

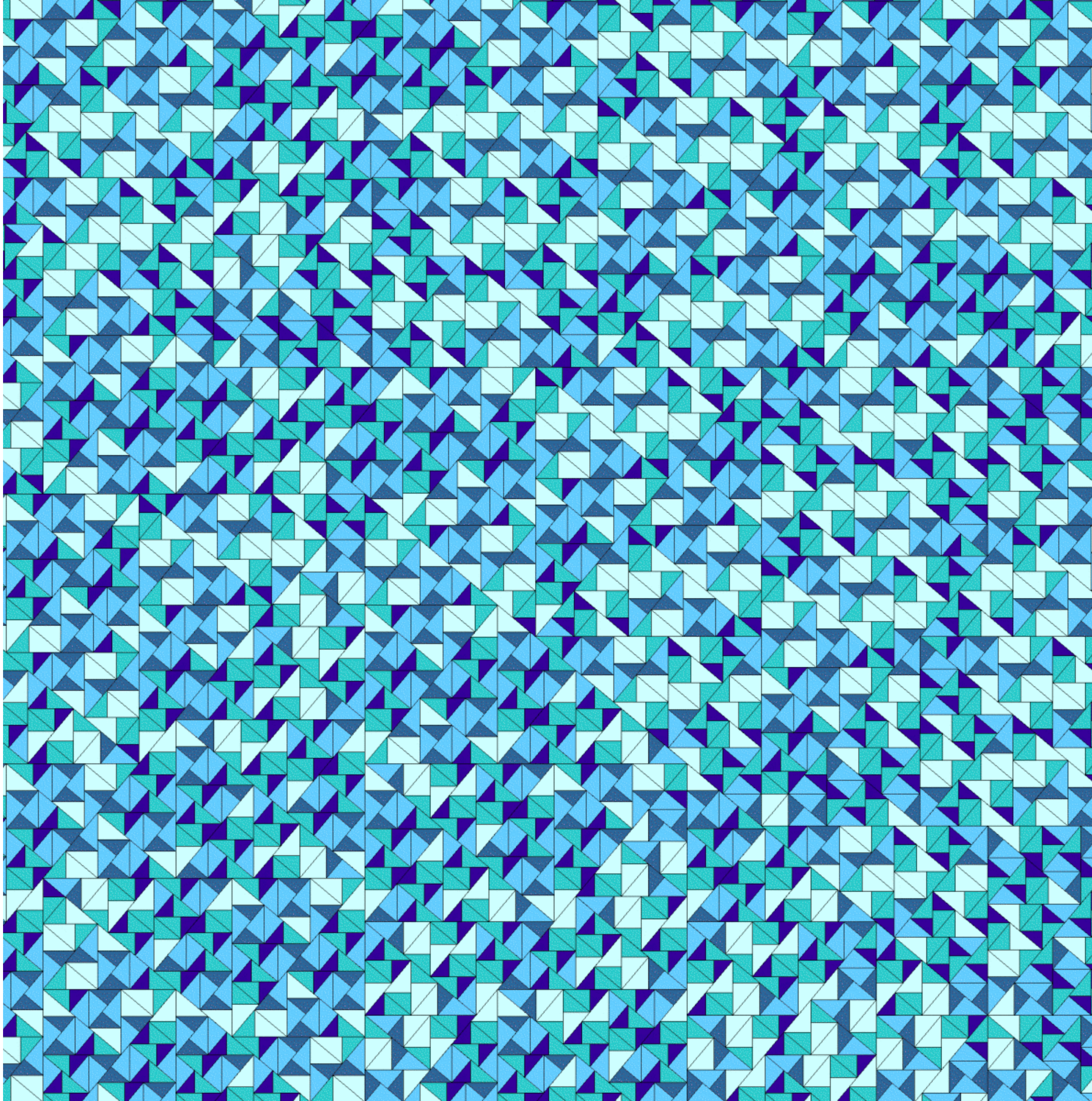




Aperiodic Tilings

"Pythagoras-3-1"
J. Pieniak

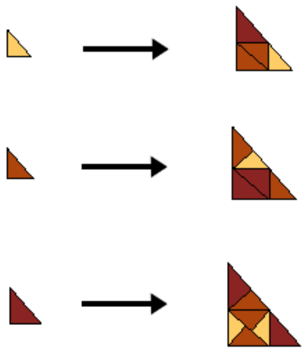




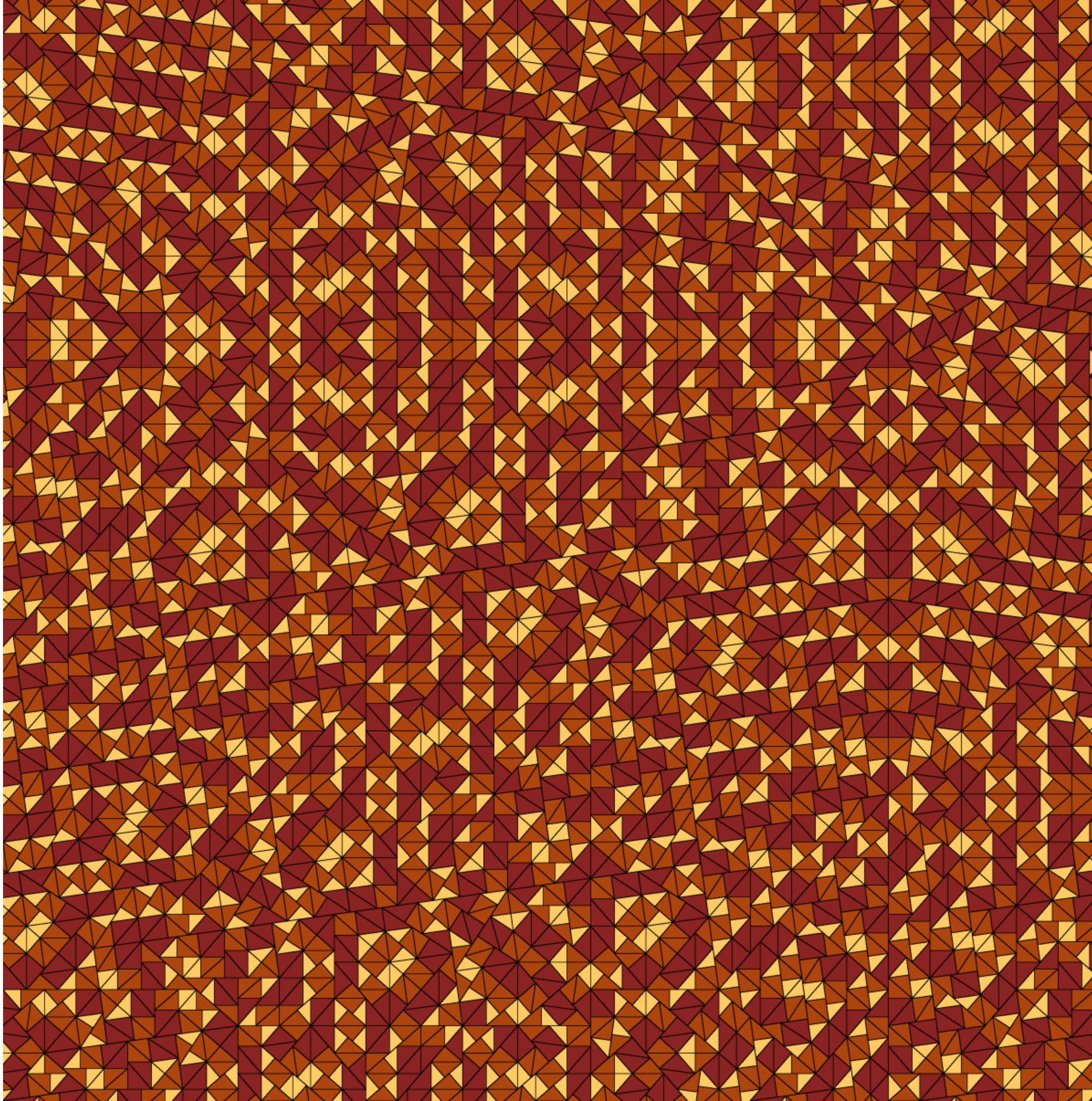
Aperiodic Tilings

"Pythia-3-1"

D. Frettlöh



Tiles occur in infinitely many orientations with statistical equidistribution !



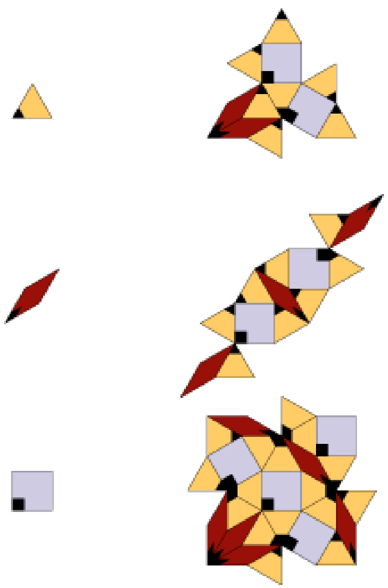
Aperiodic Tilings

"Watanabe Ito Soma 12-fold"

Y. Watanabe,

T. Soma and

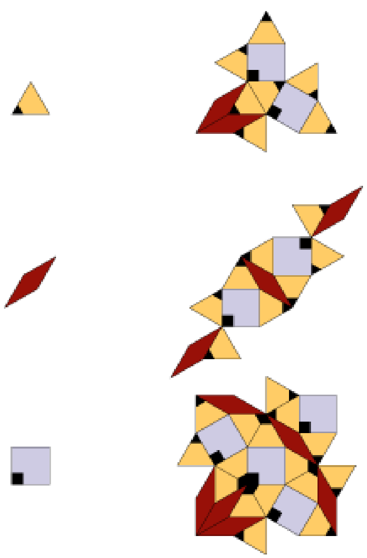
M. Ito, 1995

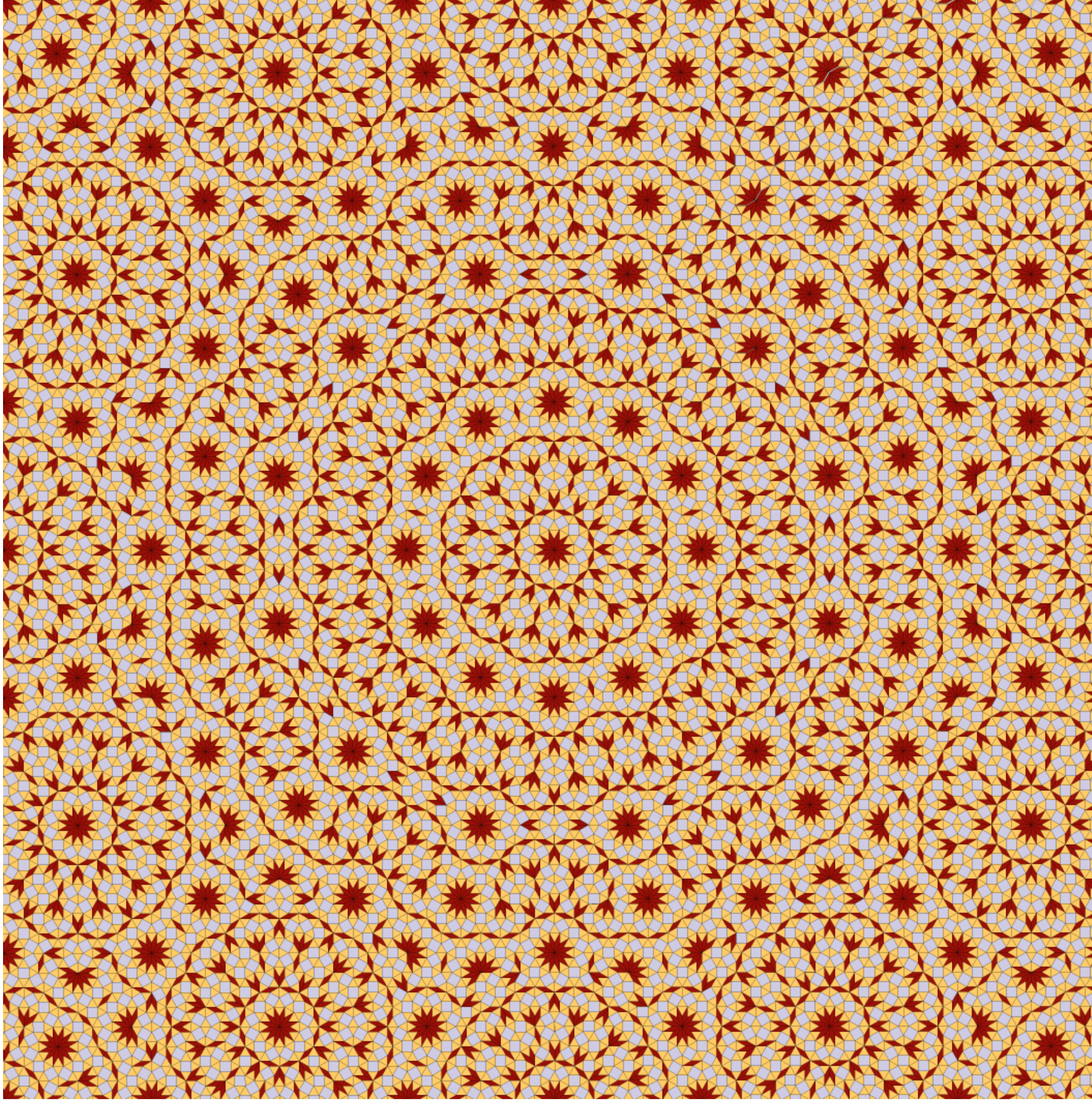


Aperiodic Tilings

"Watanabe Ito
Soma 12-fold
(variant)"

