L24

4102

11.18.2013

max flow

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1. what is the general approach to solving a max-flow problem?

finding augmenting paths in residual graphs.

2. when FF finishes, how do we know the answer is correct?

Because we can identify a cut associated with the resulting flow such that $|f| = ||S,T||$.

By max-flow-min cut thm, this $\Rightarrow$ f is max.

userid:

Max flow

Min Cut

FORD-FULKERSON

INITIALIZE $f(u,v) \leftarrow 0 \; \forall u,v$

WHILE EXISTS AN AUGMENTING PATH p IN G_f

AUGMENT f WITH $c_f(p) = \min_{(u,v) \in p} c_f(u,v)$

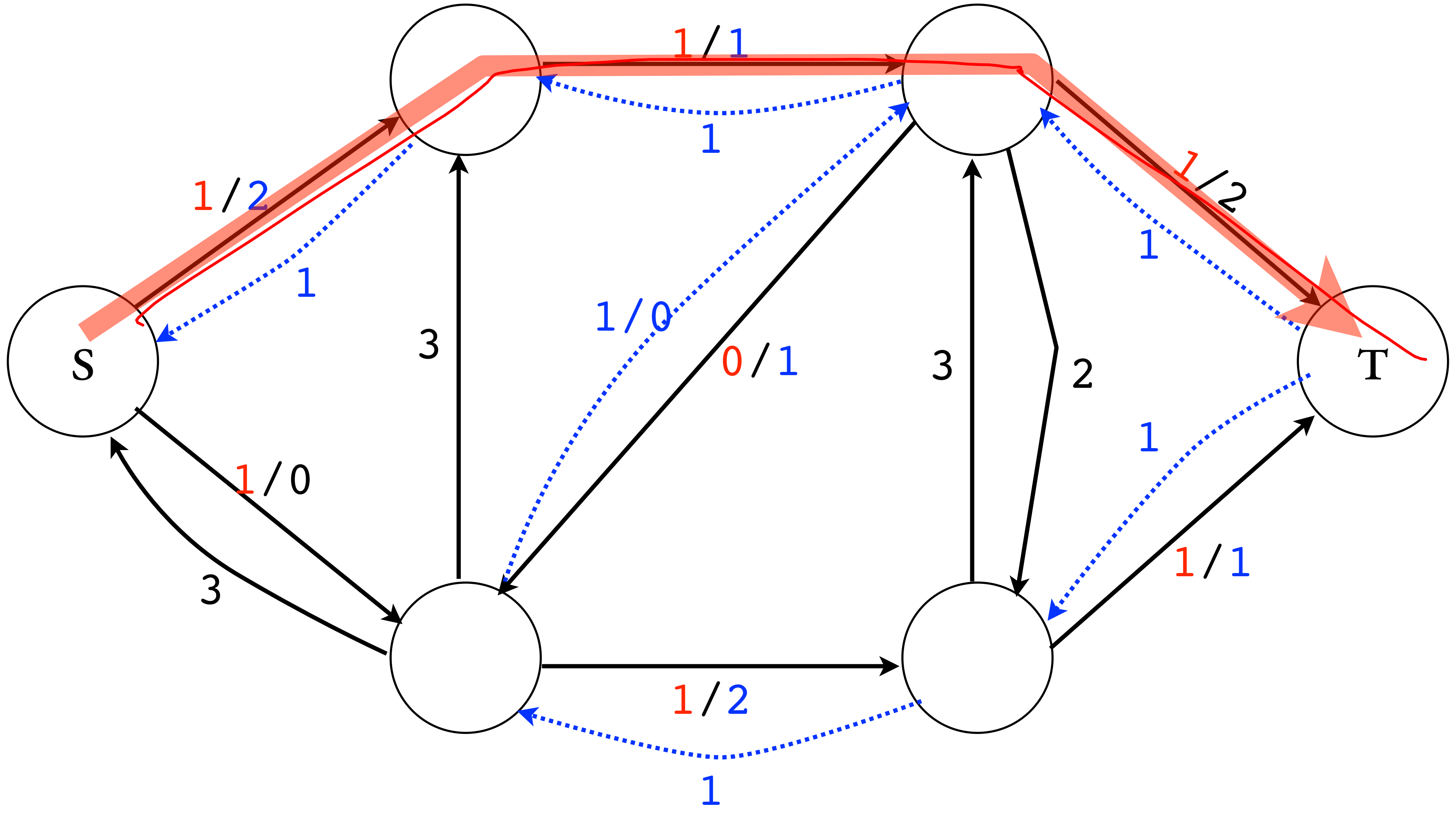
why does FF work? (high level)

EDMONDS-KARP

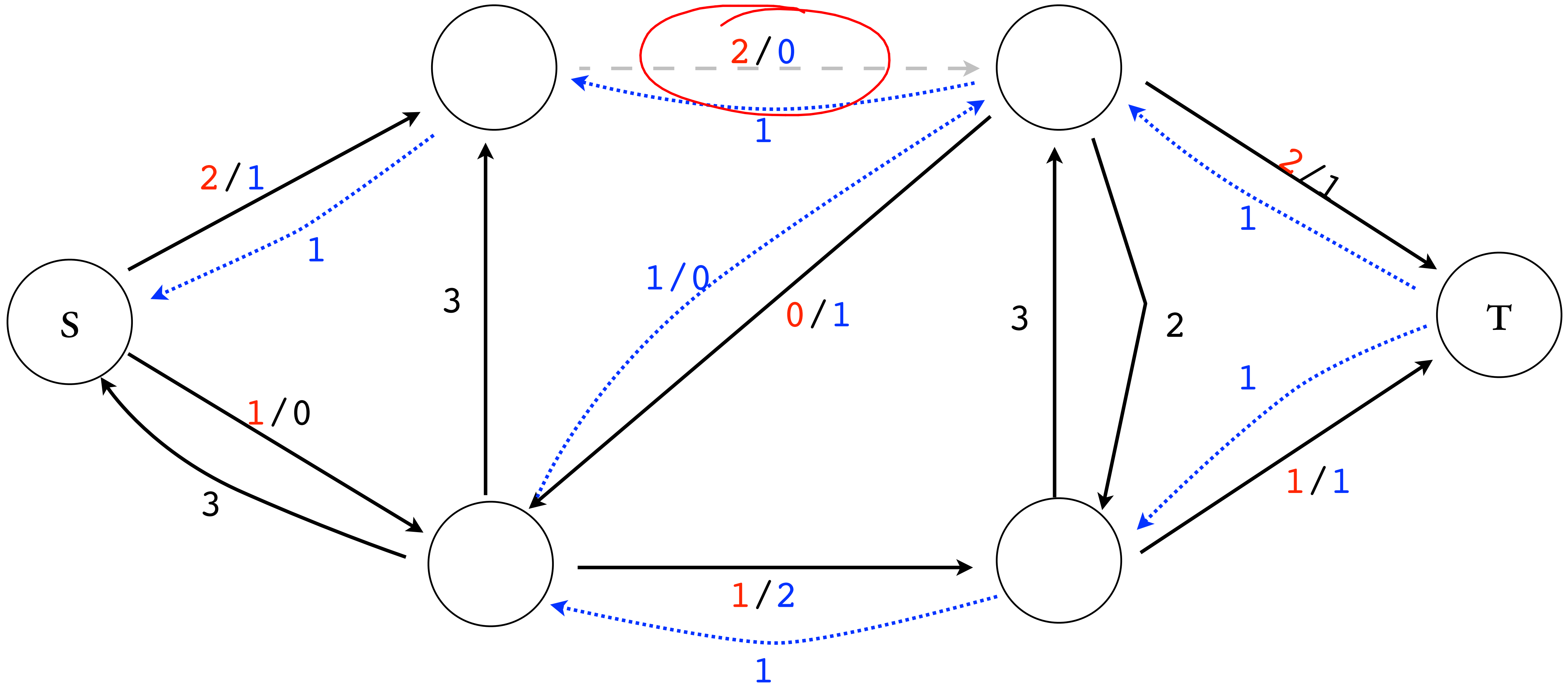
INITIALIZE $f(u,v) \leftarrow 0 \; \forall u,v$

