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Mathematics

Solution of the discrete Plateau problem

(minimal surfaces/thickness (*r*)-separation/network flows)

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ABSTRACT **We solve a discrete version of the classic Plateau problem, which asks for a minimal surface spanning a given curve. Our algorithm is based on a network-flow formulation that finds minimal slabs, intuitively corresponding to minimal "surfaces" of prescribed thickness. We let the slab thickness approach zero in order to obtain the desired minimal surface.**

The Plateau problem is to find a surface of minimum area that spans a given curve. The problem is named after J. Plateau (1801–1883), who performed an extensive series of experiments using metal wire to represent curves and soap films to model minimal spanning surfaces. Since then, many of the great mathematicians of the 19th and 20th centuries have contributed to the theory of minimal surfaces, culminating with the discovery of the general analytical solutions by Douglas (1) and Radó (2) in the 1930s (for recent reviews, see refs. 3–5). The general Plateau problem is stated as follows:

Plateau Problem (P1). Given a Jordan curve Γ^* in $\Re^3$, find a surface D^* of minimum area having boundary Γ^*.

A surface has minimal area if and only if it has zero mean curvature at each point, but this characterization is nonconstructive. In many instances, analytic solutions are known to exist but remain virtually impossible to find; solutions to specific cases of the Plateau problem have been individually discovered and proved over the last two centuries (3). Recent work has used numerical methods—e.g., the finite element approaches of refs. 6 and 7—to solve restricted instances of the Plateau problem; these are reviewed in ref. 8.

The present discussion treats the special class of instances of the Plateau problem, first described by Radó (2), which satisfy the following conditions: (*i*) the orthogonal projection Γ of the given boundary Γ^* onto the *xy* plane is simple (i.e., non-self-intersecting), and (*ii*) the solution D^* admits a functional representation $z = f(x, y)$, where f is continuous and has domain D equal to the subset of the *xy* plane bounded by Γ. Thus, we have the following:

Restricted Plateau Problem (P2). Given a Jordan curve Γ^* in $\Re^3$ which has projection Γ onto the *xy* plane that is homeomorphic to a circle, find a surface D^* [having functional representation $z = f(x, y)$] of minimum area having boundary Γ^*.

Conceptually, we may visualize the Jordan curve Γ^* as being embedded in the lateral wall of a cylinder that has D as its cross section. More precisely, this cylinder is the solid given by $cyl(\Gamma^*) \stackrel{\text{def}}{=} \{(x, y, z) \mid (x, y) \in D, |z| \leq M\}$ for some M sufficiently large. The minimal surface D^* therefore separates the cylinder into two parts, T and B, lying respectively above and below D^*. We generalize standard formulations in that we do not search for a minimal (zero-thickness) surface. Rather, we compute a minimal *r-separating slab* that has prescribed thickness $r > 0$ and that separates the cylinder into T and B, with every point of T at least distance r from every point of B (Fig. 1). The term slab intuitively corresponds to a "thick surface" (e.g., think of an orange peel) that has non-zero volume. Our method will solve for the *r*-separating slab that contains the curve Γ^* and has minimum volume. As r approaches zero, this problem naturally reduces to the restricted Plateau problem (P2).

In addition to allowing the solution to have thickness $r > 0$, we further generalize the classic Plateau formulation by allowing a positive weight function $w : cyl(\Gamma^*) \rightarrow \Re^+$ to be defined at every point within the cylinder. We therefore have the following:

Thickness-*r* Plateau Problem (P3). Given a Jordan curve Γ^*, a weight function $w : cyl(\Gamma^*) \rightarrow \Re^+$, and a thickness $r > 0$, find an *r*-separating slab $D^* \subset cyl(\Gamma^*)$ that contains Γ^* and has minimum total weight.

Note that a minimal slab of thickness r_1 will not necessarily be contained in a minimal slab of thickness $r_2 > r_1$. (These intuitive ideas are formalized in ref. 8. The idea of *r*-separation is discussed in ref. 9.)

The thickness-*r* Plateau problem (P3) minimizes the integral of the weight function over the volume of the slab. However, note that with numerical approaches (e.g., refs. 6 and 7) the space is discretized relative to a given fixed grid or sampling granularity h. The Plateau formulation easily extends to finding a minimal discrete *r*-separating slab, as we have formally shown in ref. 8. Intuitively, as the granularity quantum h of the grid approaches zero, the solution of the discrete instance will converge to the solution for the corresponding continuous (P3) instance.