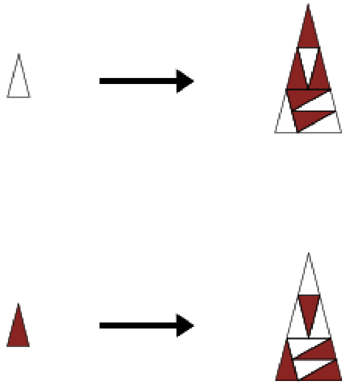
Y. Watanabe,
T. Soma and
M. Ito, 1995

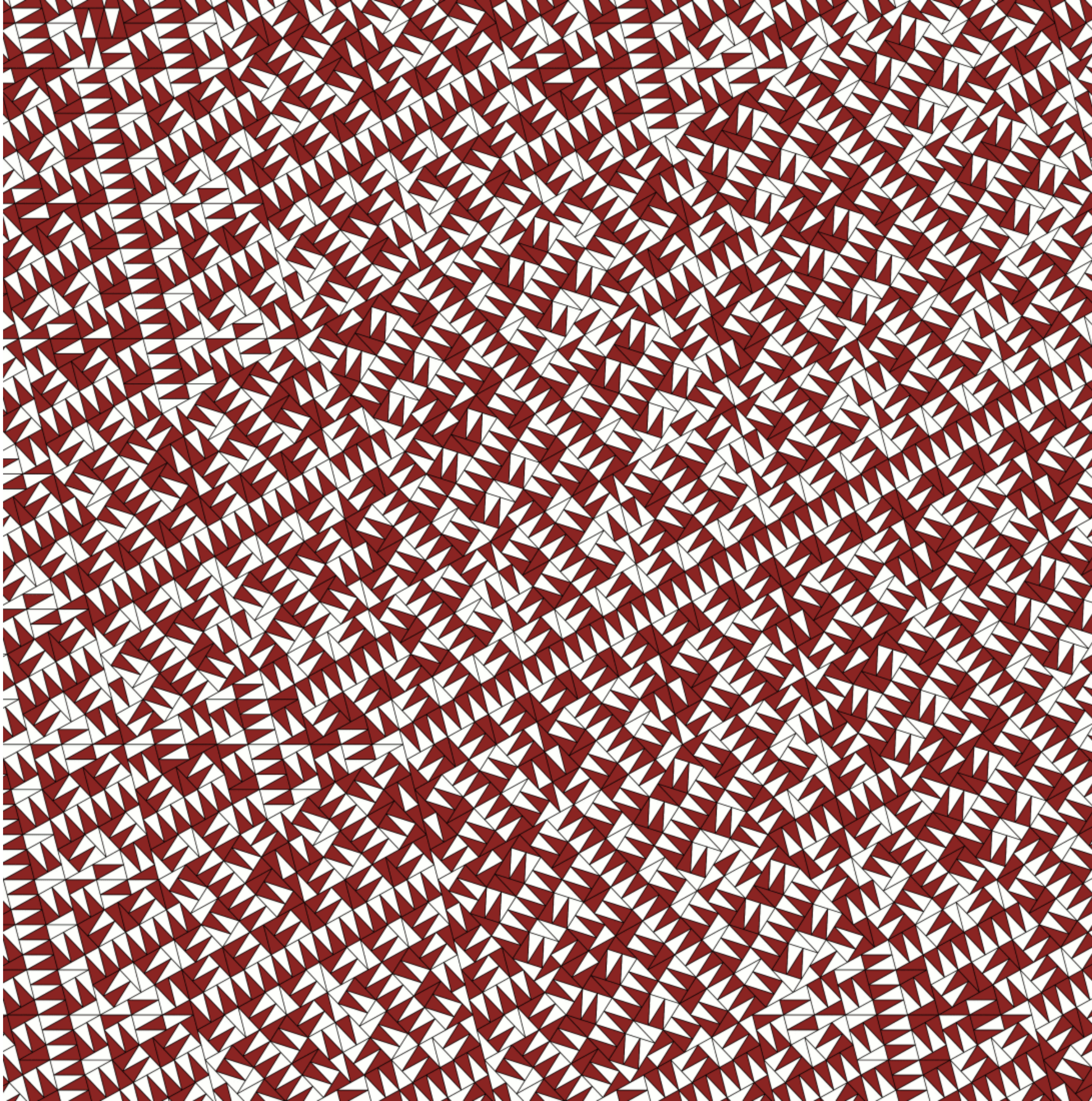




Aperiodic Tilings

“Viper”

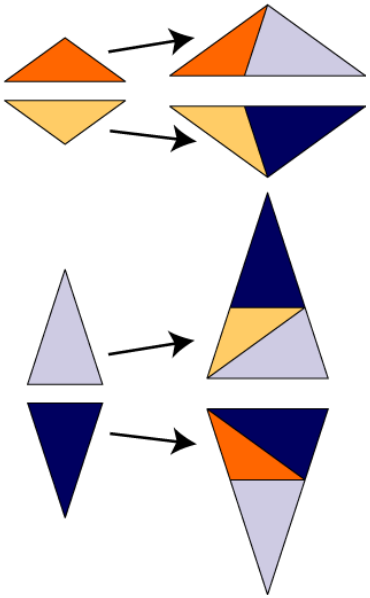


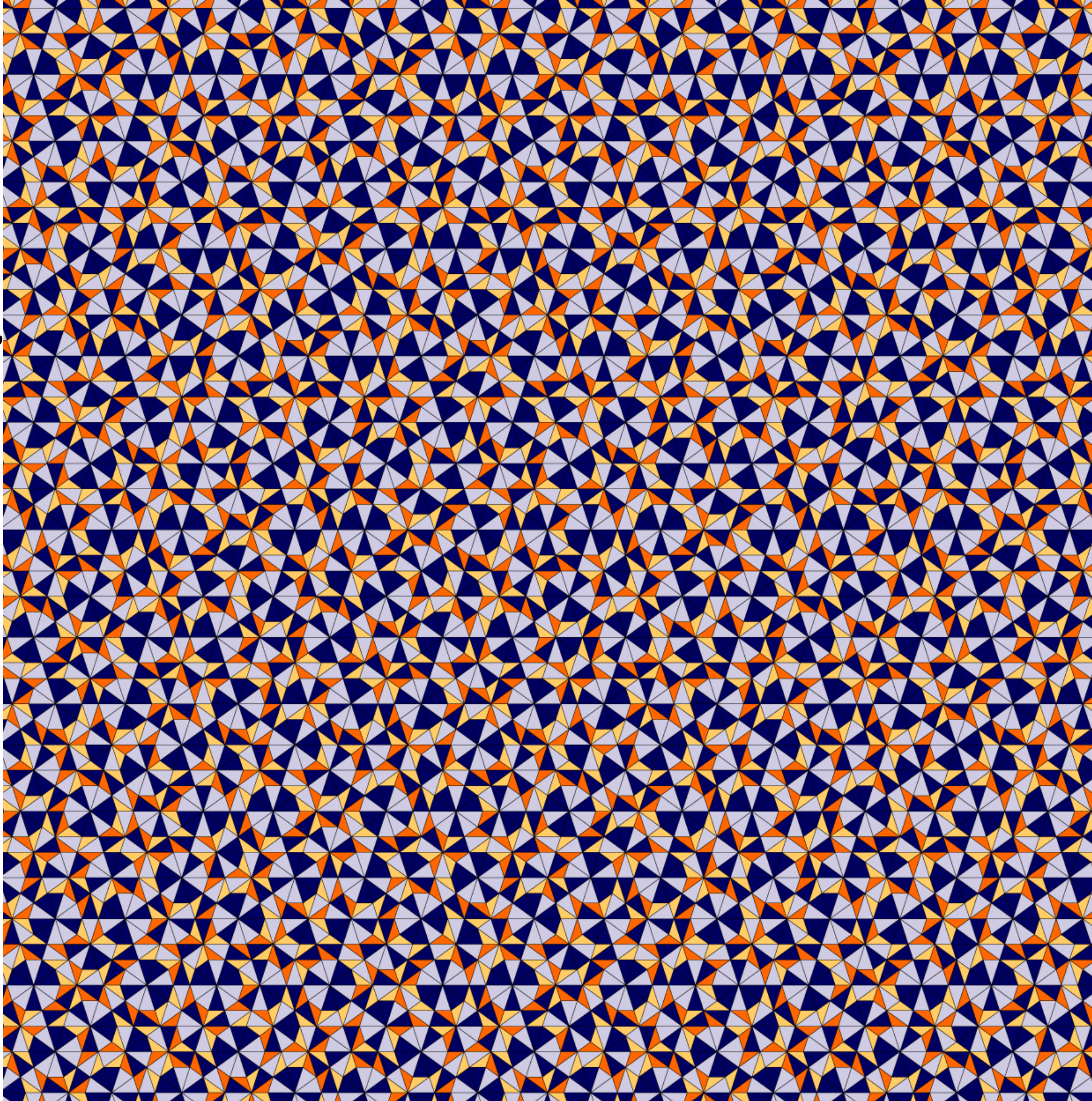


Aperiodic Tilings

"Tuebingen Triangle"

R. Lück, M. Baake, M. Schlottmann, 1990

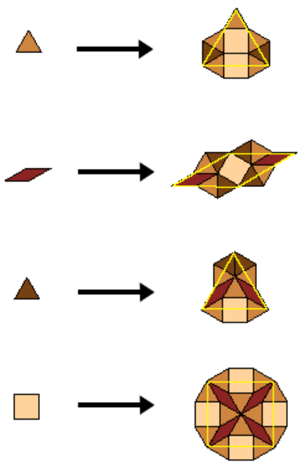


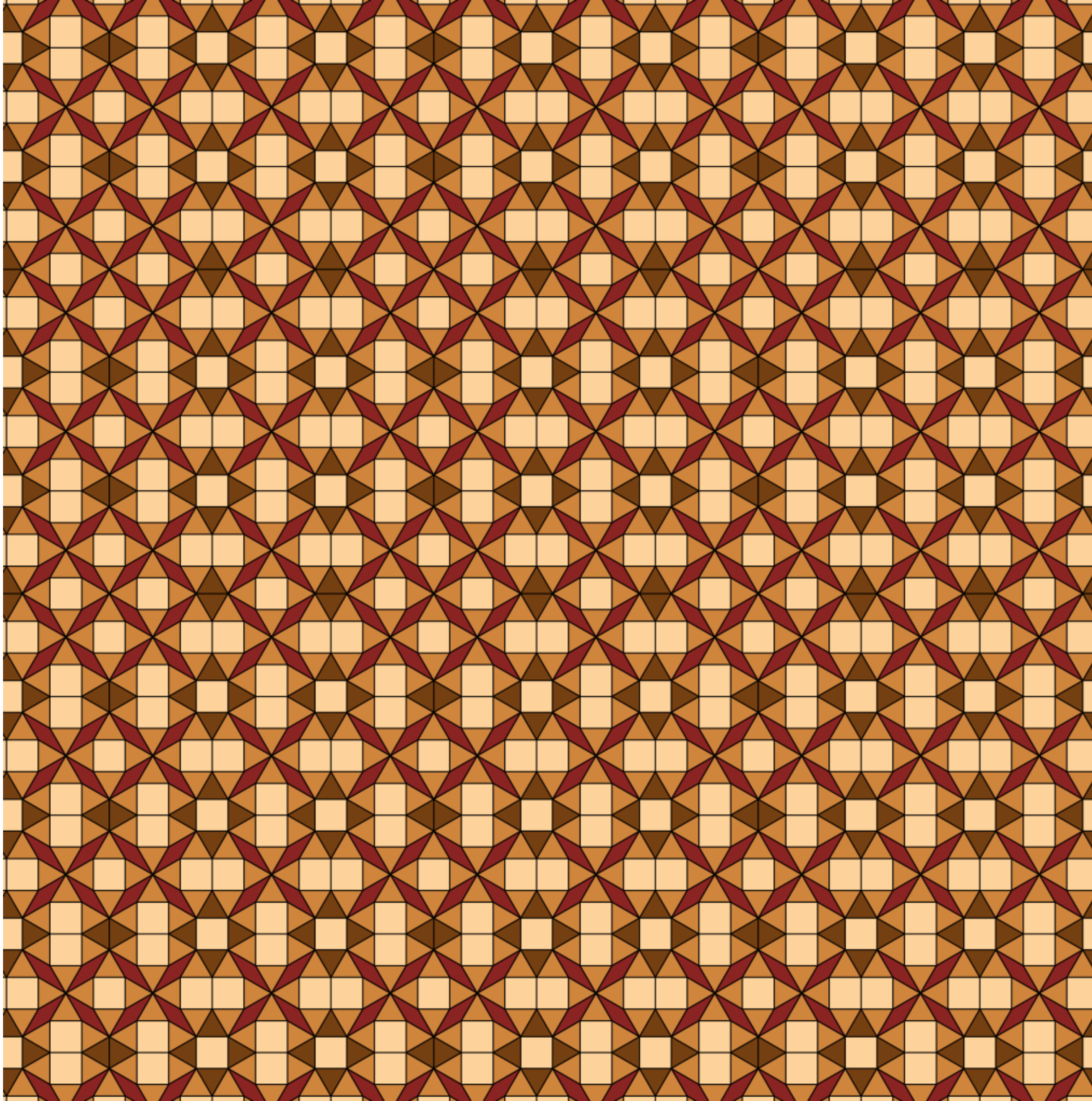


Aperiodic Tilings

“Rorschach”