WHILE EXISTS AN AUGMENTING PATH p IN G_f **(use BFS to find it)**

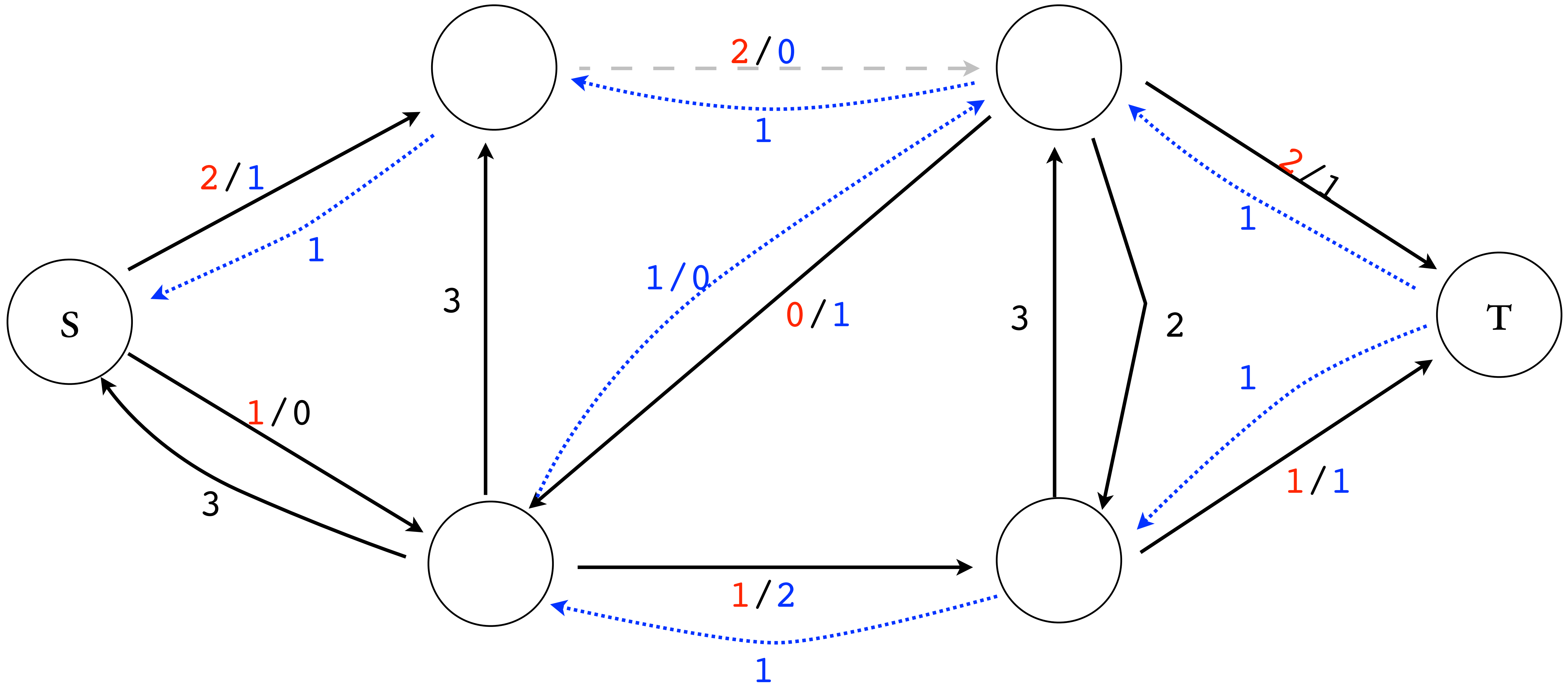
AUGMENT f WITH $c_f(p) = \min_{(u,v) \in p} c_f(u,v)$



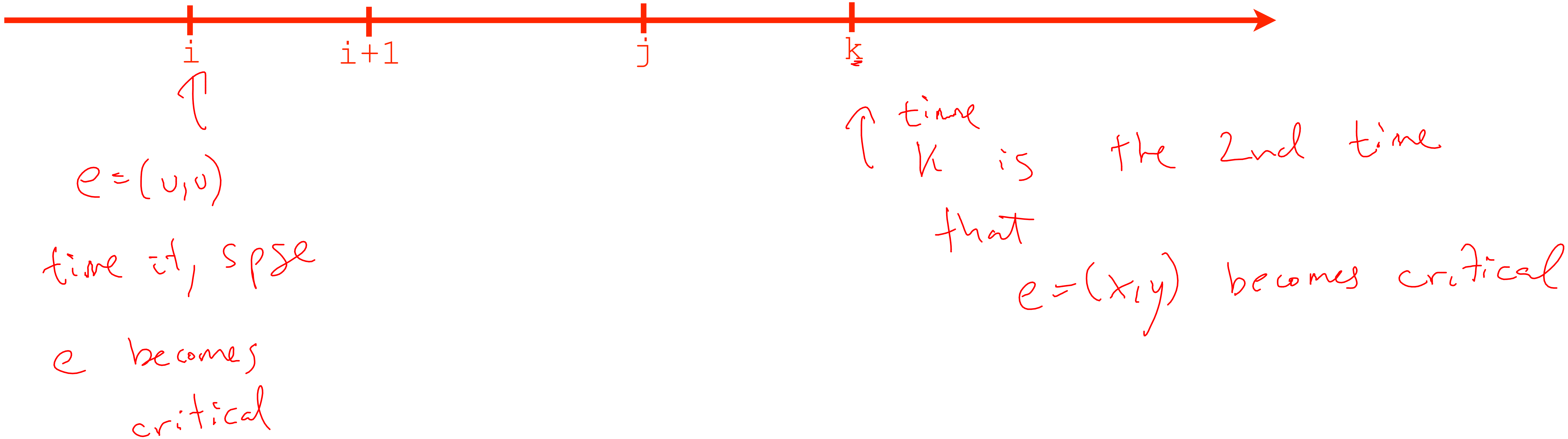
FOR EVERY AUGMENTING PATH, SOME EDGE IS CRITICAL.



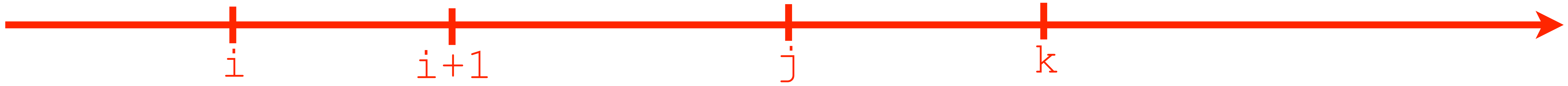
CRITICAL EDGES ARE REMOVED IN NEXT RESIDUAL GRAPH.



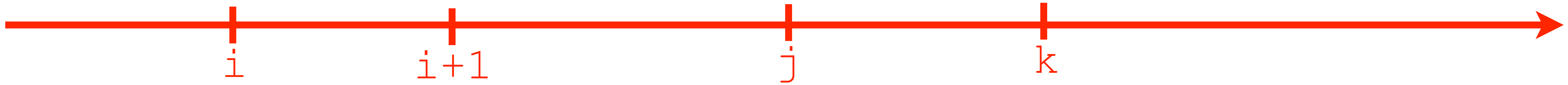
KEY IDEA: HOW MANY TIMES CAN AN EDGE BE CRITICAL?