We now sketch our network-flow-based solution for such discrete Plateau instances. Recall [cf. ref. 10] that a *flow network* $\eta = (N, A, s, t, c, c')$ is a directed graph with node set N; a set of directed arcs $A \subseteq N \times N$; a distinguished source $s \in N$ and a distinguished sink $t \in N$; an *arc capacity* function $c : A \rightarrow \Re^+$ that specifies the capacity $c_{ij} \geq 0$ of each arc $a_{ij} \in A$; and a *node capacity* function $c' : N \rightarrow \Re^+$ that specifies the capacity $c_i' \geq 0$ of each node $n_i \in N$. An infinite capacity may be modeled by specifying a sufficiently large finite capacity. A *flow* in η assigns to each arc a_{ij} a value ϕ_{ij}, with $0 \leq \phi_{ij} \leq c_{ij}$. An arc a_{ij} is called *saturated* if $\phi_{ij} = c_{ij}$. Flow is conserved at every node except s and t and cannot violate node capacities:

$$\sum_i \phi_{ij} = \sum_k \phi_{jk} \leq c_j', \; n_j \neq s, t. \tag{1}$$

A node n_i is called *saturated* if $\Sigma_i \phi_{ij} = c_j'$. Because flow is conserved, the total amount of flow from the source must be equal to the total flow into the sink; we call this quantity the *value* Φ of the flow:

$$\Phi = \sum_i \phi_{si} = \sum_j \phi_{jt}. \tag{2}$$

A flow with the maximum possible value is called a *maximum flow*. An *s–t cut* in a network is a set (N', A') of nodes $N' \subseteq N$ and arcs $A' \subseteq A$, such that every path from s to t uses at

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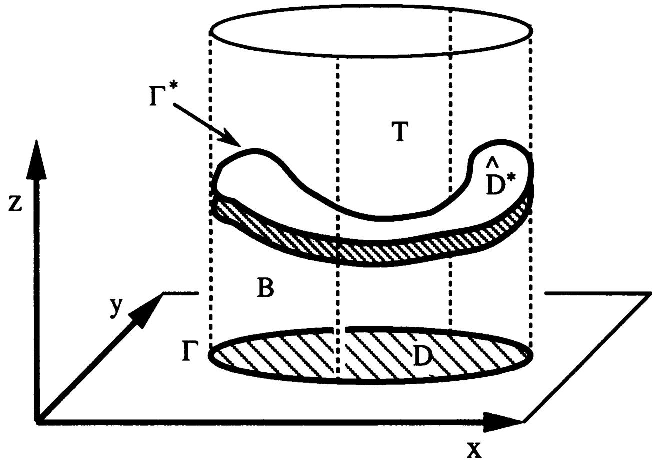

FIG. 1. An r-separating slab $\hat{D}^*$ relative to a given contour Γ^*.

least one node of N' or at least one arc of A'. The *capacity* $c(N', A')$ of a cut is the sum of the capacities of all nodes and arcs in the cut. The *max-flow min-cut* theorem (10) states that the maximum flow value is equal to the minimum cut capacity. Moreover, the nodes and arcs of any minimum s–t cut are a subset of the saturated nodes and saturated arcs in some maximum s–t flow.

To solve the discrete Plateau problem, we partition $cyl(\Gamma^*)$ into small cubes of side h and represent every cube by a node with weight equal to the total weight of the cube. We then form a flow network over this grid of nodes by (*i*) giving each node a flow capacity equal to its weight and (*ii*) introducing an infinite-capacity arc between two nodes if the distance between these nodes is r or less. This ensures that the set of nodes in any minimum s–t cut will have thickness of at least r—i.e., the cut will correspond to a discrete r-separating slab. Note that $r > h > 0$, and as $h \to 0$, $r \to 0$, and $h/r \to 0$, the minimum r-separating slab in this discrete representation approaches the minimum surface in the continuous problem.

We may use the max-flow min-cut theorem to solve the discrete Plateau problem after applying two transformations (see ref. 8 for details): (*i*) we introduce a source s connected to all nodes on the lateral wall of the cylinder below the Jordan curve Γ^*, and we introduce a sink t connected to all nodes on the cylinder wall above Γ^*; (*ii*) we convert the node-capacitated network into an equivalent arc-capacitated flow network.

The first transformation forces the minimum surface to span the curve Γ^*, while the second transformation enables us to apply standard algorithms to compute a maximum s–t flow. The maximum s–t flow will determine a minimum s–t cut—i.e., a minimum-weight source–sink separator in the original node-weighted representation—which by the above discussion corresponds to the desired minimal r-separating slab.

Computational experience has confirmed the validity of our approach (8). The algorithm finds a minimal surface within polynomial time—e.g., using the network-flow method of ref. 11 yields $O(n^2)$ time complexity, where n is the number of nodes in the discrete grid representation of $cyl(\Gamma^*)$. Moreover, our methodology solves a generalized problem formulation—namely, with arbitrary weight function and prescribed solution thickness.

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