B. Sing, 2007

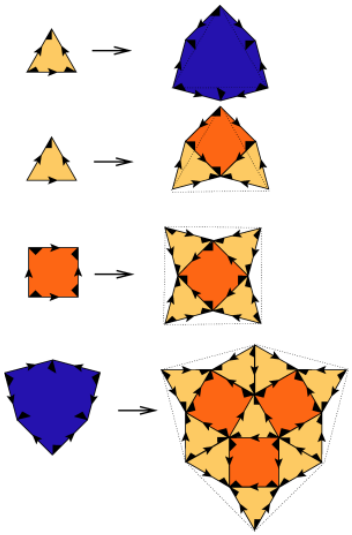


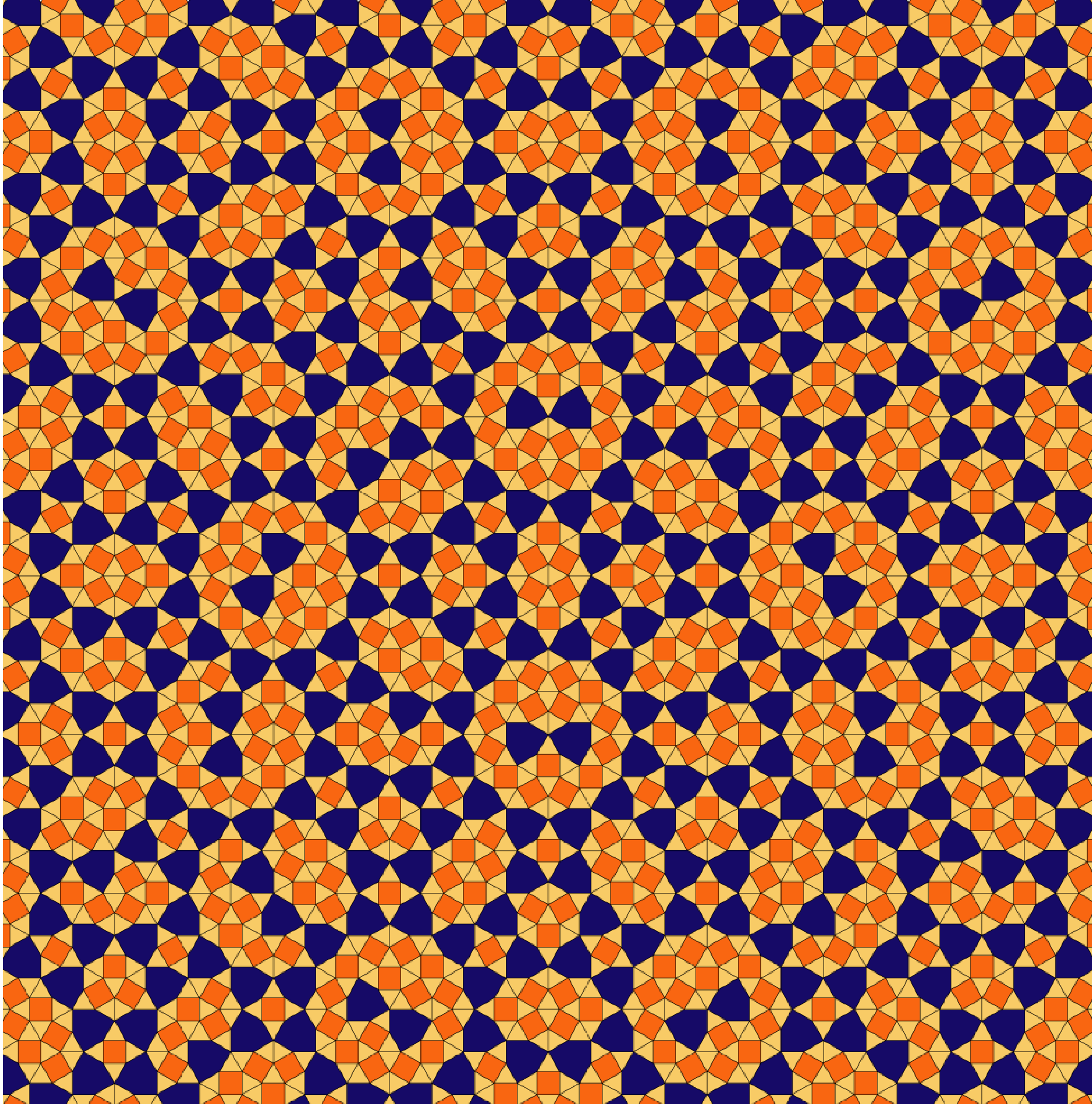


Aperiodic Tilings

"Shield"

F. Gähler, 1988

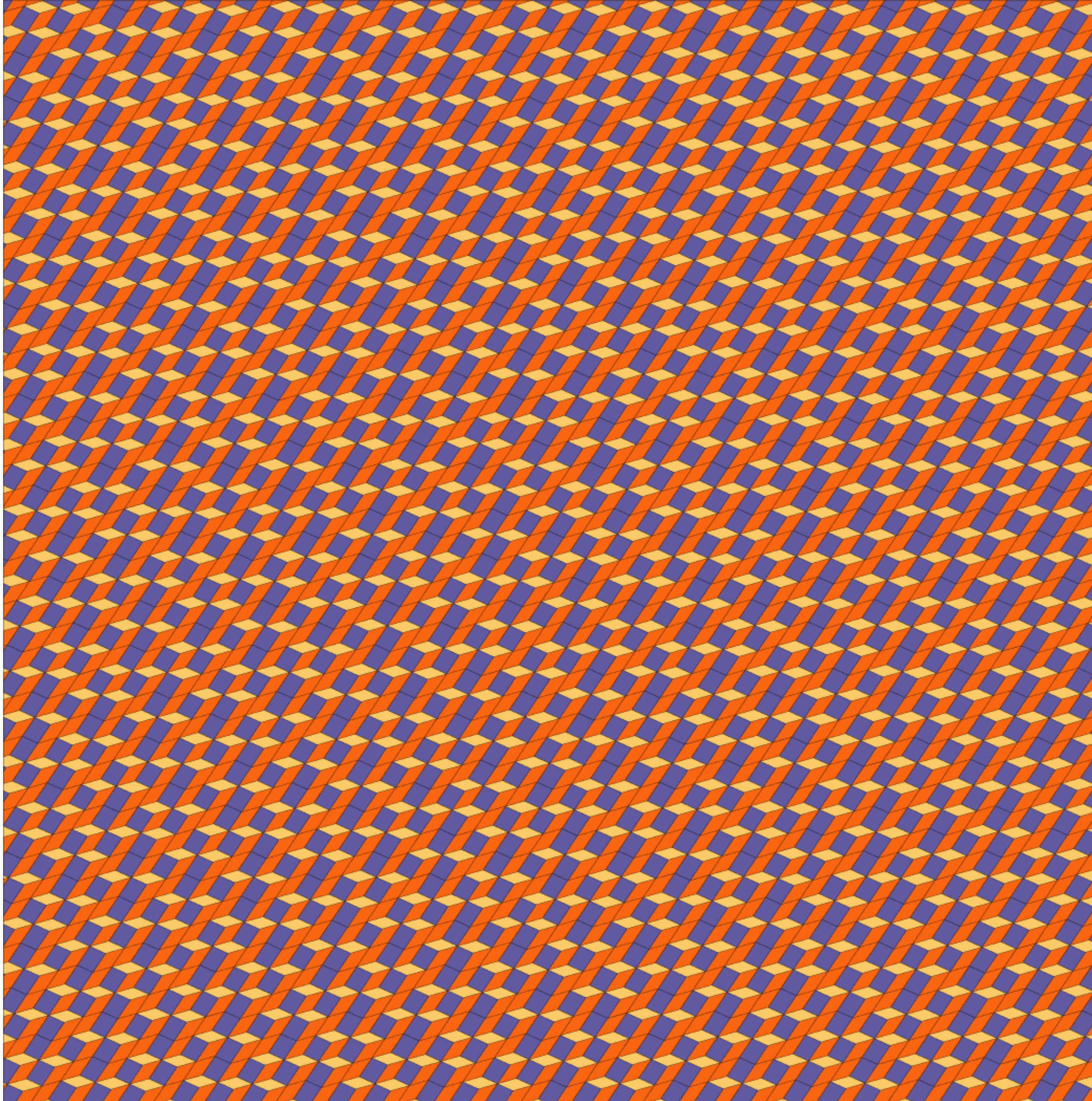




Aperiodic Tilings

“Smallest Pisot (dual)”

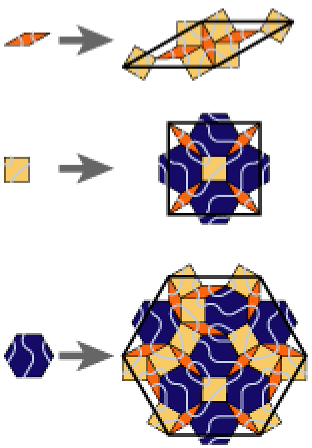
E. Harriss

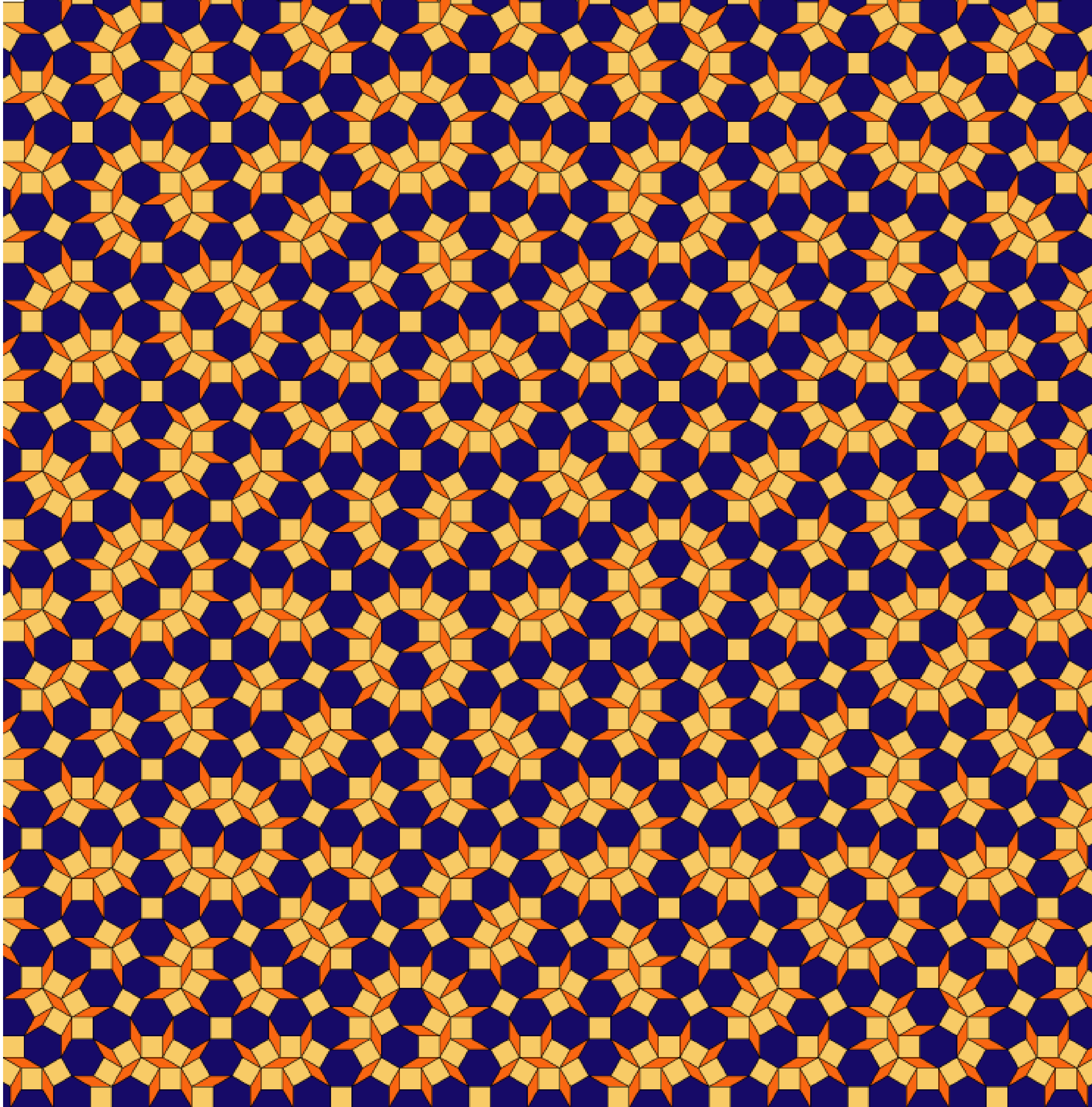


Aperiodic Tilings

“Socolar”