Outline of the argument

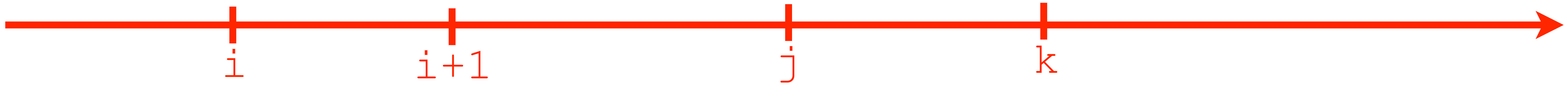


first time (u,v) is critical:



time i+1: (u,v) is critical: $\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$

time j: Edge (u,v) STRIKES BACK



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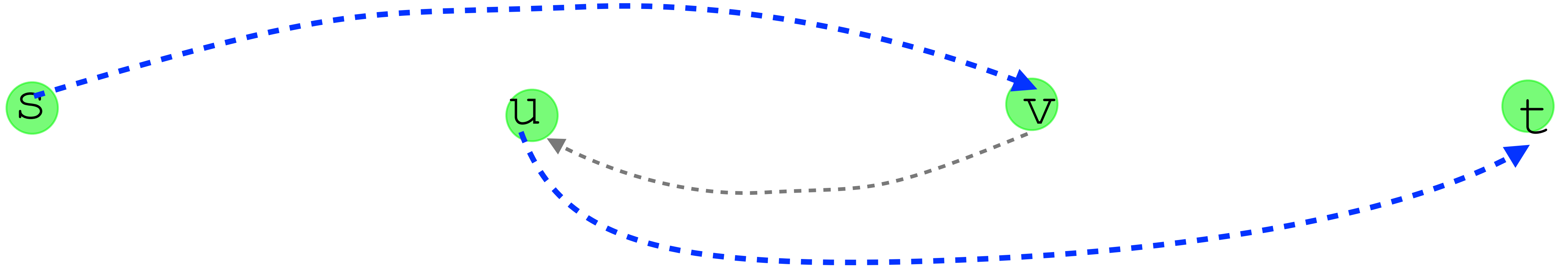
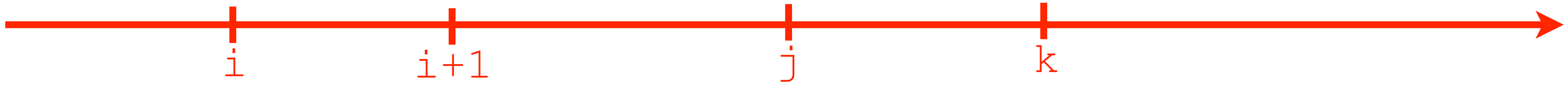


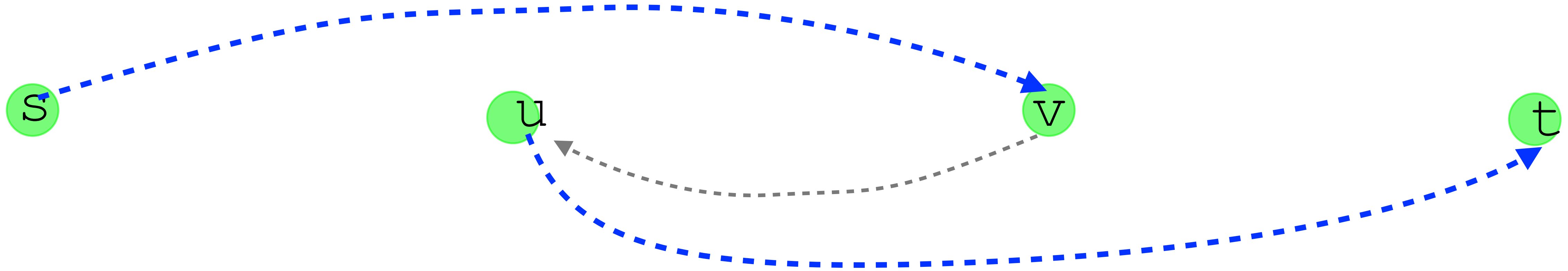
$$\delta_j(s,u) = \delta_j(s,v) + 1$$

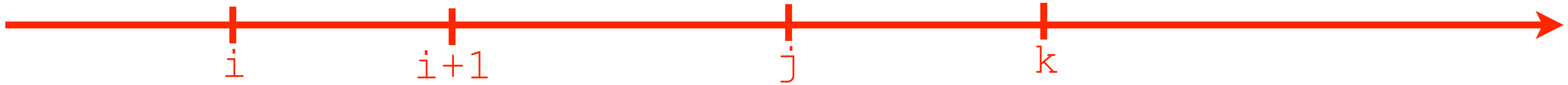


time j: Edge (u,v) STRIKES BACK

$$\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$$

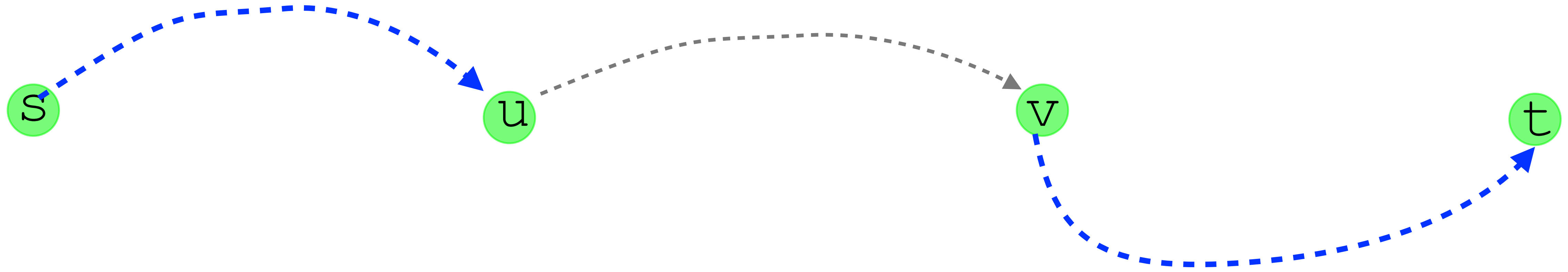
$$\delta_j(s,u) = \delta_j(s,v) + 1$$





time k: RETURN OF THE (u,v) critical

$$\delta_k(s,u) \geq \delta_i(s,u) + 2$$



QUESTION: How many times can (u,v) be critical?

⇒ edge e can become critical $\leq \frac{V}{2}$ times

b/c after $\frac{V}{2}$ times $\delta(s,u) \geq V$,

thus e cannot be on a simple path from $s \leadsto t$.

edge critical only $\frac{V}{2}$ times.

there are only E edges.

ergo, total # of augmenting paths: $\frac{EV}{2}$

time to find an augmenting path: $\Theta(E+V)$ (BFS)

total running time of E-K algorithm:

$$\Theta(E^2V)$$

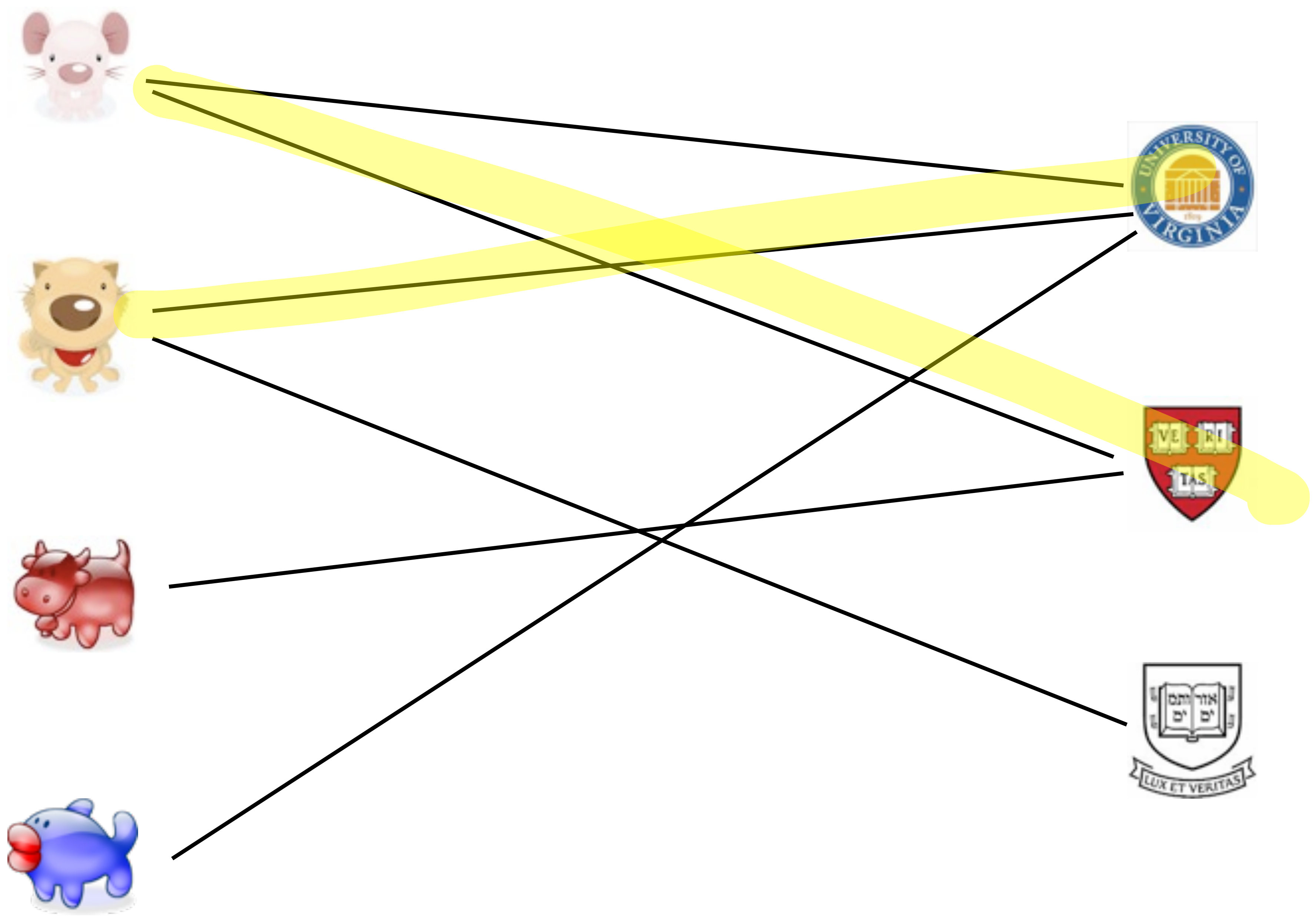
augmenting path.

