# CS661 Problem Set 1

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"Begin at the beginning," said the King very gravely,
"and go on till you come to the end; then stop."

1. True or false:
   a. $\emptyset \supseteq \emptyset$
   b. $\emptyset \supset \emptyset$
   c. $\emptyset \in \emptyset$
   d. $\{1,2\} \in 2^{\{1,2\}}$
   e. $2^{\{1,2\}} \supseteq \{1,2\}$
   f. $\{x,y\} \in \{\{x,y\}\}$

2. Write the following set explicitly: $2^{\{1, 2\}} \times \{v,w\}$

3. Show that $2^S$ and $\{0,1\}^{|S|}$ are isomorphic for an arbitrary finite set S.

4. Which of the following sets are closed under the specified operations:
   a) $\{x \mid x \text{ is an odd integer}\}$, multiplication
   b) $\{y \mid y=2n, n \text{ some integer}\}$, subtraction
   c) $\{2m+1 \mid m \text{ some integer}\}$, division
   d) $\{z \mid z=a+bi \text{ where a and b are real and } i=\sqrt{-1}\}$, exponentiation

5. Is the transitive closure of a symmetric closure of a binary relation necessarily reflexive?

6. Show that a countable union of countable sets is countable.

7. Show that if T is countable, then the set $\{S \mid T \supseteq S, S \text{ finite}\}$ is also countable.

8. Give a simple bijection for each one of the following pairs of sets:
   a) the integers, and the odd integers.
   b) the integers, and the positive integers.
   c) the naturals, and the rationals crossed with the integers.

9. Is there a bijection between $\{x \mid x \in \mathbb{R}, 0 \leq x \leq 1\}$ and $\mathbb{R}$?

"Why," said the Dodo, "the best way to explain it is to do it."

10. Generalize $|2^S| > |S|$ to arbitrary sets (not necessarily countable ones).

11. What is the cardinality of each of the following sets ?
    a. The set of all polynomials with rational coefficients.
    b. The set of all functions mapping reals to reals.
    c. The set of all possible Pascal programs.
    d. The set of all finite strings over the alphabet {0,1,2}.
    e. The set of all 5x5 matrices over the rationals.
    f. The set of all points in 3-dimensional Euclidean space.
    g. The set of all valid English words.
    h. $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$
    i. $\mathbb{N} \times \mathbb{Z} \times \mathbb{Q}$
    j. $\mathbb{R} - \mathbb{Q}$
    k. $S = \{\mathbb{Q} \times \mathbb{R}, \emptyset, 2^{|S|}, S, 2^S\}$

"And thick and fast they came at last, and more, and more, and more"

12. Show that $n^4-4n^2$ is divisible by 3 for all $n \geq 0$.

13. How many distinct boolean functions on N variables are there? In other words, what is the value of $|\{f \mid f:\{0,1\}^N \rightarrow \{0,1\}\}|$ ?

14. How many distinct N-ary functions are there from finite set A to finite set B? Does this generalize the previous question?

15. Show that in any group of people, there are at least two people with the same number of acquaintances within the group. Assume that the "acquaintance" relation is symmetric but non-reflexive.

16. Show that in any group of six people, there are either 3 mutual strangers or 3 mutual acquaintances.

17. Show that the difference of an uncountable set and a countable set is uncountable.

18. Show that the intersection of two uncountable sets can be empty, finite, countably infinite, or uncountably infinite.

"Is that all?" Alice timidly asked.