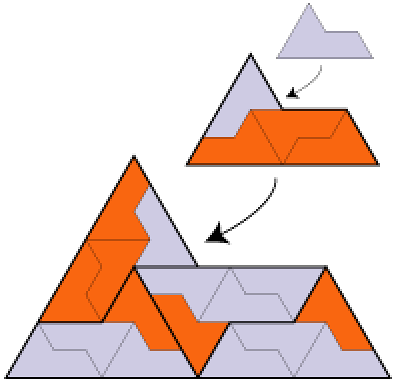
J. E. S. Cocolar,
1989

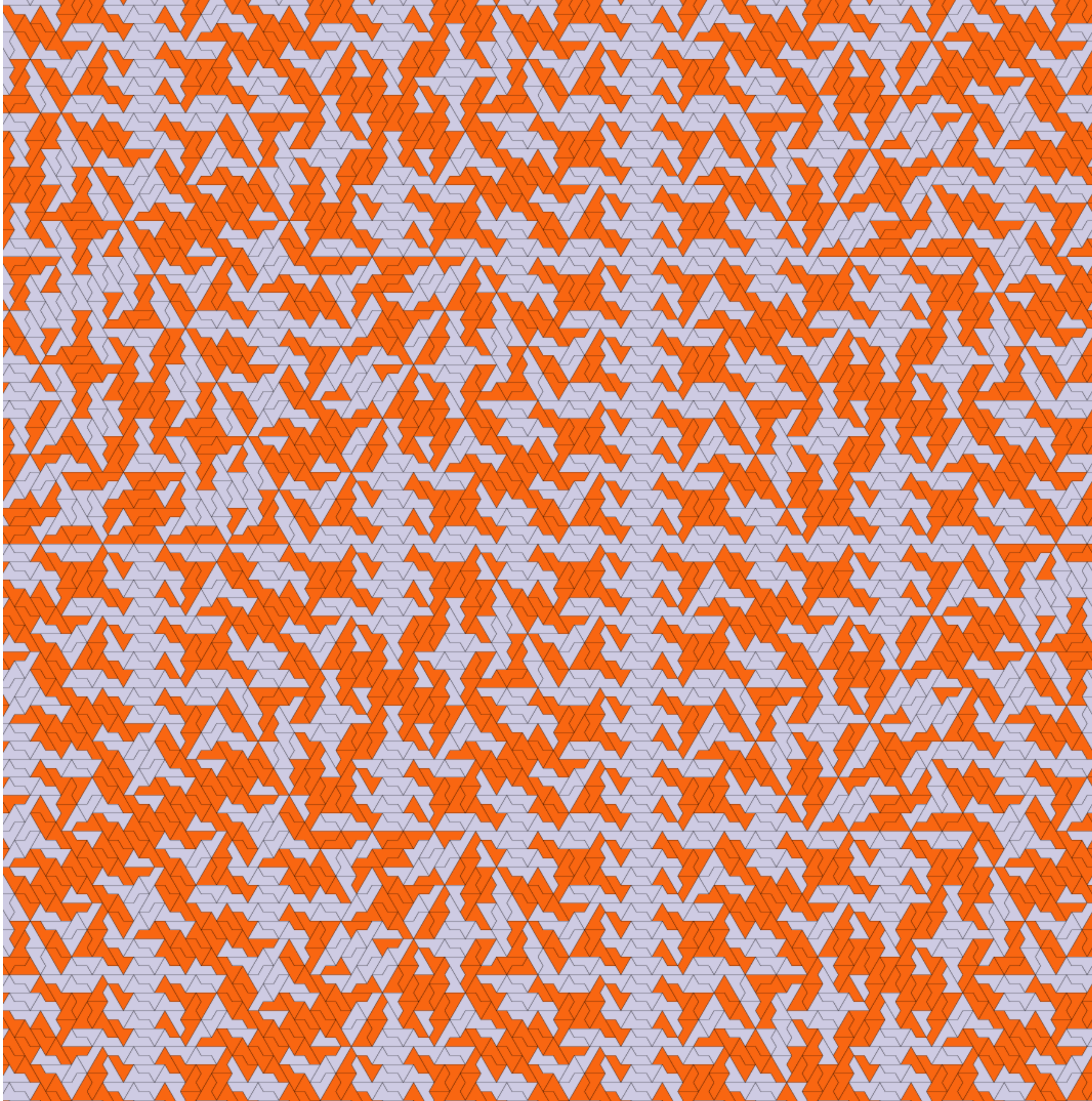




Aperiodic Tilings

"Sphinx"
J.-Y. Lee, and
R. V. Moody

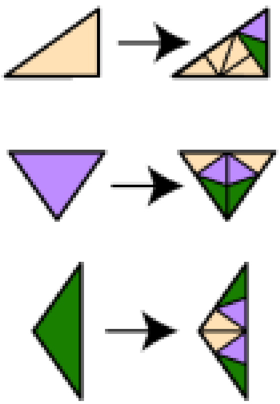




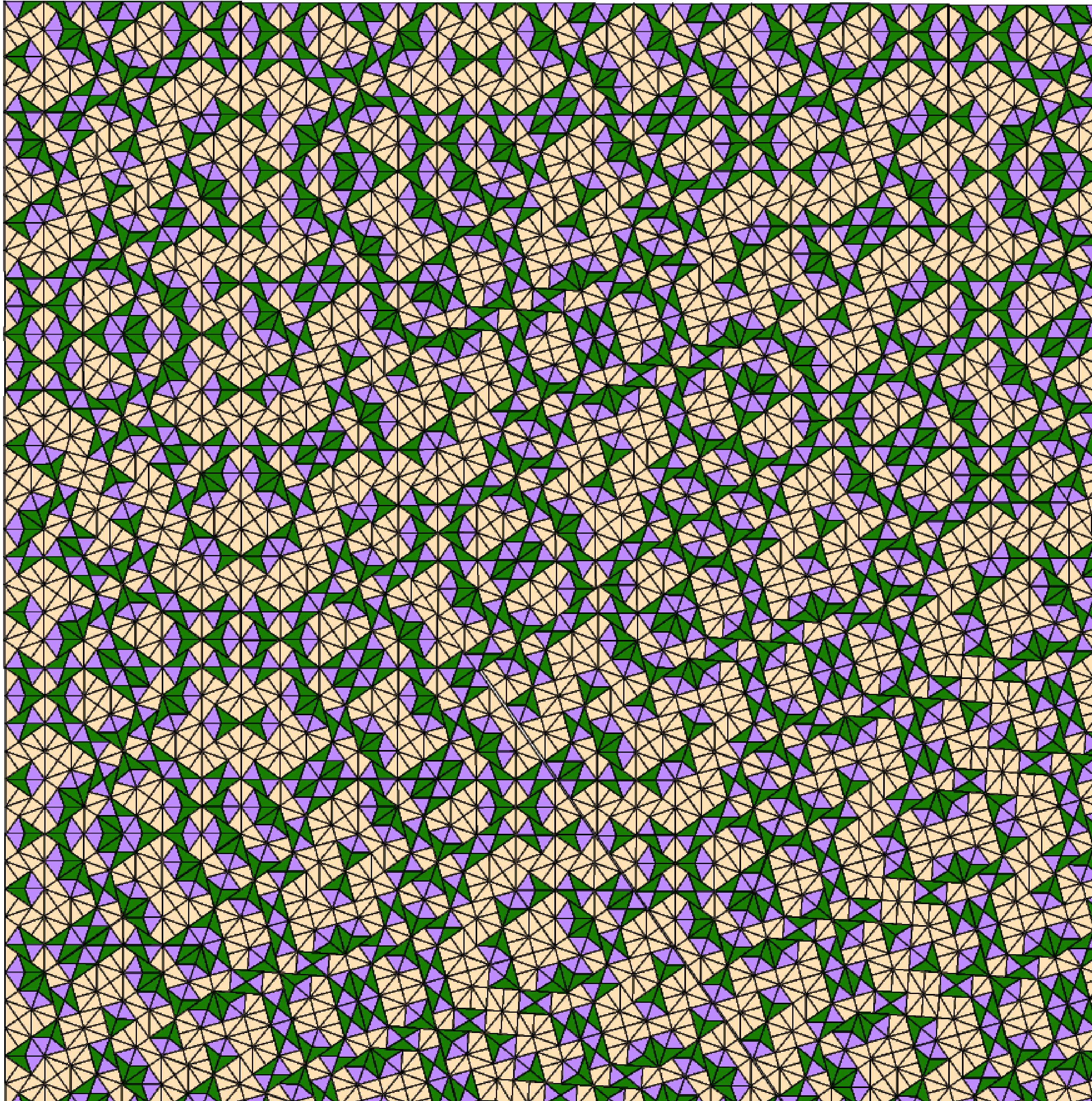
Aperiodic Tilings

“Sqrt6 Triangles”

D. Walton

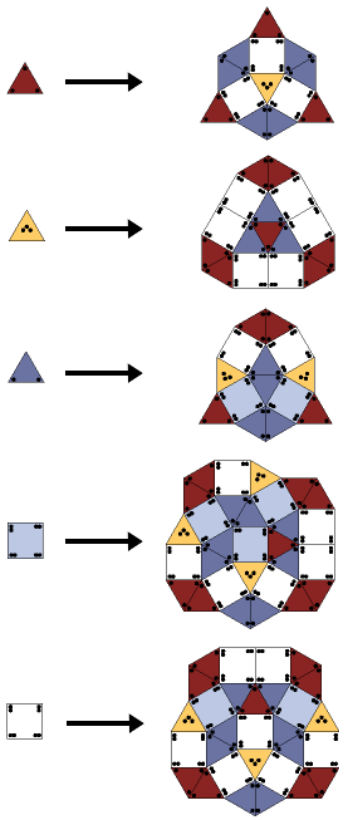


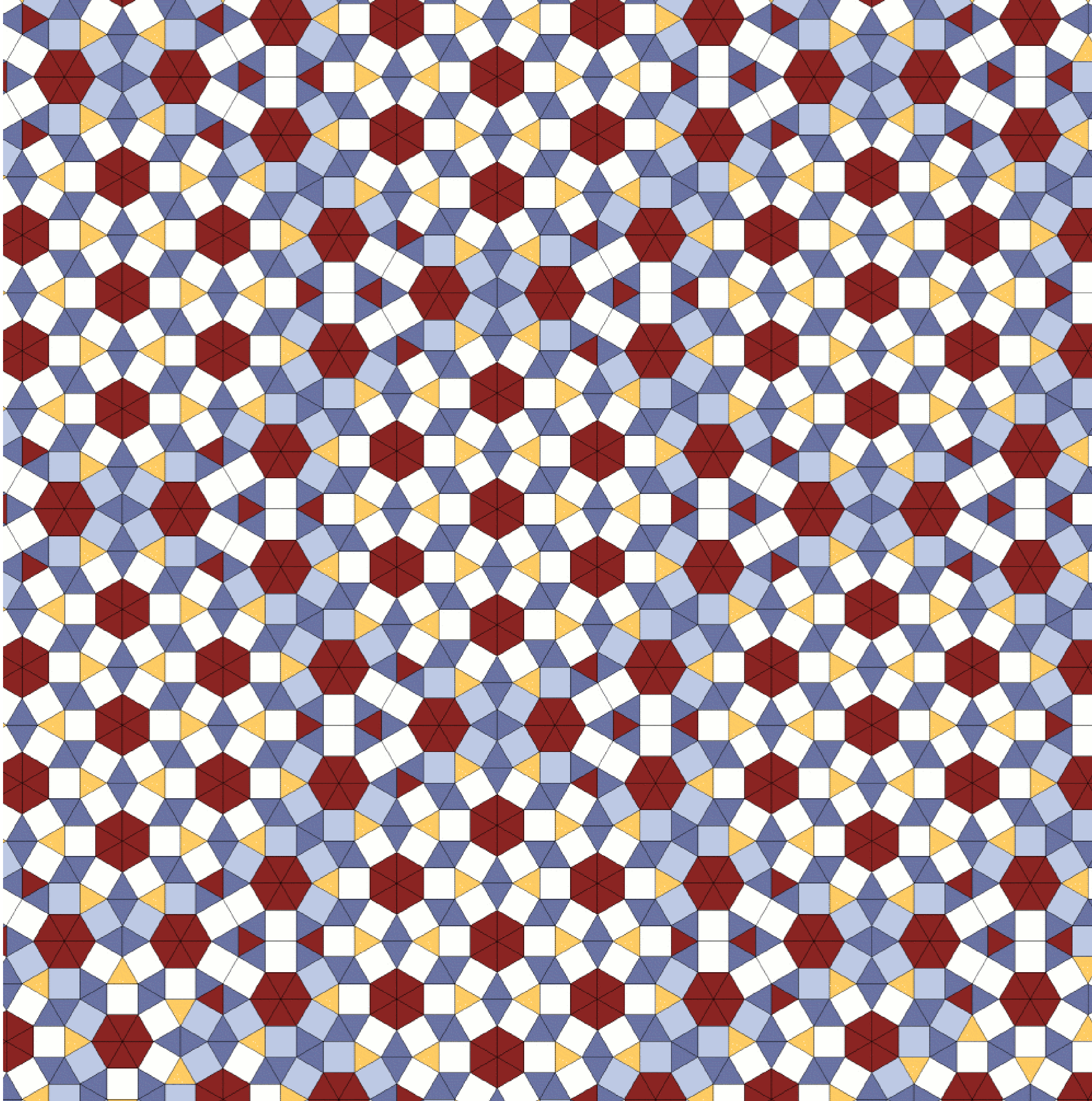
Tiles occur in infinitely many orientations with statistical equidistribution !



Aperiodic Tilings

"Square-triangle"
M. Schlottmann

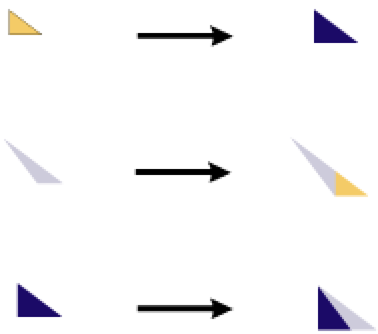


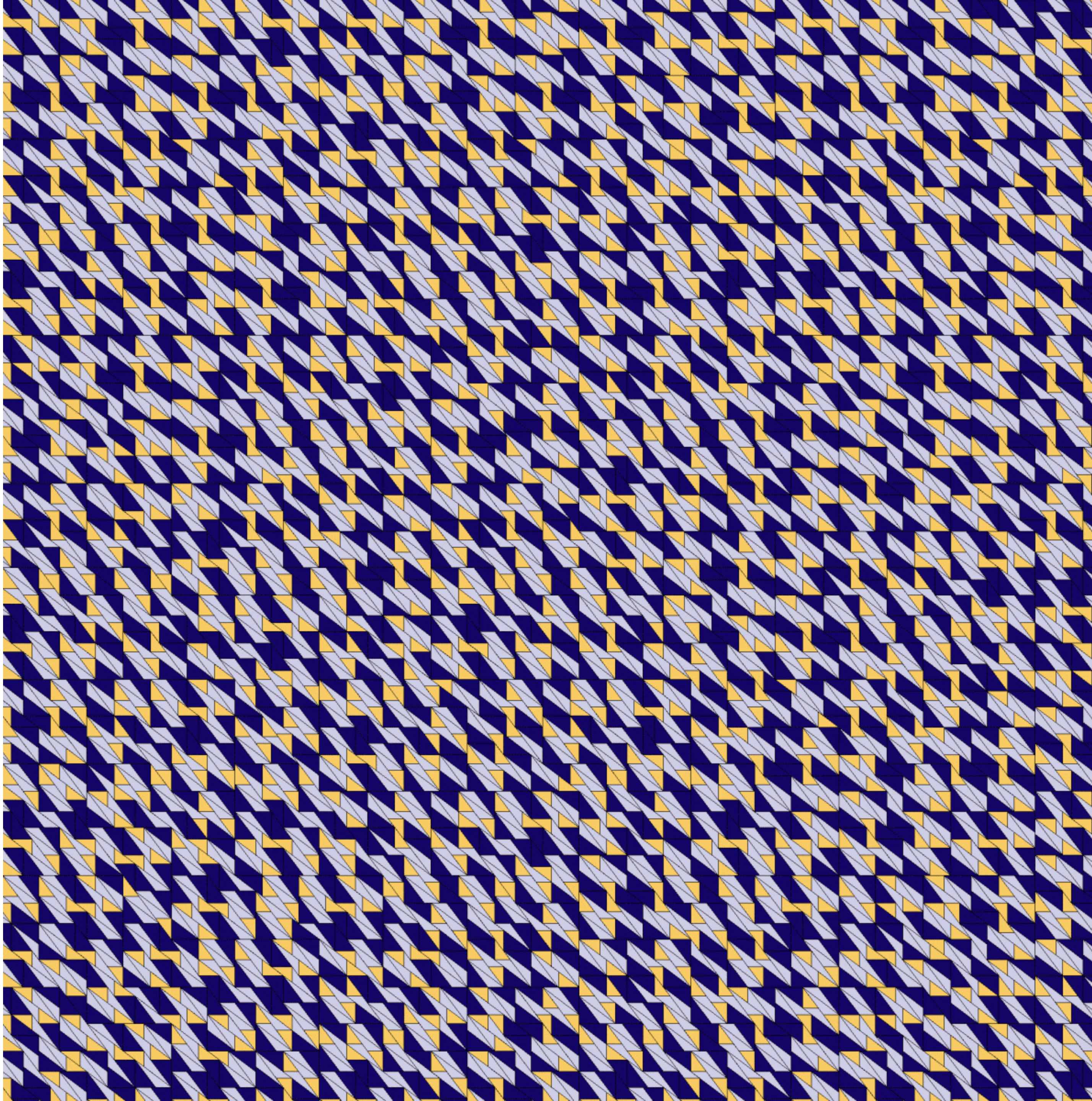


Aperiodic Tilings

“Squeeze”