ff	$O(E\|f^*\|)$
ek2	$\Theta(E^2V)$
push-relabel	$\Theta(EV^2)$
faster push-relabel	$\Theta(V^3)$

APPLICATIONS OF MAX FLOW

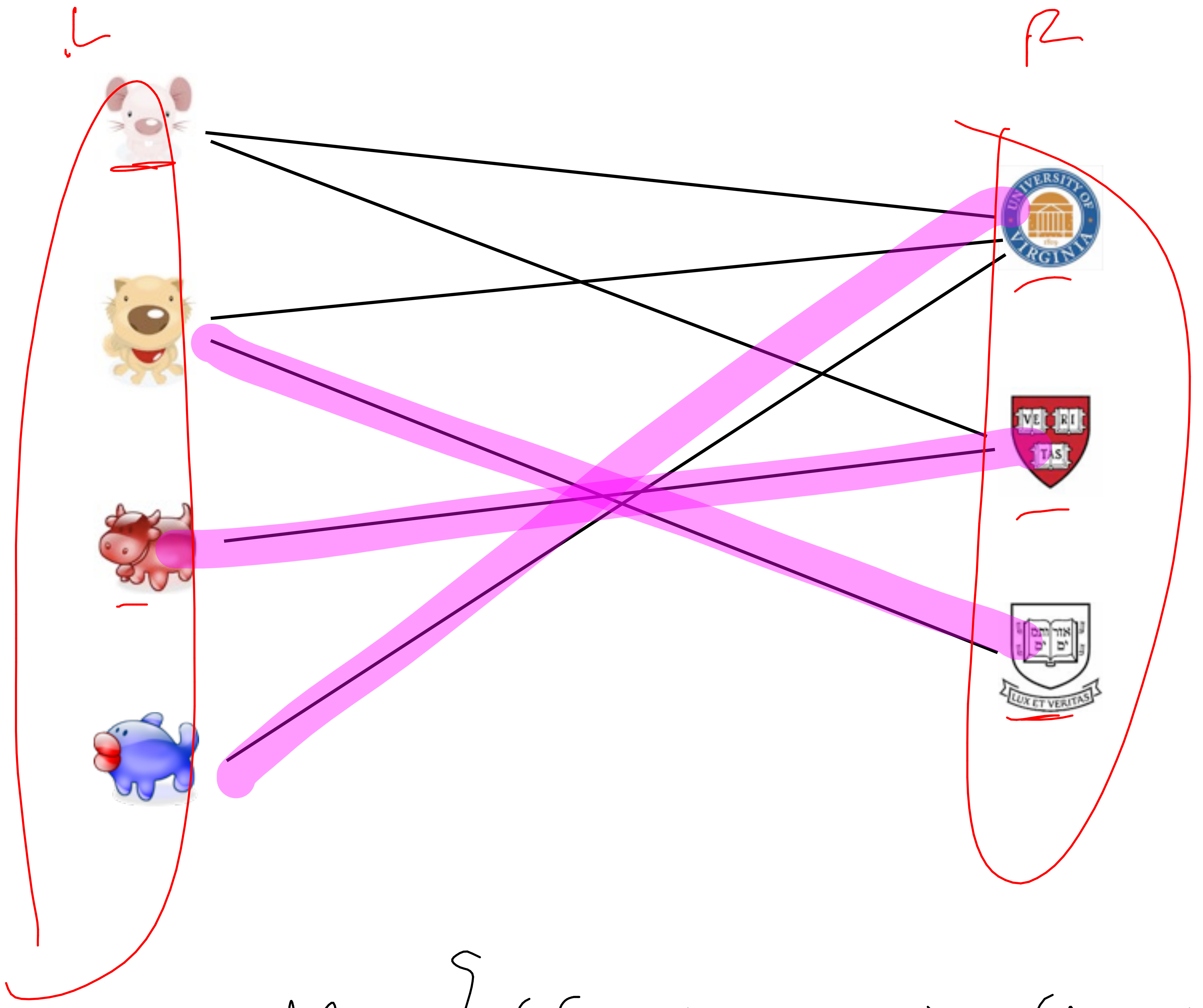
Bipartite Matchings

MAXIMUM BIPARTITE MATCHING



2 people matched

MAXIMUM BIPARTITE MATCHING



3 people are matched.

$M = \{ (f, UVA)\ (c, H)\ (d, y) \}$

BIPARTITE MATCHING

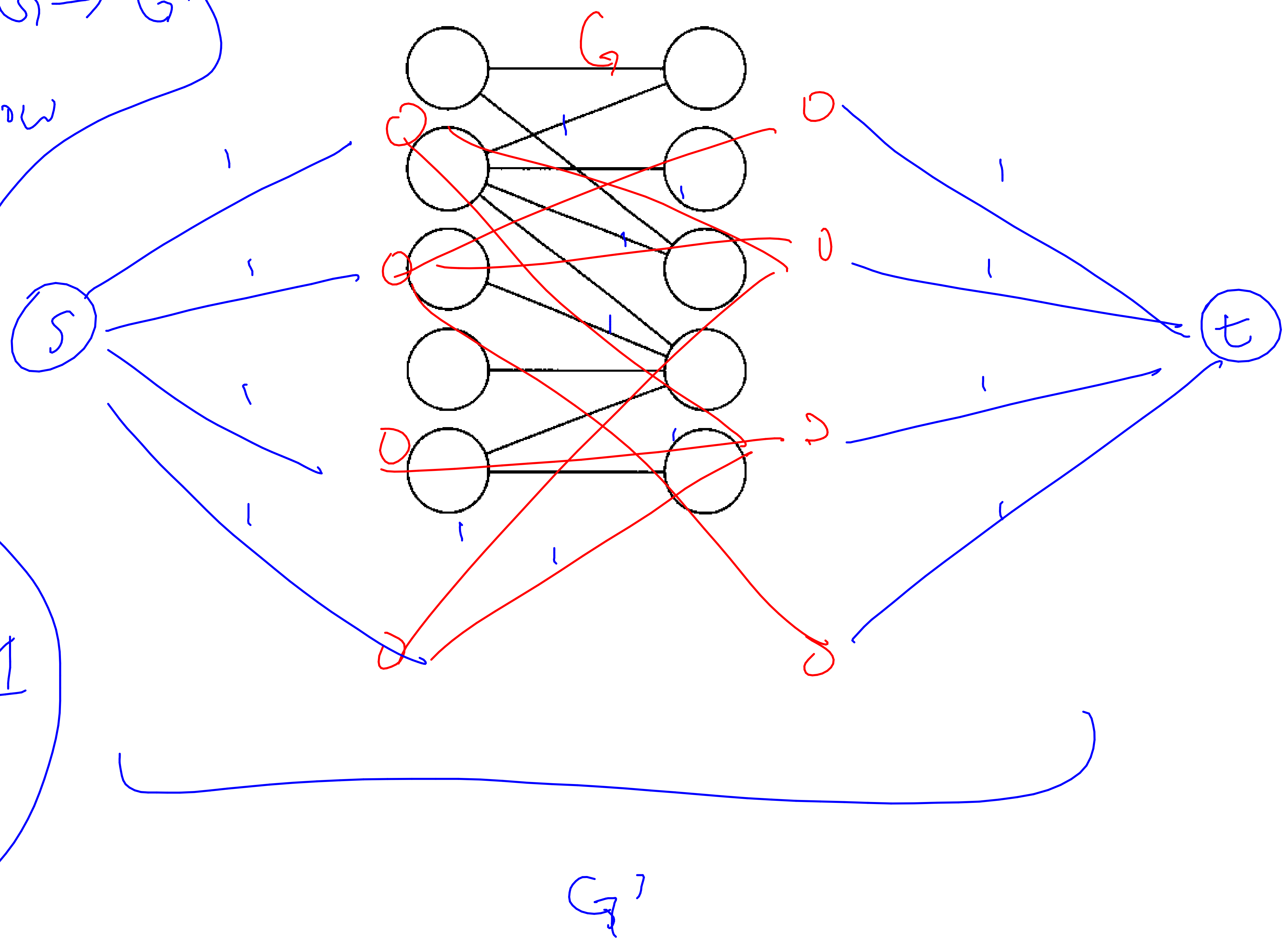
PROBLEM: Given a graph (L, R, E), find the largest set
nodes
of edges $M \subseteq E$ such that each vertex is incident to at most one edge in M.

