1. Problem 1.

   - a). A point on the side of a cylinder is uniquely located by height h and angle ϕ (see fig.1). To sample points in a area uniform manner, choose $h = HU_1 = U_1$ (height=1) and $\phi = 2\pi U_2$ where U_1 and U_2 are uniformly distributed random variables. Corresponding xyz coordinates are $x = R\cos\phi = \cos\phi$ (radius=1), $y = \sin\phi$ and $z = h$.
   - b). A point on the side of a cone is uniquely located by height h and angle ϕ (see fig.1). Since the surface area of a unit cone is $\pi\sqrt{2}$, the differential probability is $\frac{dA}{\pi\sqrt{2}}$. Suppose the probability density functions for the two variables are $\rho(h)$ and $\rho(\phi)$, to sample points in a area uniform manner, the differential probability must satisfy $\rho(h)\rho(\phi)\,dh\,d\phi = \frac{dA}{\pi\sqrt{2}} = \frac{1}{\pi\sqrt{2}}\sqrt{2}\,dh\,(1-h)\,d\phi$, so we have $\rho(h)\rho(\phi) = \frac{1-h}{\pi}$. Integrate the equation w.r.t. h and ϕ respectively, consider the fact that $\int_0^1 \rho(h)\,dh = 1$ and $\int_0^{2\pi} \rho\phi\,d\phi = 1$, we have $\rho(h) = 2(1-h)$ and $\rho(\phi) = \frac{1}{2\pi}$. So the C.D.F. is $P(h) = 2h - h^2$ and $P(\phi) = \frac{\phi}{2\pi}$. So choose $h = 1 - \sqrt{U_1}$ and $\phi = 2\pi U_2$ to sample points.
   - c). A point on a unit sphere is uniquely located by the two angles θ and ϕ. Using the same derivation of b), we get $\rho(\theta)\rho(\phi) = \frac{\sin\theta}{4\pi}$, where 4π is the surfaces area of unit sphere. Integrate the eq with respect to θ and ϕ we get $\rho(\theta) = \frac{\sin\theta}{2}$ and $\rho = \frac{1}{2\pi}$, and then $P(\theta) = \frac{1-\cos\theta}{2}$, $P(\phi) = \frac{\phi}{2\pi}$. So we choose $\theta = \arccos(1 - 2U_1)$, $\phi = 2\pi U_2$ to sample points.

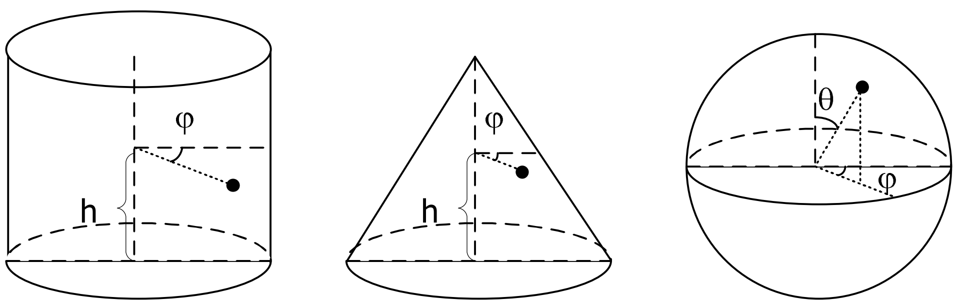

Figure 1: Three figures for problem 1.

2. Problem 2. Suppose the normalized micro-facet distribution is $\rho(\alpha) = c\,D(\alpha)$ where c is constant. The integral $\int_\omega D(\alpha)\cos\alpha\,d\omega = \int_\theta\int_\phi e^{-\cos^2\theta/\cos^2\beta}\cos\theta\sin\theta\,d\theta d\phi = \pi\cos^2\beta(1 - e^{-\frac{1}{\cos^2\beta}})$. (The integration turns out to be very easy to work out since you can substitute integral variables several times: $\cos\theta \to x \to x^2$). The normalization constant c is the reciprocal of the integration. Obviously the distribution is symmetric w.r.t. ϕ, so we sample $\phi = 2\pi U_2$, now $\rho(\theta) = 2\pi c\,e^{-\cos^2\theta/\cos^2\beta}\cos\theta\sin\theta$, hence $P(\theta) = \int_{\theta=0}^{\Theta}\rho(\theta) = \pi c\cos^2\beta(e^{-\cos^2\Theta/\cos^2\beta} - e^{-1/\cos^2\beta})$, and after cleaning up this mess, we get θ is chosen according to $(e^{-\cos^2\Theta/\cos^2\beta} - e^{-1/\cos^2\beta})/(1 - e^{-1/\cos^2\beta}) = U_1$ (Just compute the reverse function). As stated before, ϕ is chosen according to $\phi = 2\pi U_2$.

3. Problem 3. The normalization constant c is equal to the reciprocal of the integration $\int_\omega D(\alpha)\cos\alpha\,d\omega = \frac{2\pi}{7}$. Still since the distribution is symmetric w.r.t. ϕ, we work out $P(\theta) = 1 - \cos^7\theta$, hence to correctly sample points, we choose $\theta = \arccos\sqrt[7]{U_1}$ and $\phi = 2\pi U_2$.