C. Goodmann-Straus

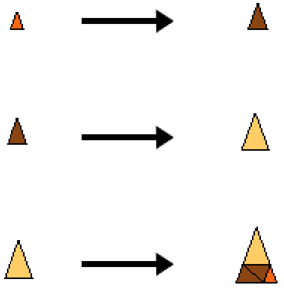


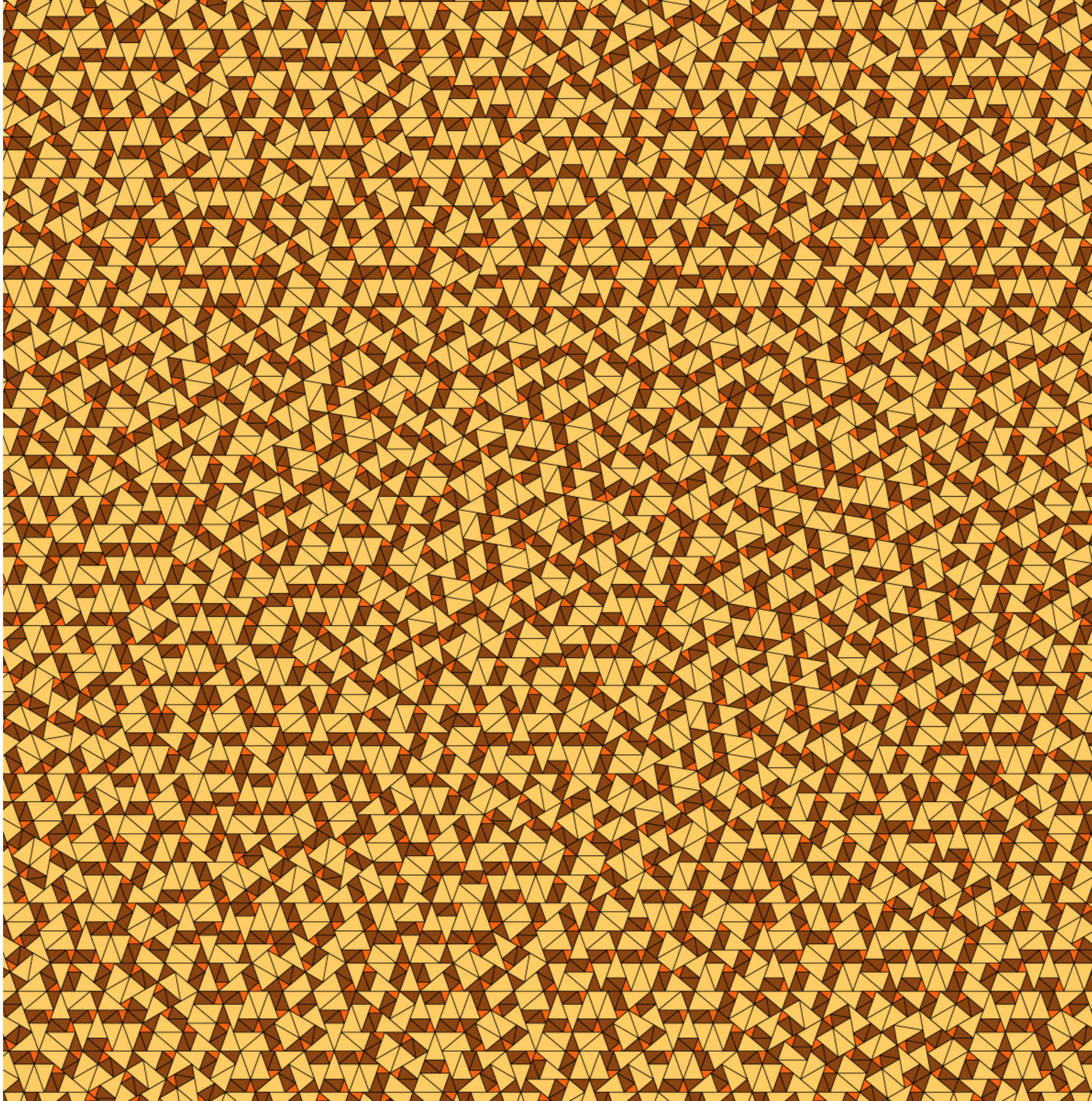


Aperiodic Tilings

“Tipi-3-1”

D. Frettlöh



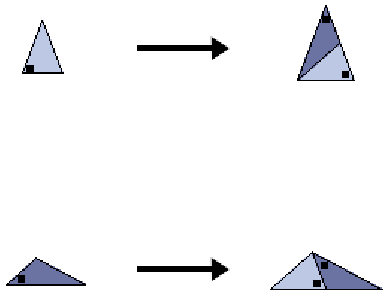


Aperiodic Tilings

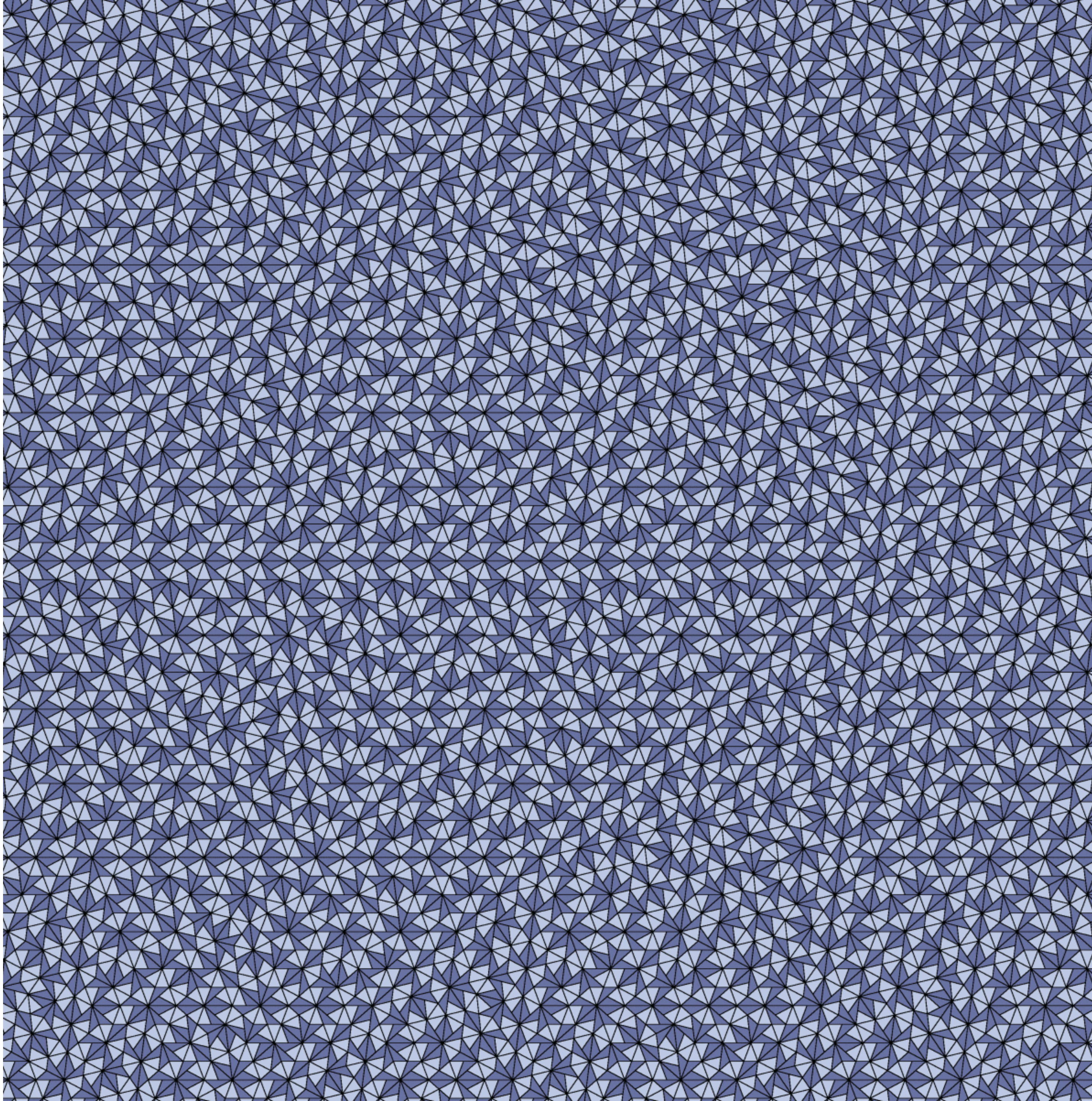
"Triangle Due"

L. Danzer and

C. Goodman-
Strauss

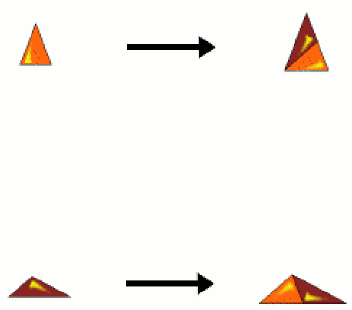


Tiles occur in infinitely many orientations!



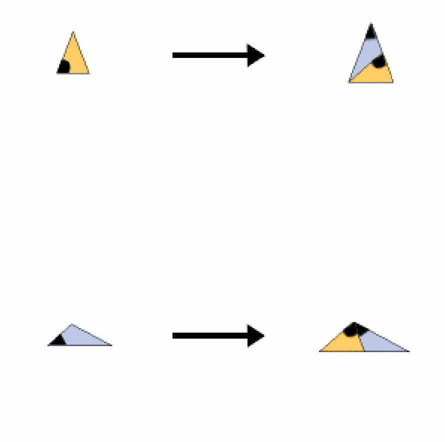
Aperiodic Tilings

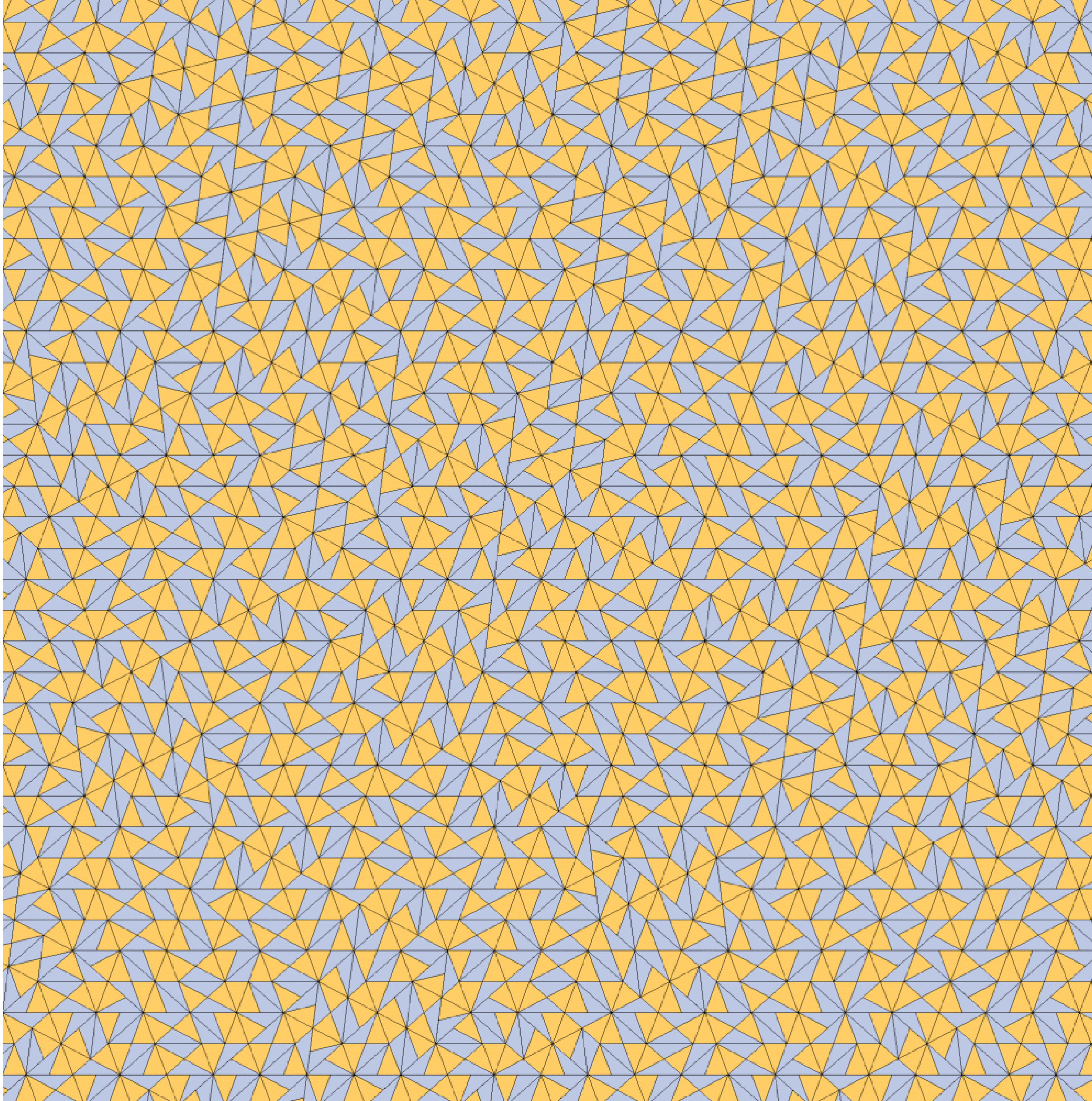
"Triangle Due
(single mirror)"