ALGORITHM

① Transform G → G'

② RUN MAXFLOW

③ output the middle edges with flow f(e) = 1



① Θ(V)

②

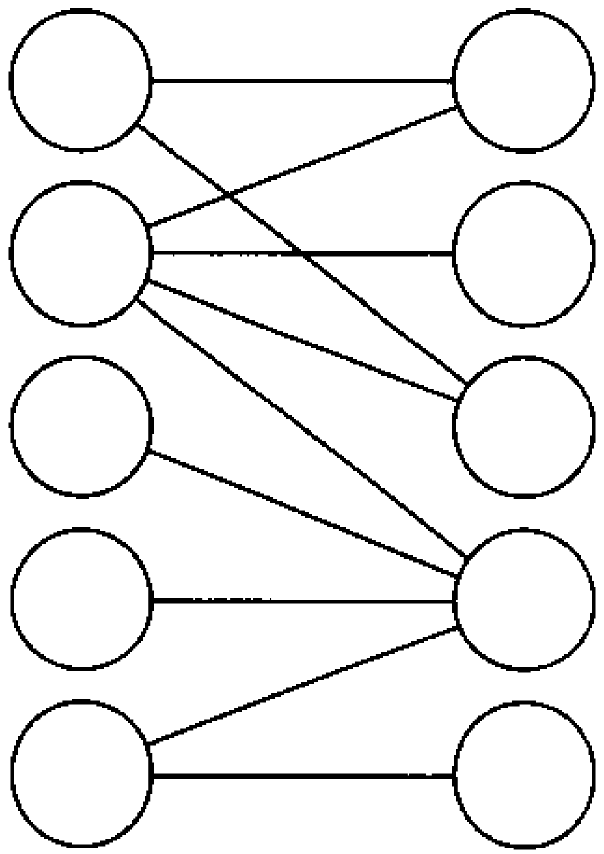
③ Θ(E+V)

ALGORITHM

1. MAKE NEW G'
FROM INPUT G.

2. RUN FF ON G'

3. OUTPUT ALL MIDDLE EDGES
WITH FLOW F(E)=1.



Why does this work??

Need to show:

G has a maximal matching $|M| = K \iff G'$ has MAXFLOW K.

CORRECTNESS

IF G HAS A MATCHING (M) OF SIZE K, THEN ⟹ G' has a flow of K.

① for each edge $e=(x,y) \in M$, set $f(e) = 1$ in G'.

$f(s,x) = 1$

$f(y,t) = 1$

② Verify that this is A FLOW.

flow constraint ✓ capacity constraint

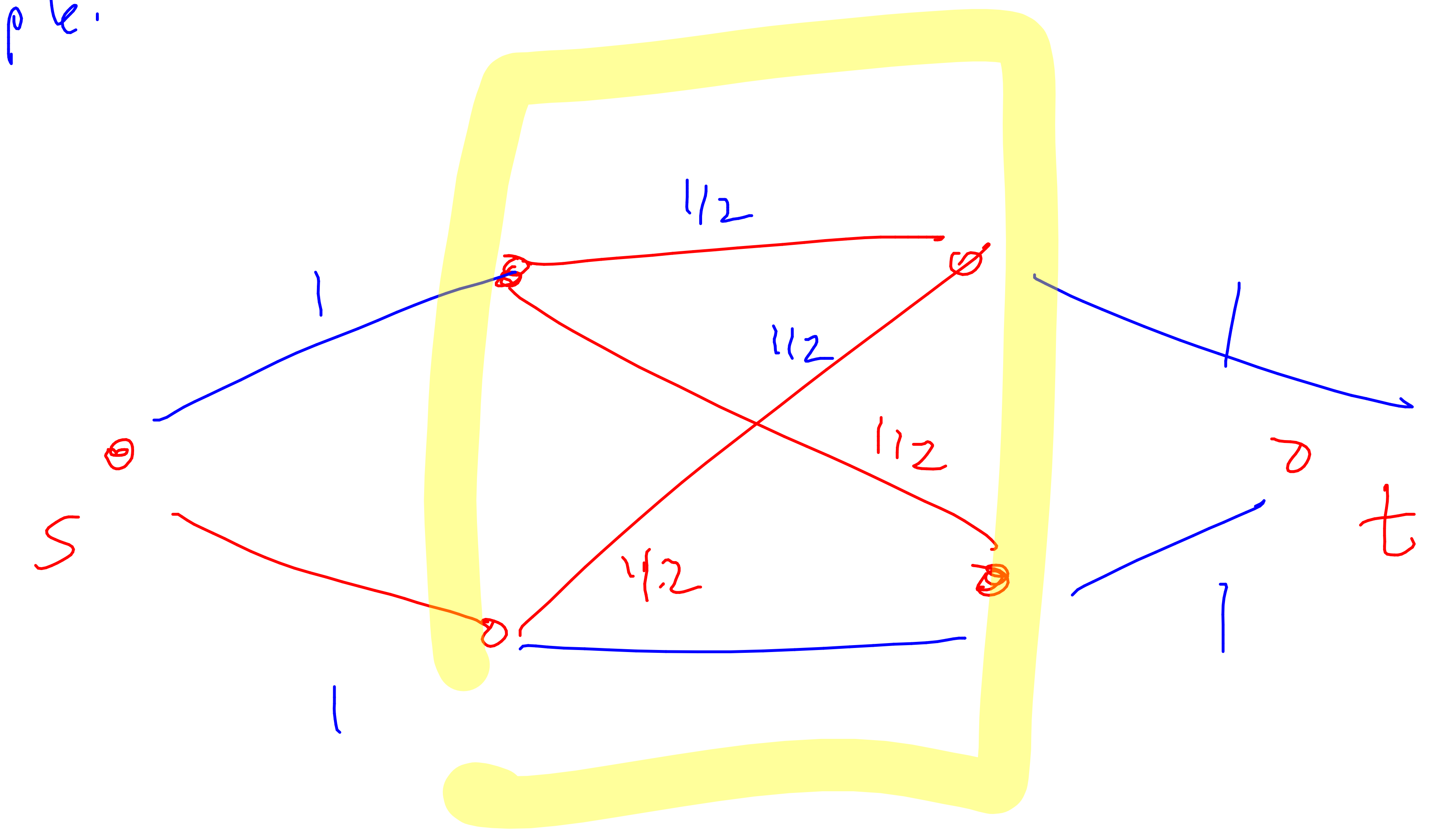
$|M| = K$ ⟹ outgoing flow from S is K

⟹ $|f| = K$.

CORRECTNESS

IF G' HAS A FLOW OF K, THEN G has a matching of size K.

tricky example:



flow of 1 unit.

C(e)=1 for all edges.