Aperiodic Tilings

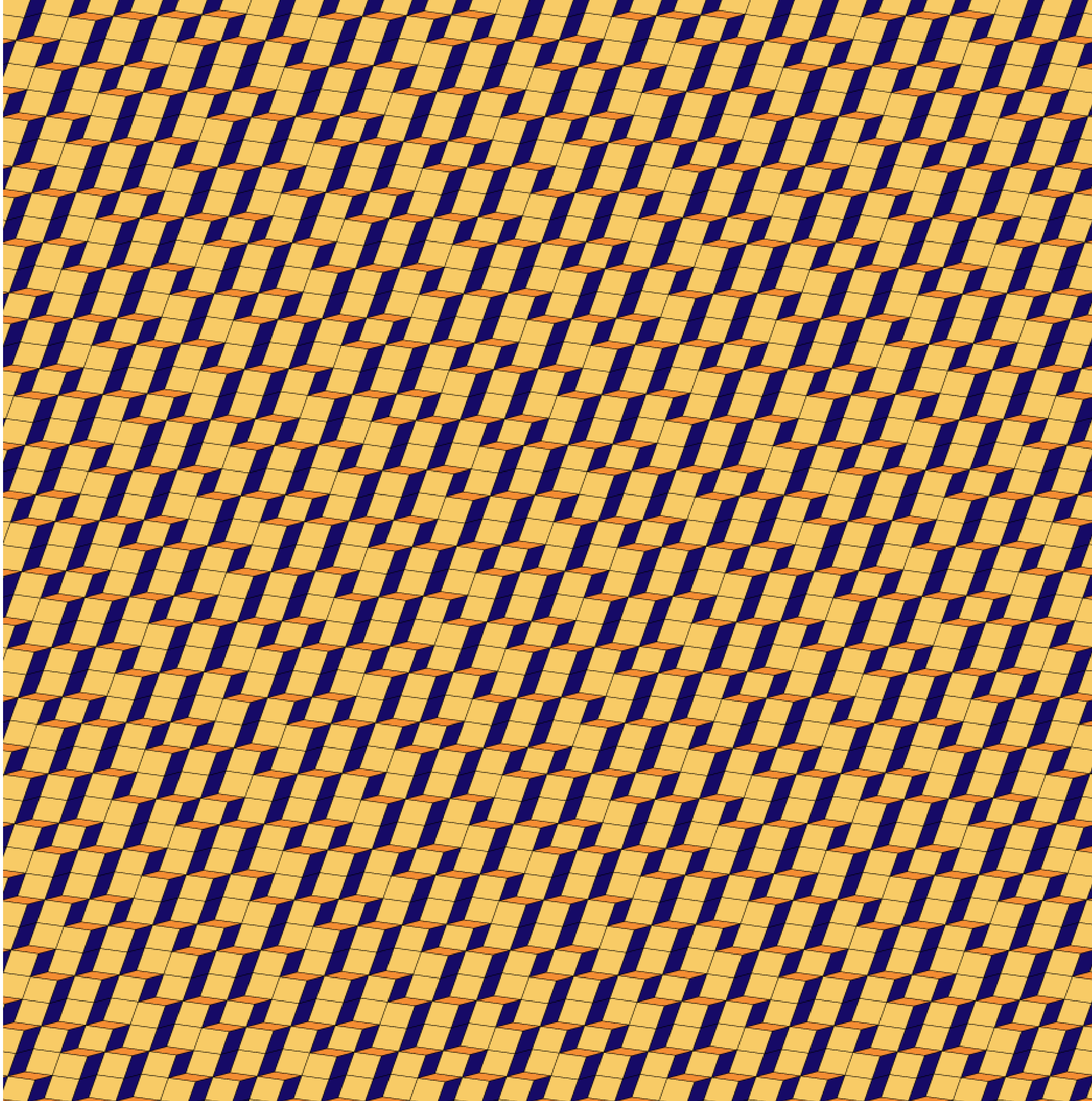
“Triangle Due
(twin mirror)”





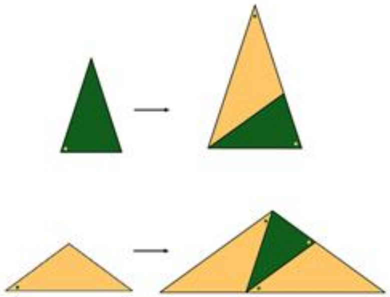
Aperiodic Tilings

"Tribonacci Dual"
G. Rauzy

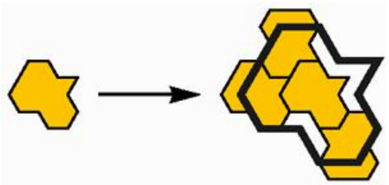


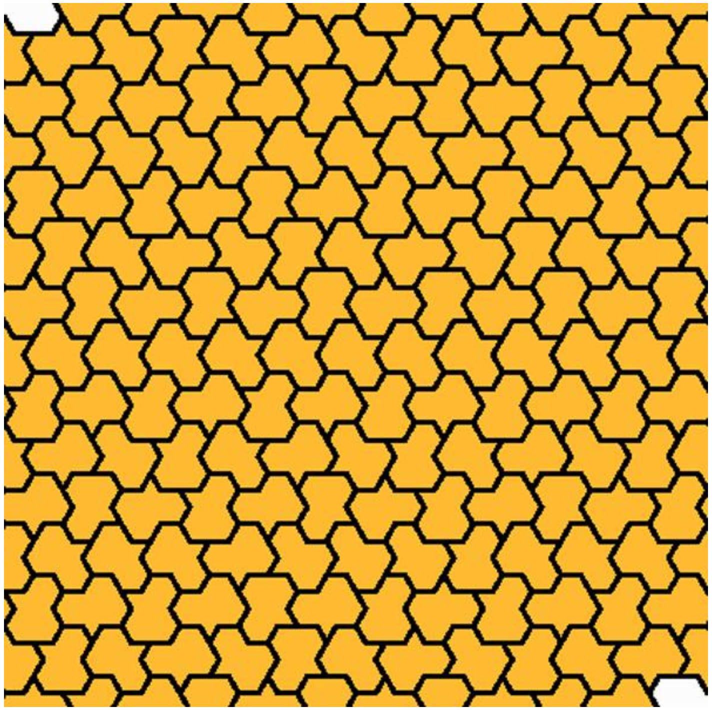
Aperiodic Tilings

“Penrose triangle”
Roger Penrose



“Limhex”
J. Socolar

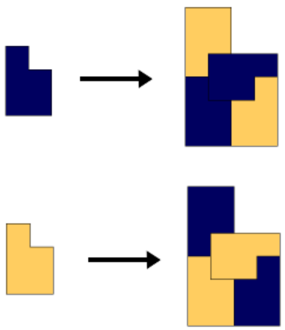


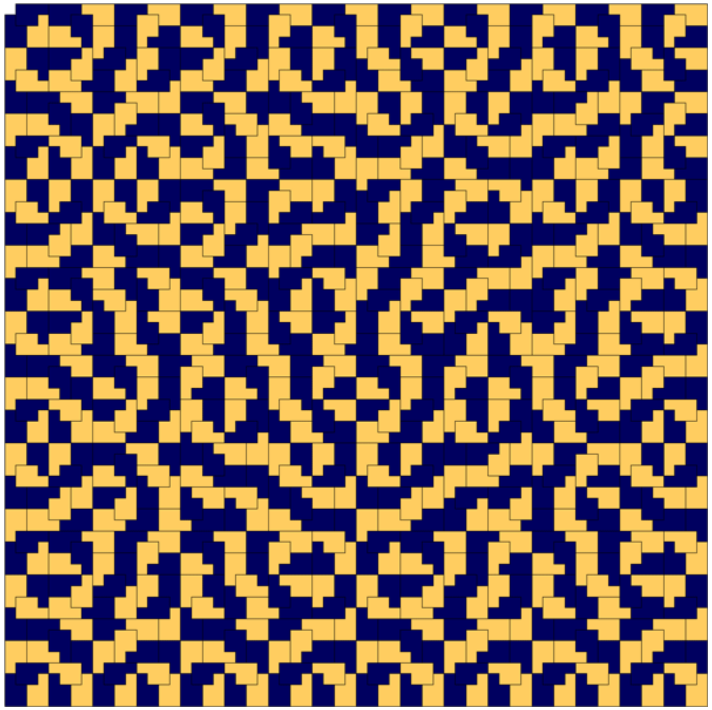


Aperiodic Tilings

“Pentomino”

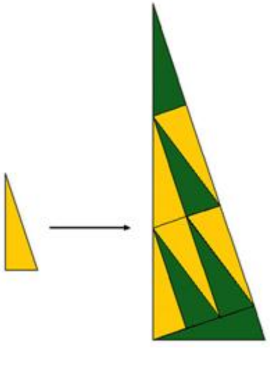
J. Pieniak

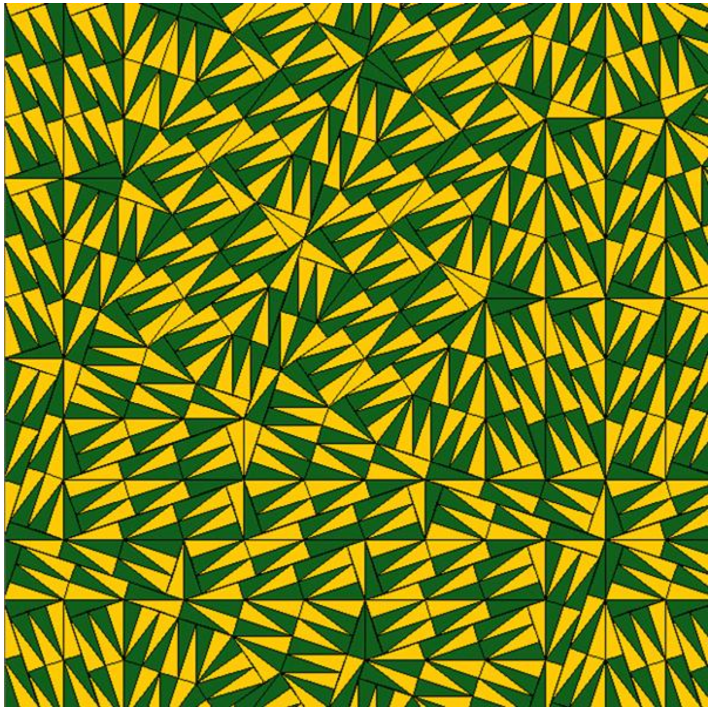




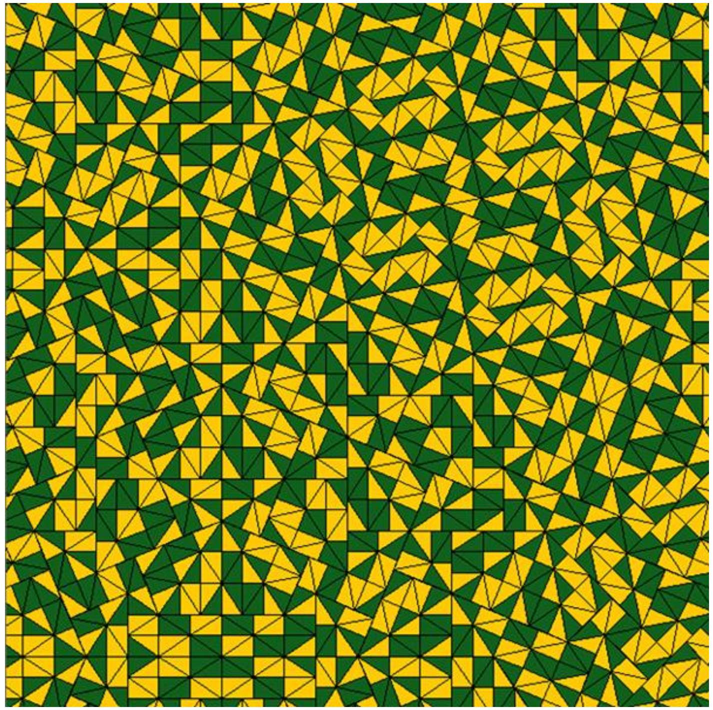
“Pinwheel variant”

I. Suschko



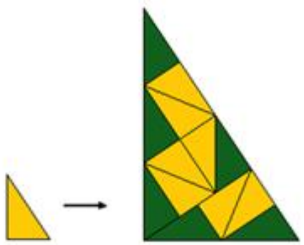


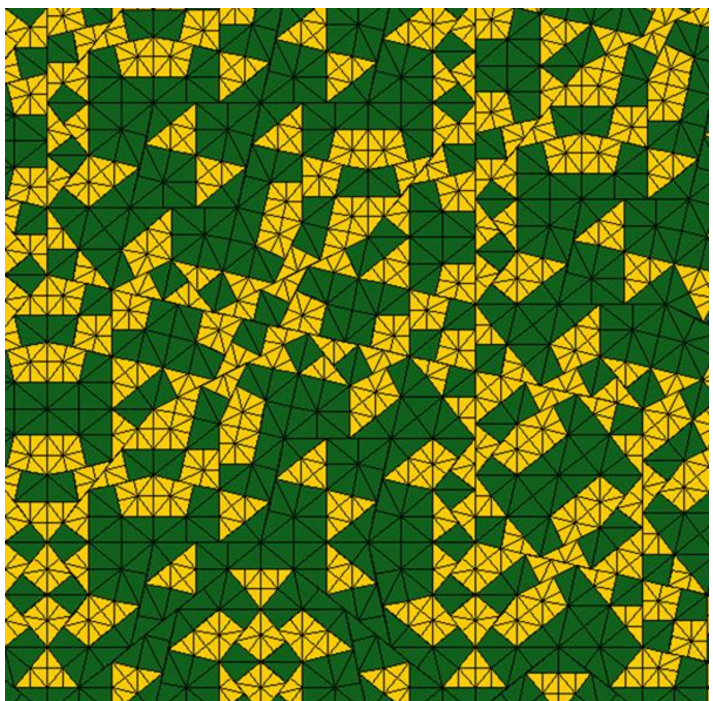
Aperiodic Tilings



"Pinwheel variant (13 tiles)"