INTEGRALITY THEOREM

IF CAPACITIES ARE ALL INTEGRAL, THEN ∃ a MAXFLOW that will be integral

(i.e all flow values will be integers)

Why is this true??

Consider what FF does. At the start, capacities + flow are integral.

Spse this is true for the first k iterations of the FF loop.

On the next loop, the augmenting path will have an integer as the bottleneck edge.

⇒ next residual graph will have integral capacities.

⇒ flow will be integral

CORRECTNESS

IF G' HAS A FLOW OF K, THEN G HAS K-MATCHING.

(1) $\exists$ an integral flow with value K.

$\Rightarrow$ since capacities are 1, then the $f(e) = 0$ or 1

$\Rightarrow$ Now, consider all middle edges $e = (x, y)$ s.t. $f(e) = 1$

$$M = \{ e \mid f(e) = 1 \ \& \ e = (x, y) \}$$

$|M| = K$ & that M is a matching

properties of flow,

flow constraint.

only one edge from s to x $\Rightarrow$ only one edge in M that touches x.

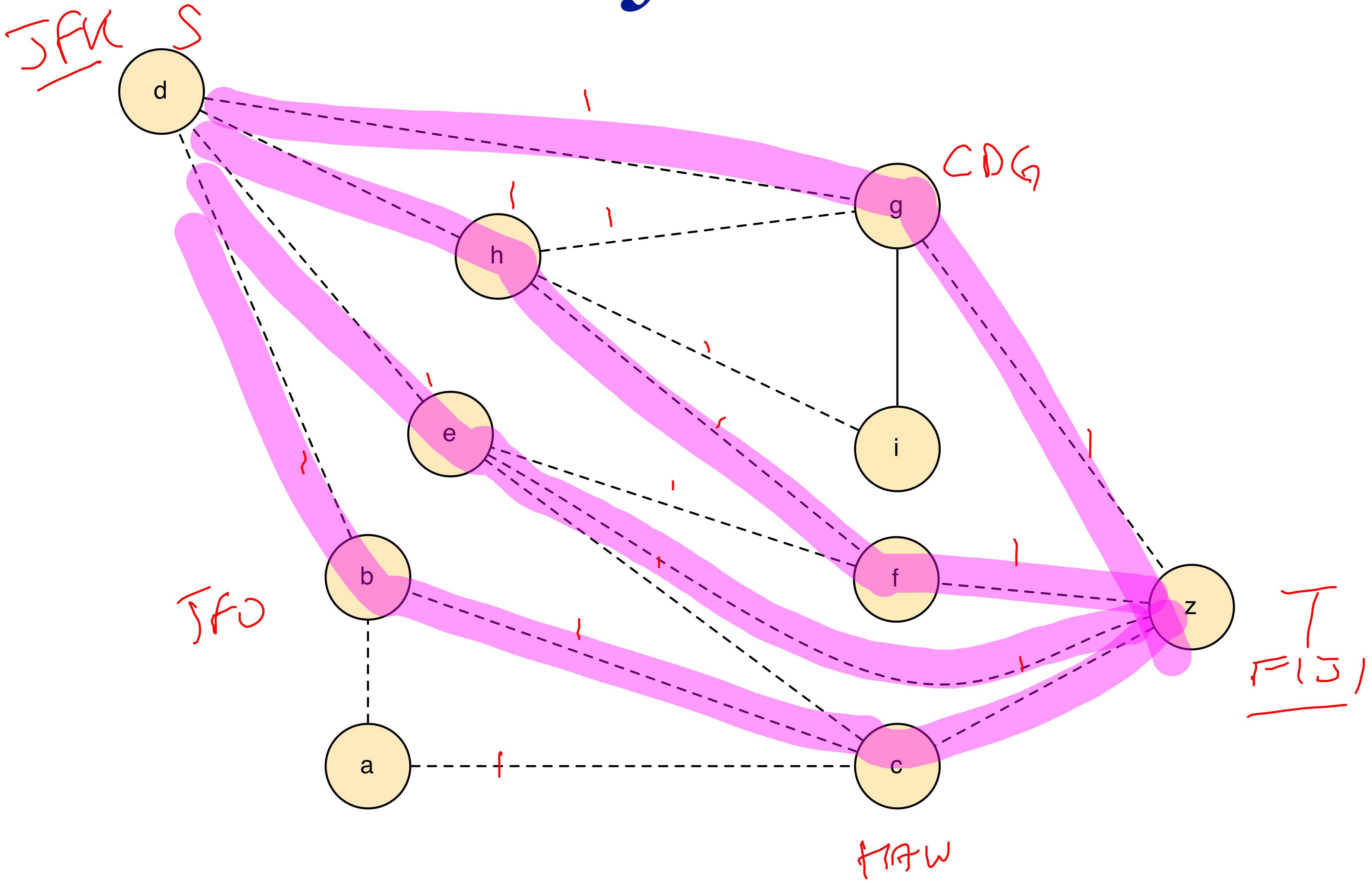
RUNNING TIME

$\Theta(EV)$ EV E^2V

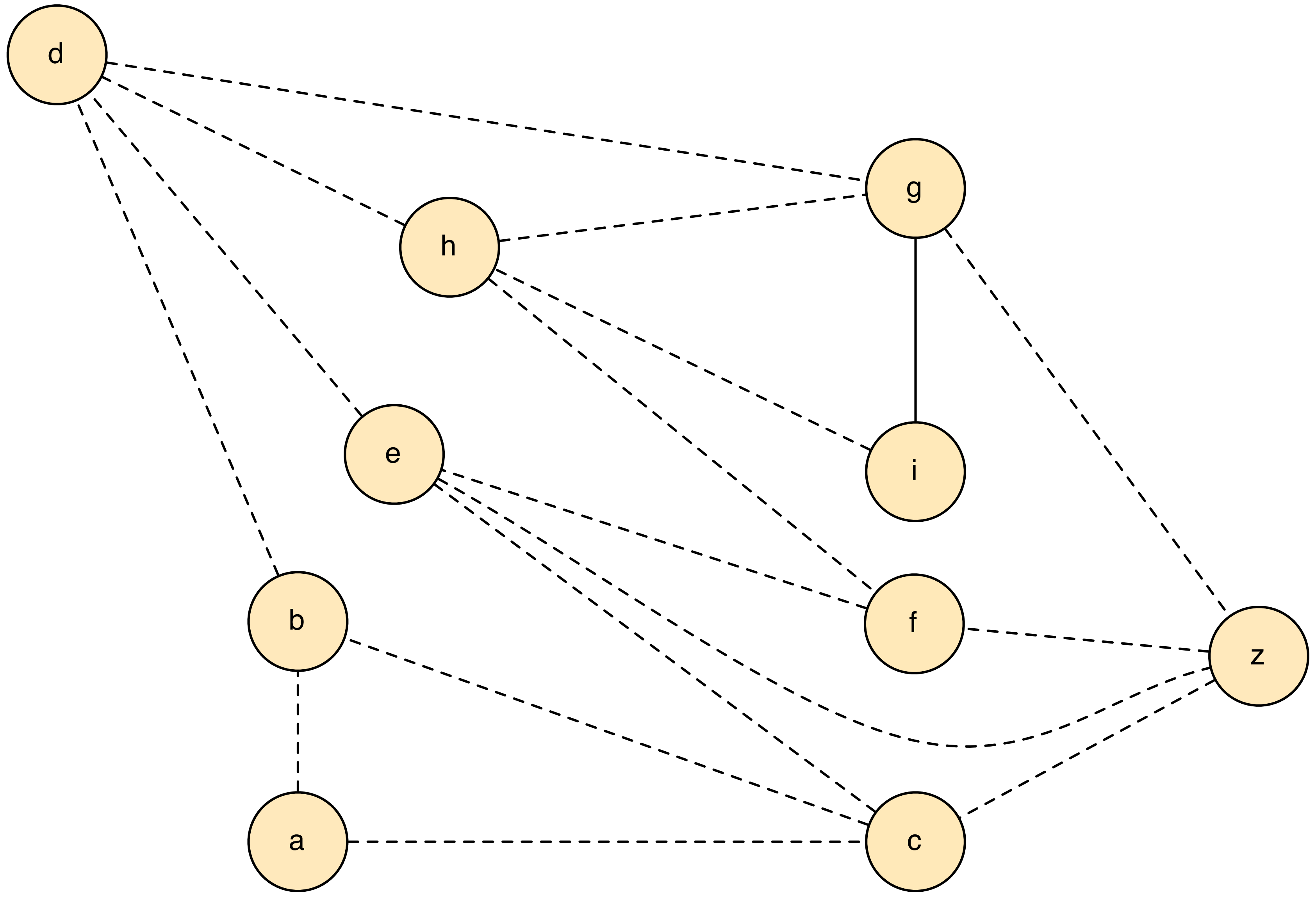
MAX flow $|f| \le V$, by FF analysis, the running time is $\Theta(E \cdot |f^*|)$

$\Rightarrow \Theta(E \cdot V)$

EDGE-DISJOINT PATHS



ALGORITHM



ANALYSIS

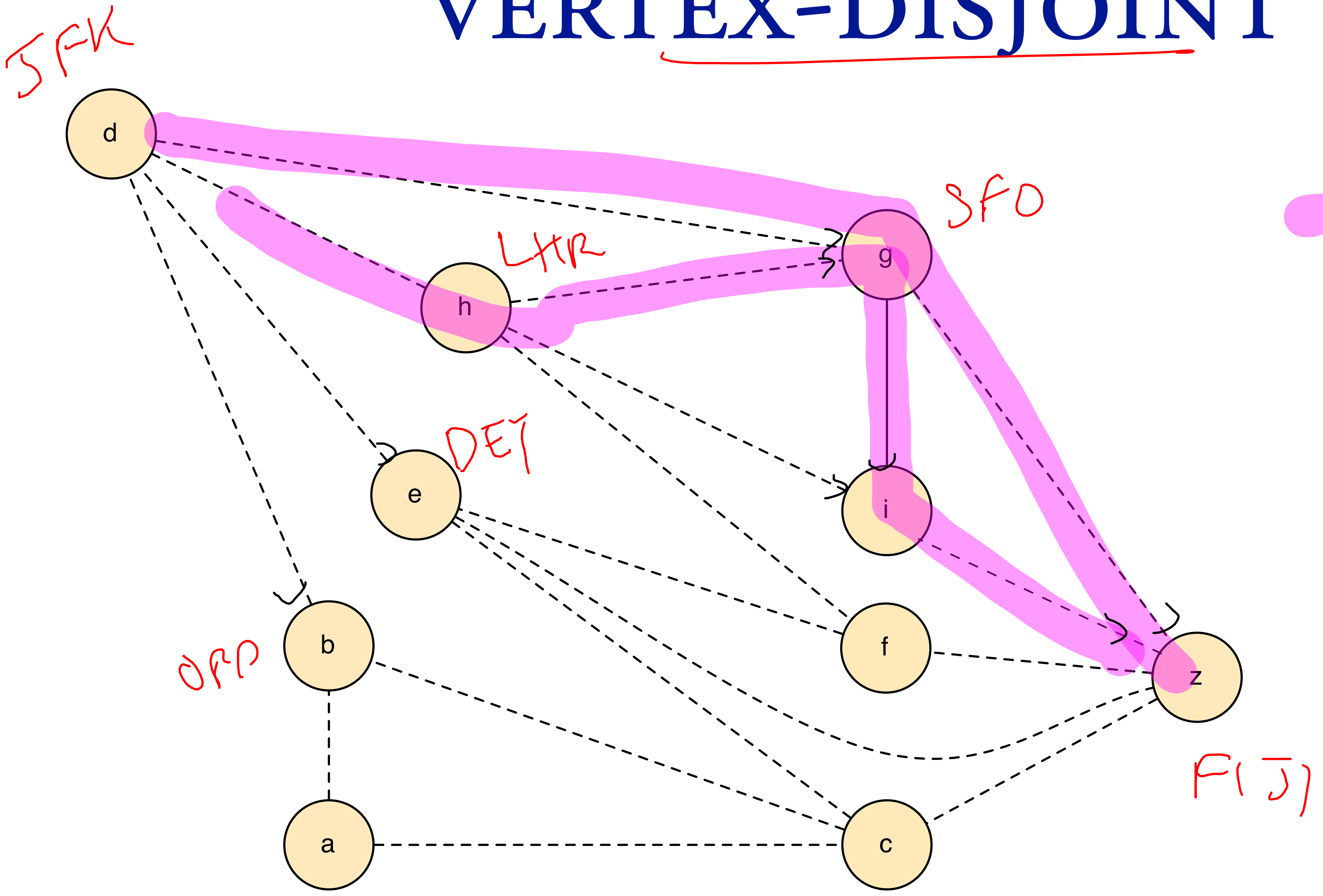
IF G HAS K DISJOINT PATHS, THEN G' has a flow with value K.

If G' has a flow of value K, ⇒ ∃ K disjoint paths from s to t.

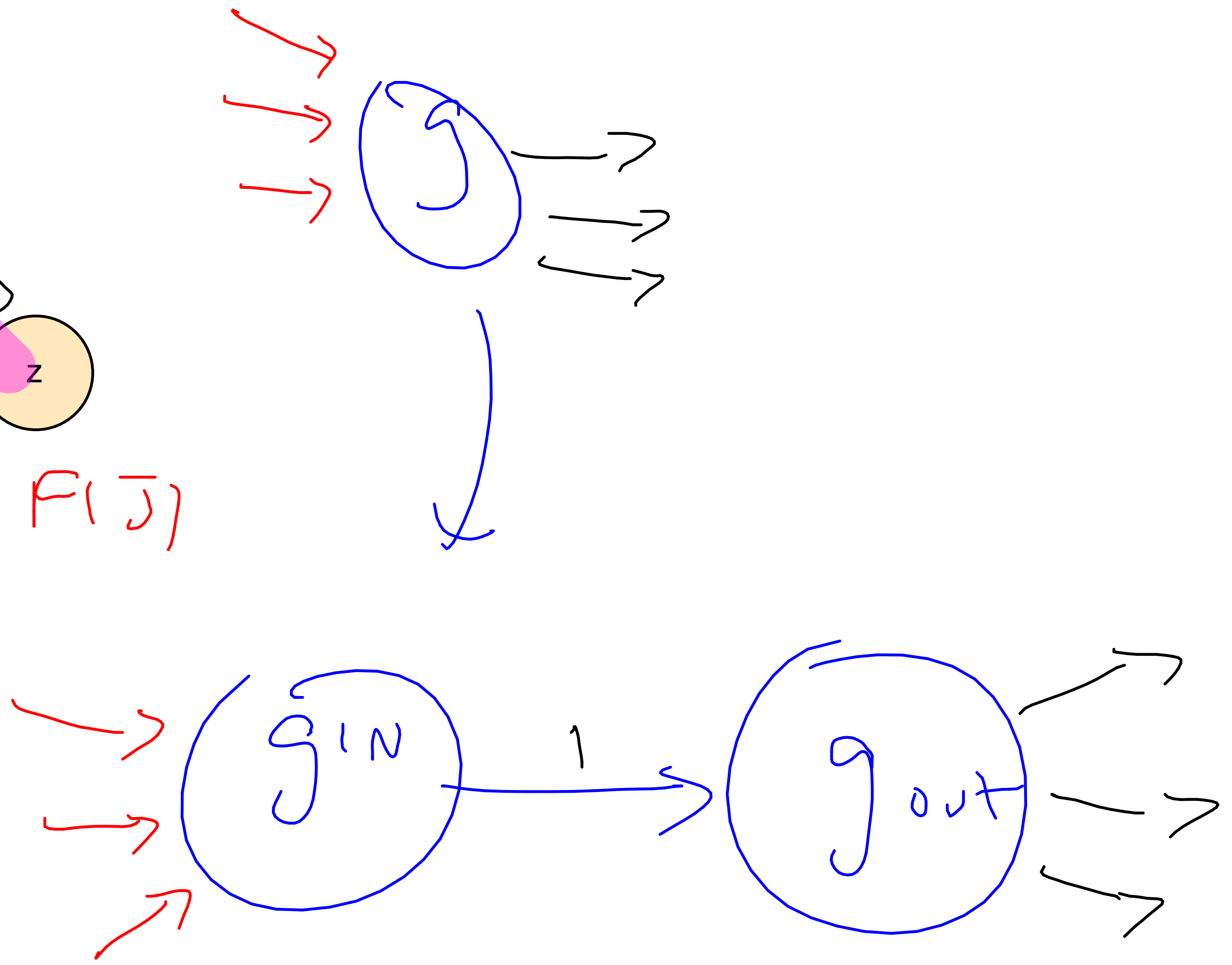
ANALYSIS

IF G' HAS A FLOW OF K, THEN

VERTEX-DISJOINT PATHS



edge disjoint,
but not vertex disjoint



BASEBALL ELIMINATION

	W	L	Left	Against: A	P	N	M
ATL	83	71	8	-	1	6	1
PHL	80	79	3	1	-	0	2
NY	78	78	6	6	0	-	0
MONT	77	82	3	1	2	0	-