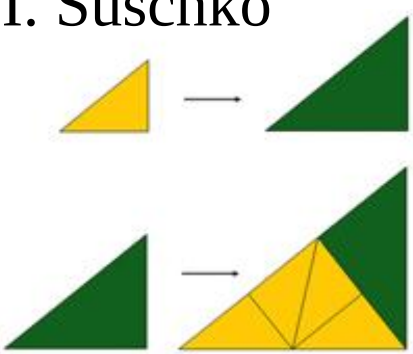
I. Suschko





"Pinwheel-1-2"

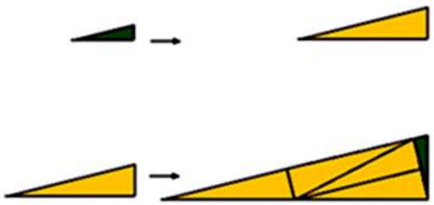
I. Suschko

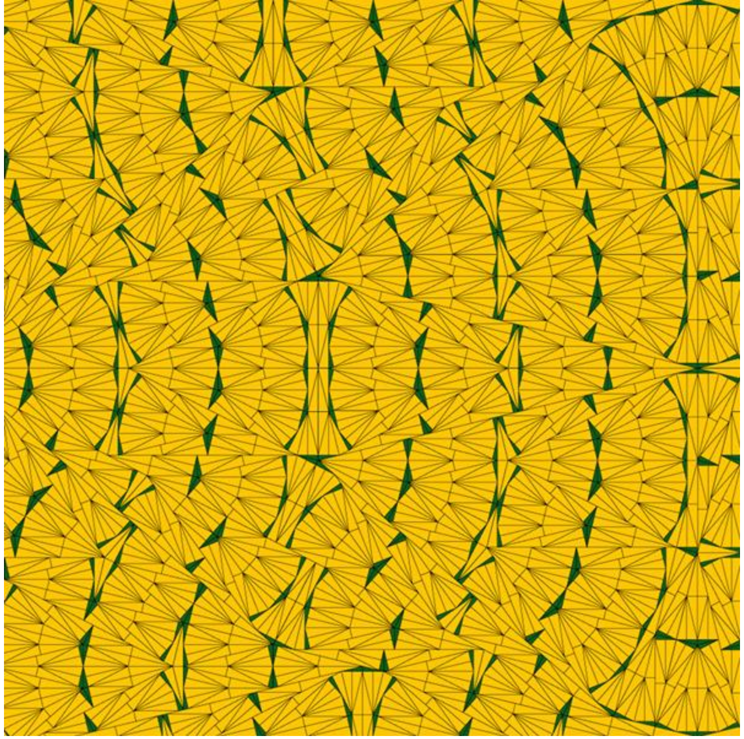


Aperiodic Tilings

“Pinwheel-2-1”

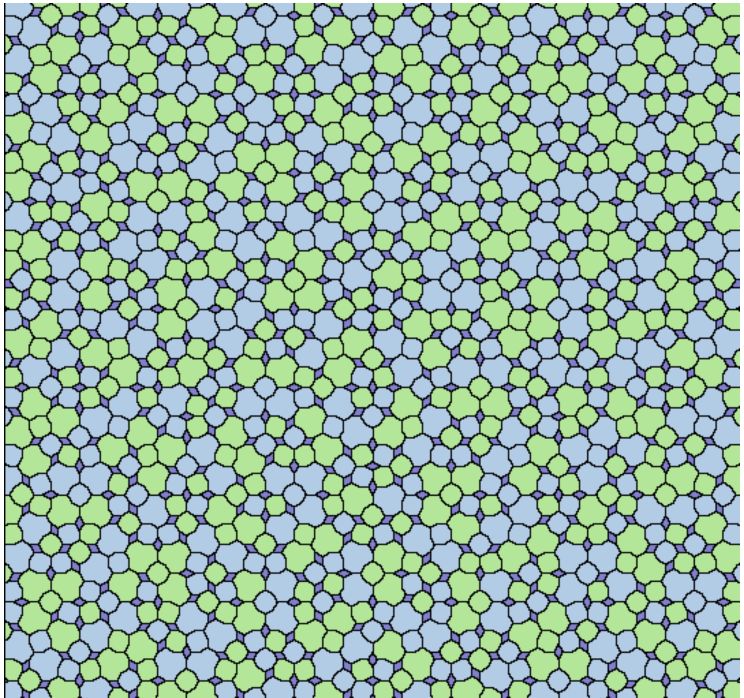
I. Suschko



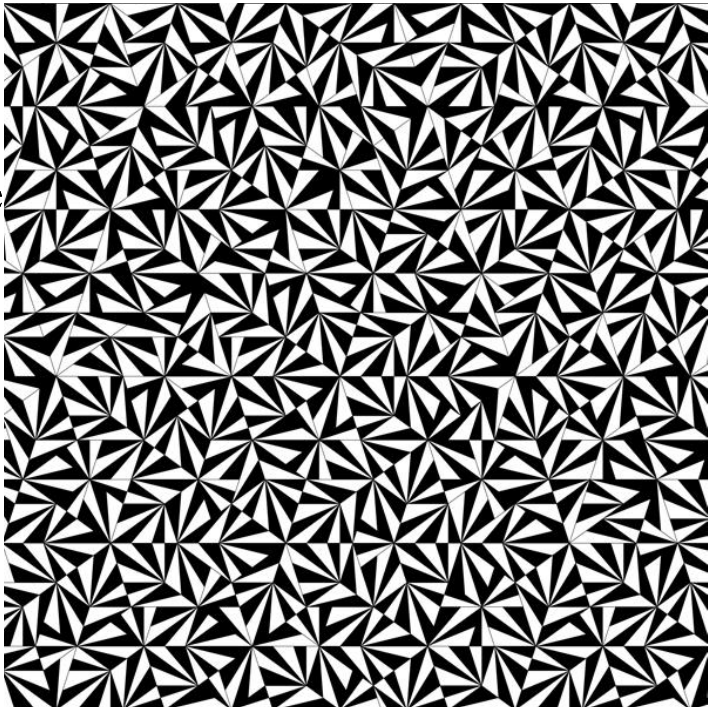


“Plate Tiling”

H. U. Nissen

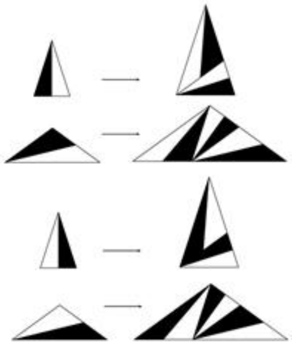


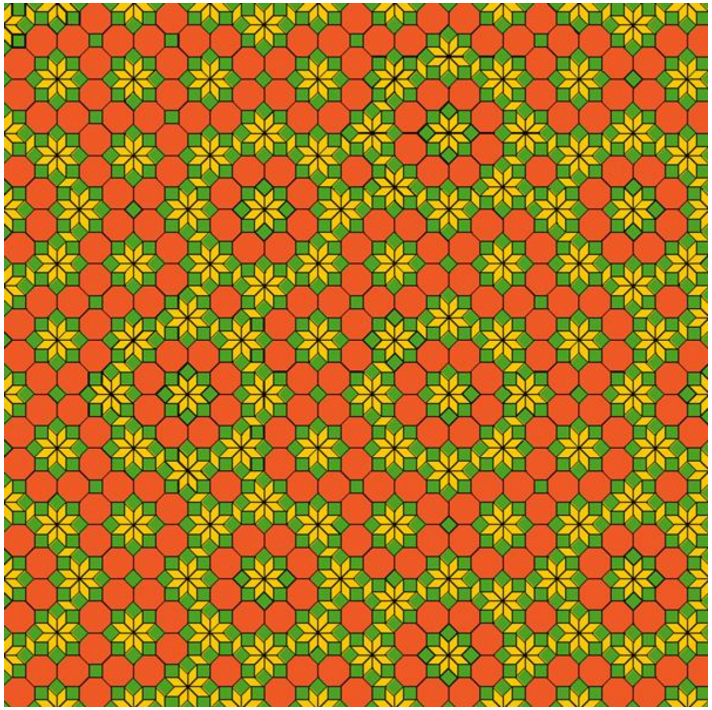
Aperiodic Tilings



“Psychedelic Penrose variant I”

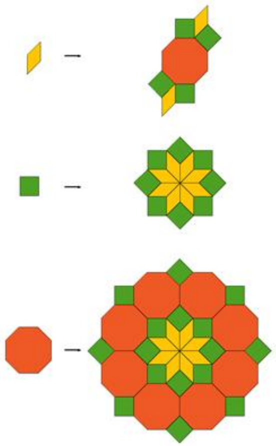
I. Suschko



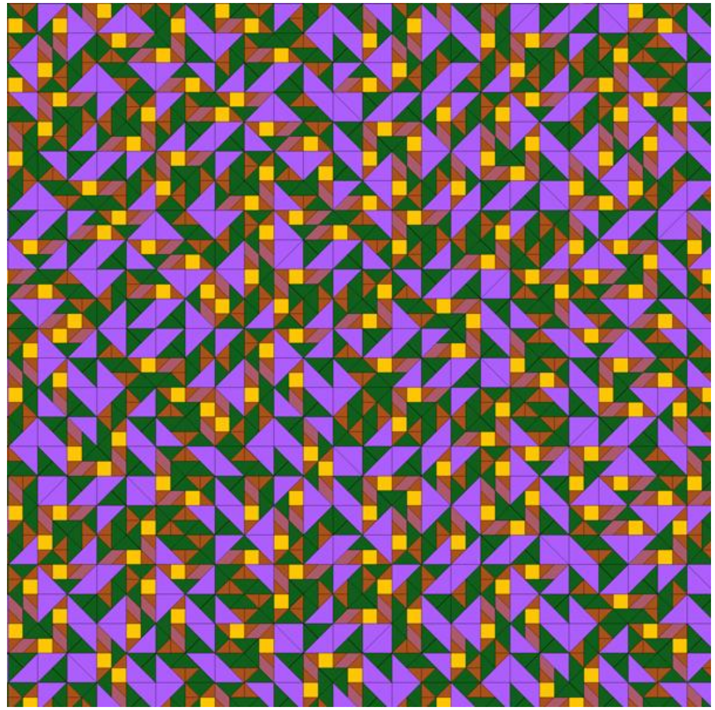


“Rhomb square oktagon”

I. Suschko

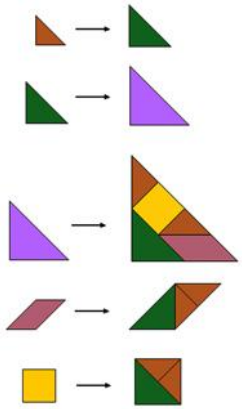


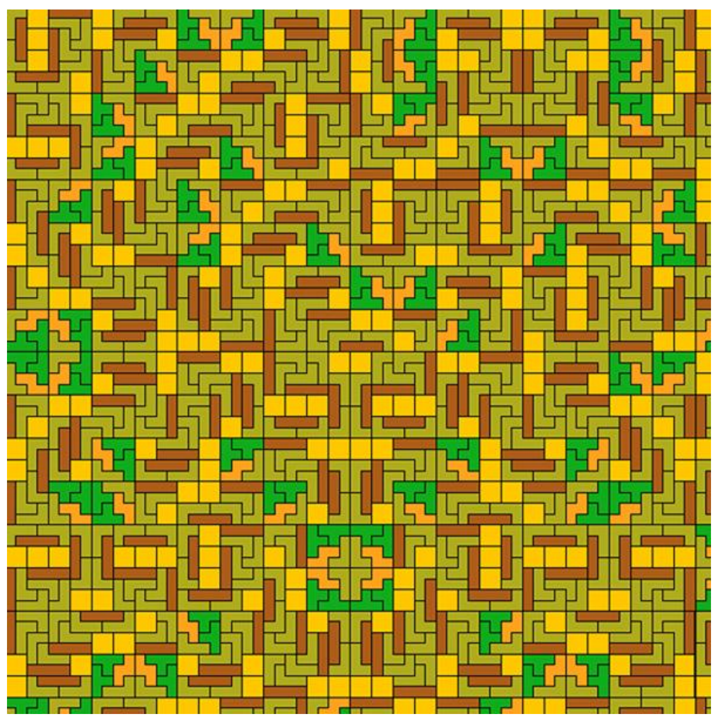
Aperiodic Tilings



"Tangram"

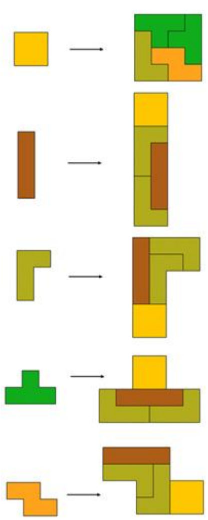
I. Suschko





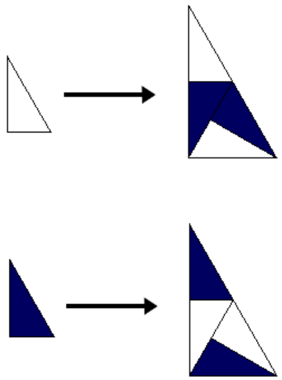
"Tetris"

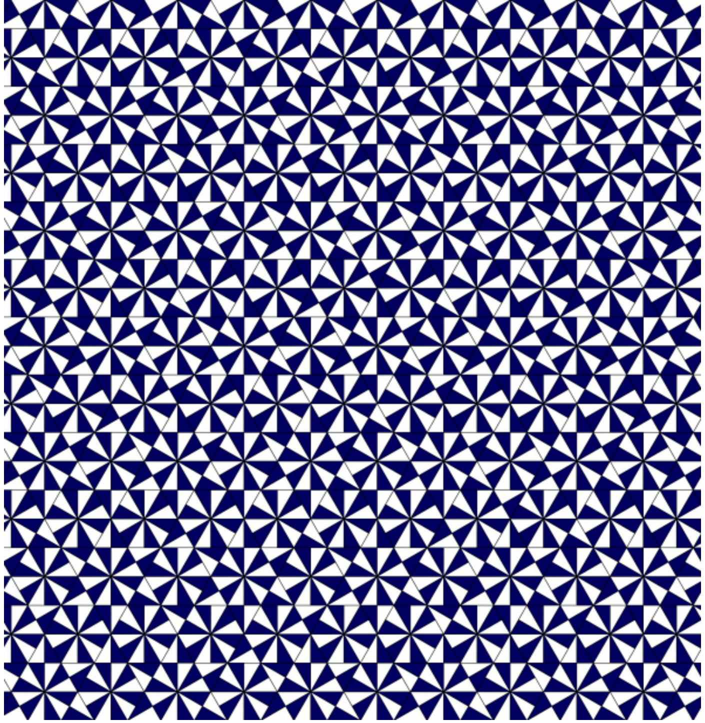
I. Suschko



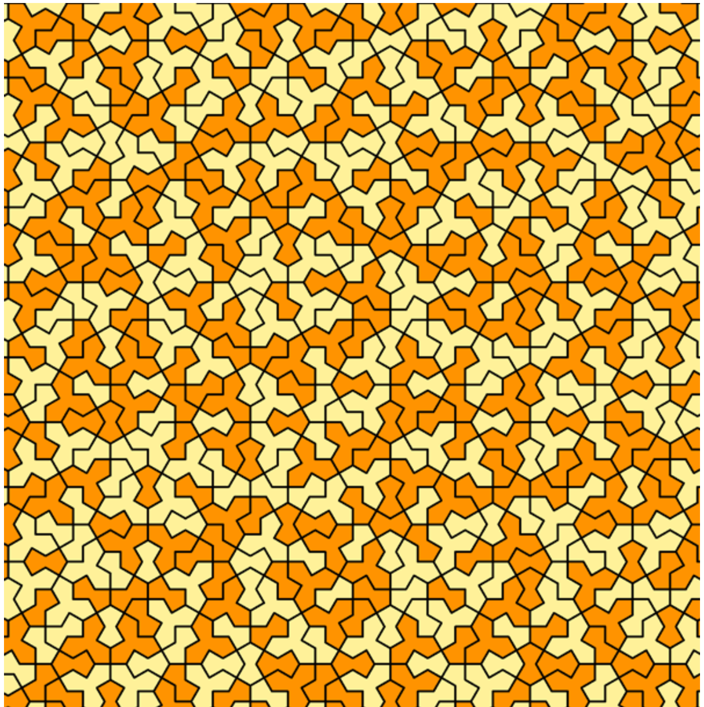
Aperiodic Tilings

“Trihex”
Folklore





“Wheel Tiling”
H.U. Nissen



Hilbert's Problems

Problem 19: Are solutions of Lagrangians always analytic?

Status: Resolved in the affirmative by Bernstein (1904).

Problem 20: Do all variational problems with certain boundary conditions have solutions?

Status: Resolved in the affirmative.

Problem 21: Proof of the existence of linear differential equations having a prescribed monodromic group

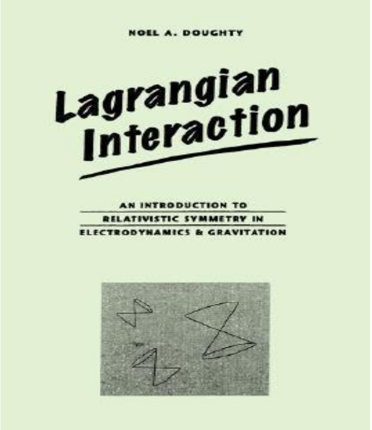
Status: Resolved by Plemelj (1908), Schlesinger (1964), Dekkers (1978), and Bolibrukh (1989).

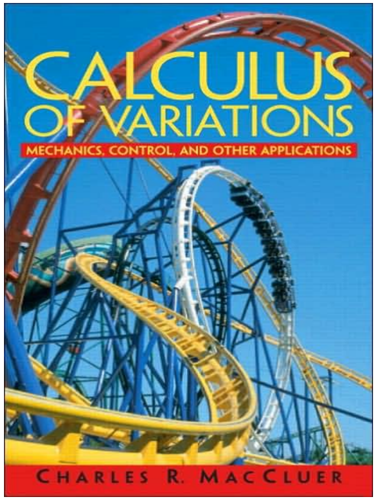
Problem 22: Uniformization of analytic relations by means of automorphic functions

Status: Resolved.

Problem 23: Further development in calculus of variations

Status: Unresolved.





EDGE of DISCOVERY

updated 5:33 p.m. EDT, Tue October 14, 2008

DARPA invests in math

STORY HIGHLIGHTS

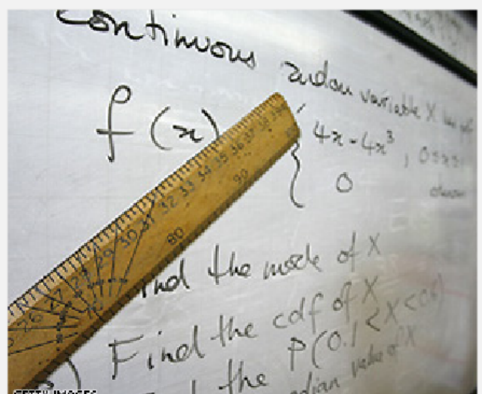
- Mathematicians being offered new challenges designed to "revolutionize" math
- One challenge is solving the Reimann Hypothesis, unsolved since before 1900
- New math could propel other sciences, including biology, computing, physics

Next Article in Technology »

DARPA Mathematical Challenges

It's rare for mathematicians to be publicly challenged with solving the problems of the universe. In 1900, David Hilbert issued 23 iconic problems; in 2000, the Clay Mathematics Institute offered the Millennium Prize Problems; and DARPA's were

The latest set of challenges



In 2007, the Defense Advanced Research Projects Agency (DARPA) issued 23 mathematical challenges in order to "dramatically [revolutionize] mathematics and thereby [strengthen] the scientific and technological capabilities of [the Department of Defense.]" CNN spoke with John Etnyre, a professor of mathematics at the Georgia Institute of Technology, to get an inside look at some of these challenges and why the mathematical community is interested in them.

Etnyre, who specializes in low-dimensional topology and geometry, says he finds the fluids and 4D problems on DARPA's list to be especially interesting, and he is considering tackling those challenges himself.

Most Popular on CNN
STORIES
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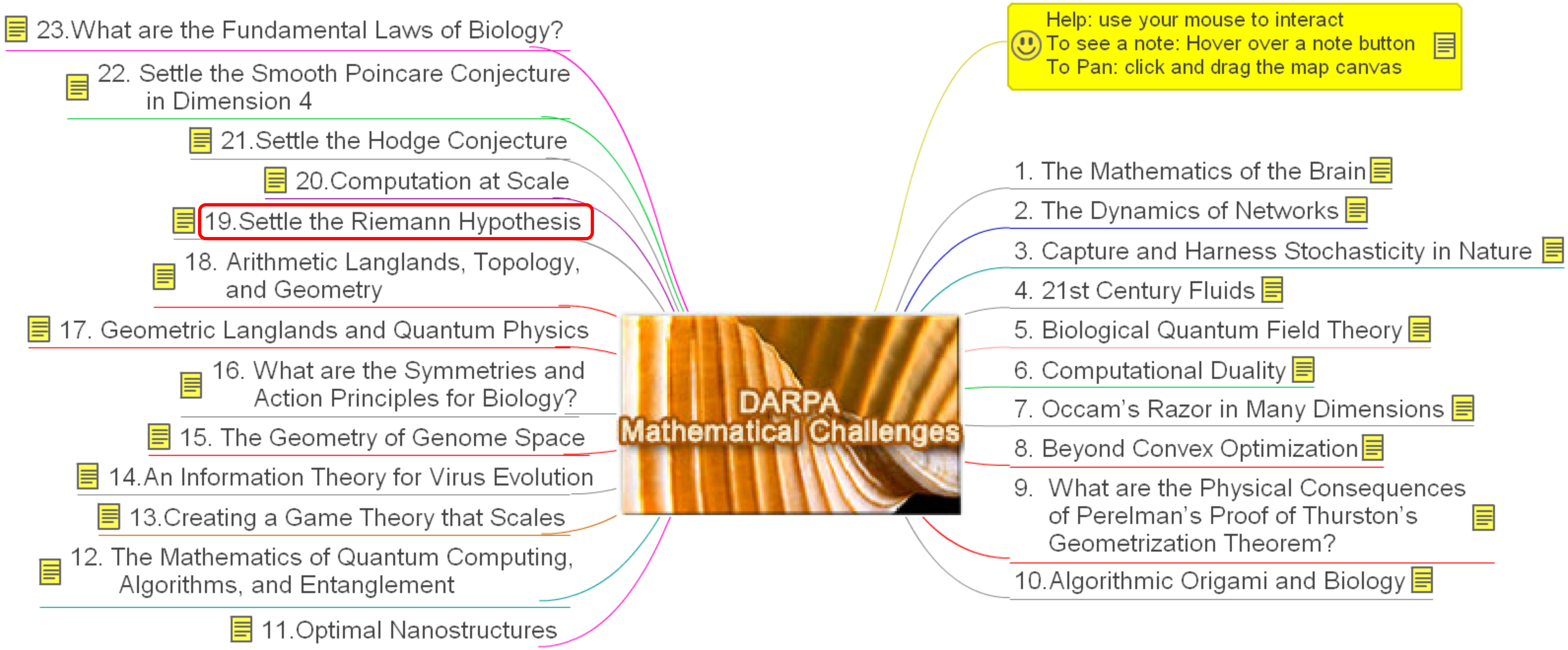
DARPA's Mathematical Challenges



"DARPA-hard" problems!

http://www.gogeometry.com/mindmap/darpa_mathematical_challenges_elearning.html
http://www.mathisfunforum.com/viewtopic.php?id=10753

DARPA's Mathematical Challenges

1: **The Mathematics of the Brain**: Develop a mathematical theory to build a functional model of the brain that is mathematically consistent and predictive rather than merely biologically inspired.

2: **The Dynamics of Networks**: Develop the high-dimensional mathematics needed to accurately model and predict behavior in large-scale distributed networks that evolve over time occurring in communication, biology and the social sciences.

3: **Capture and Harness Stochasticity in Nature**: Address Mumford's call for new mathematics for the 21st century. Develop methods that capture persistence in stochastic environments.

4: **21st Century Fluids**: Classical fluid dynamics and the Navier-Stokes Equation were extraordinarily successful in obtaining quantitative understanding of shock waves, turbulence and solitons, but new methods are needed to tackle complex fluids such as foams, suspensions, gels and liquid crystals.

5: **Biological Quantum Field Theory**: Quantum and statistical methods have had great success modeling virus evolution. Can such techniques be used to model more complex systems such as bacteria? Can these techniques be used to control pathogen evolution?

6: **Computational Duality**: Duality in mathematics has been a profound tool for theoretical understanding. Can it be extended to develop principled computational techniques where duality and geometry are the basis for novel algorithms?

DARPA's Mathematical Challenges

7: **Occam's Razor in Many Dimensions**: As data collection increases can we "do more with less" by finding lower bounds for sensing complexity in systems? This is related to questions about entropy maximization algorithms.

8: **Beyond Convex Optimization**: Can linear algebra be replaced by algebraic geometry in a systematic way?

9: **What are the Physical Consequences of Perelman's Proof of Thurston's Geometrization Theorem?** Can profound theoretical advances in understanding three dimensions be applied to construct and manipulate structures across scales to fabricate novel materials?

10: **Algorithmic Origami and Biology**: Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.

11: **Optimal Nanostructures**: Develop new mathematics for constructing optimal globally symmetric structures by following simple local rules via the process of nanoscale self-assembly.

12: **The Mathematics of Quantum Computing, Algorithms, and Entanglement**: In the last century we learned how quantum phenomena shape our world. In the coming century we need to develop the mathematics required to control the quantum world.

13: **Creating a Game Theory that Scales**: What new scalable mathematics is needed to replace the traditional Partial Differential Equations (PDE) approach to differential games?

DARPA's Mathematical Challenges

14: **An Information Theory for Virus Evolution**: Can Shannon's theory shed light on this fundamental area of biology?

15: **The Geometry of Genome Space**: What notion of distance is needed to incorporate biological utility?

16: **What are the Symmetries and Action Principles for Biology?** Extend our understanding of symmetries and action principles in biology along the lines of classical thermodynamics, to include important biological concepts such as robustness, modularity, evolvability and variability.

17: **Geometric Langlands and Quantum Physics**: How does the Langlands program, which originated in number theory and representation theory, explain the fundamental symmetries of physics? And vice versa?

18: **Arithmetic Langlands, Topology, and Geometry**: What is the role of homotopy theory in the classical, geometric, and quantum Langlands programs?

19: **Settle the Riemann Hypothesis**: The Holy Grail of number theory.

20: **Computation at Scale**: How can we develop asymptotics for a world with massively many degrees of freedom?

21: **Settle the Hodge Conjecture**: This conjecture in algebraic geometry is a metaphor for transforming transcendental computations into algebraic ones.

DARPA’s Mathematical Challenges

22: **Settle the Smooth Poincare Conjecture in Dimension 4**: What are the implications for space-time and cosmology? And might the answer unlock the secret of “dark energy”?

23: **What are the Fundamental Laws of Biology?** This question will remain front and center for the next 100 years. DARPA places this challenge last as finding these laws will undoubtedly require the mathematics developed in answering several of the questions listed above.