BASEBALL ELIMINATION

	W	L	Left	N	B	Bo	T	D
				Against				
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
TOR	63	72	27	7	7			
DET	49	86	27	3	4			

BASEBALL ELIMINATION

	W	L	Left	N	B	Bo	T	D
							Against	
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
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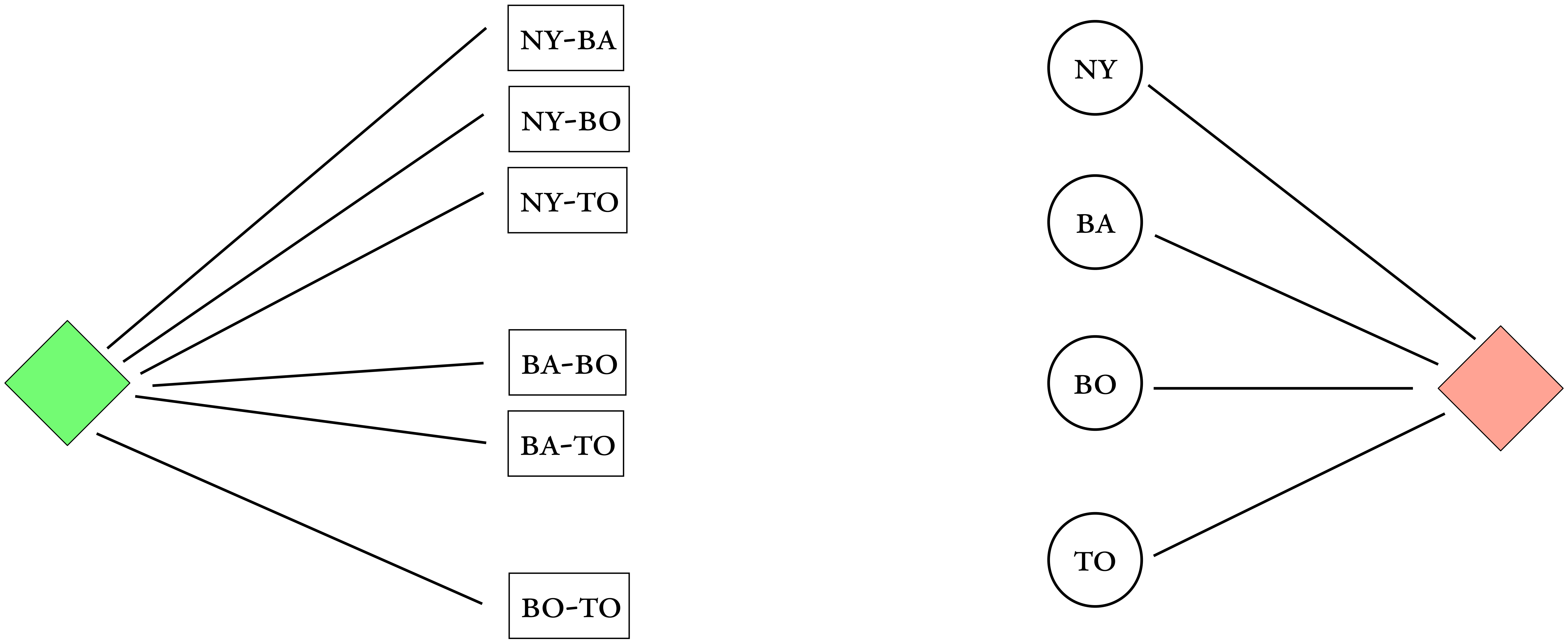
76 ✓

75
71
69
63
27
305

$\frac{305}{4} = 76.25$

$\Rightarrow$ some team must win 77.



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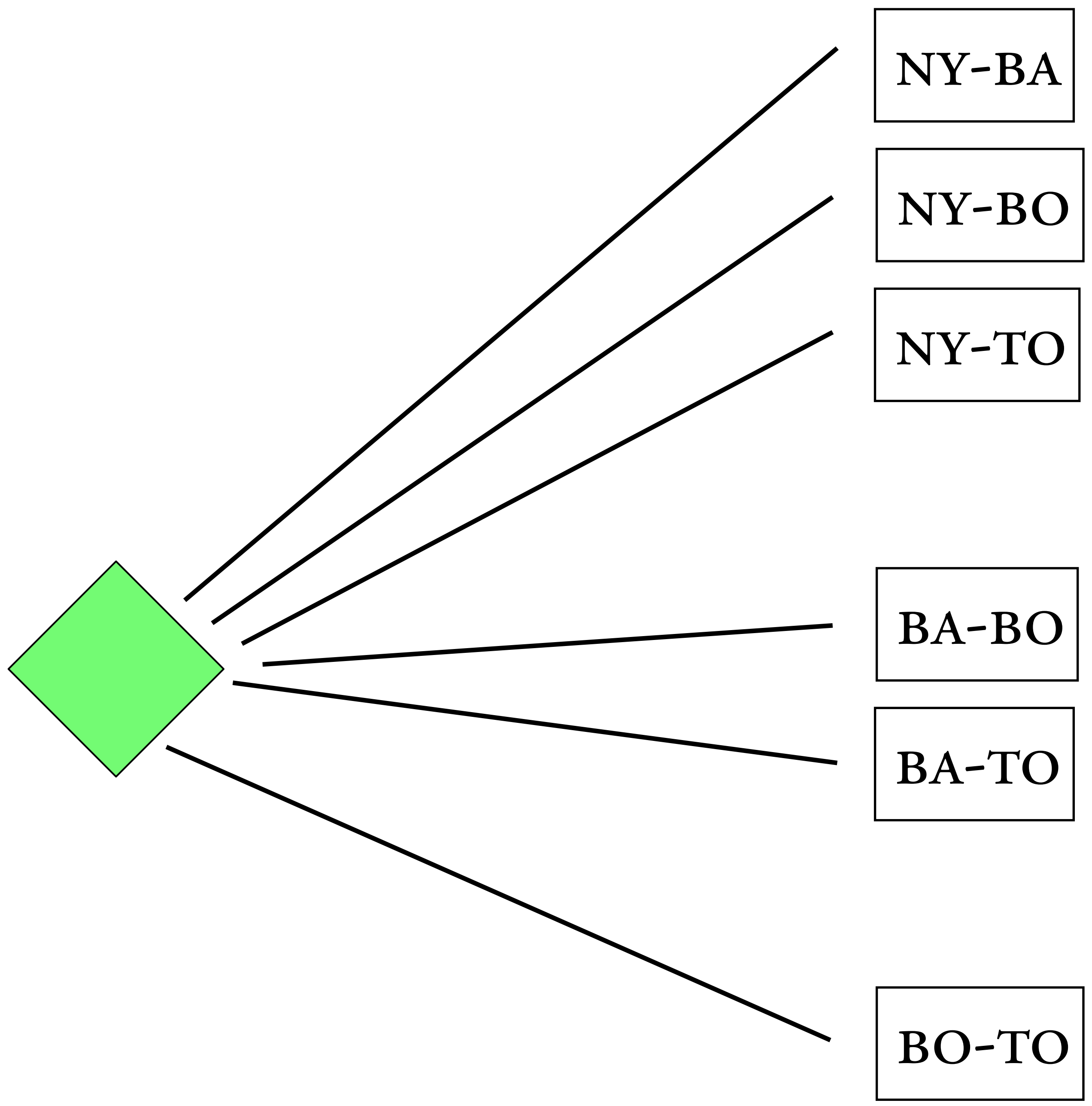

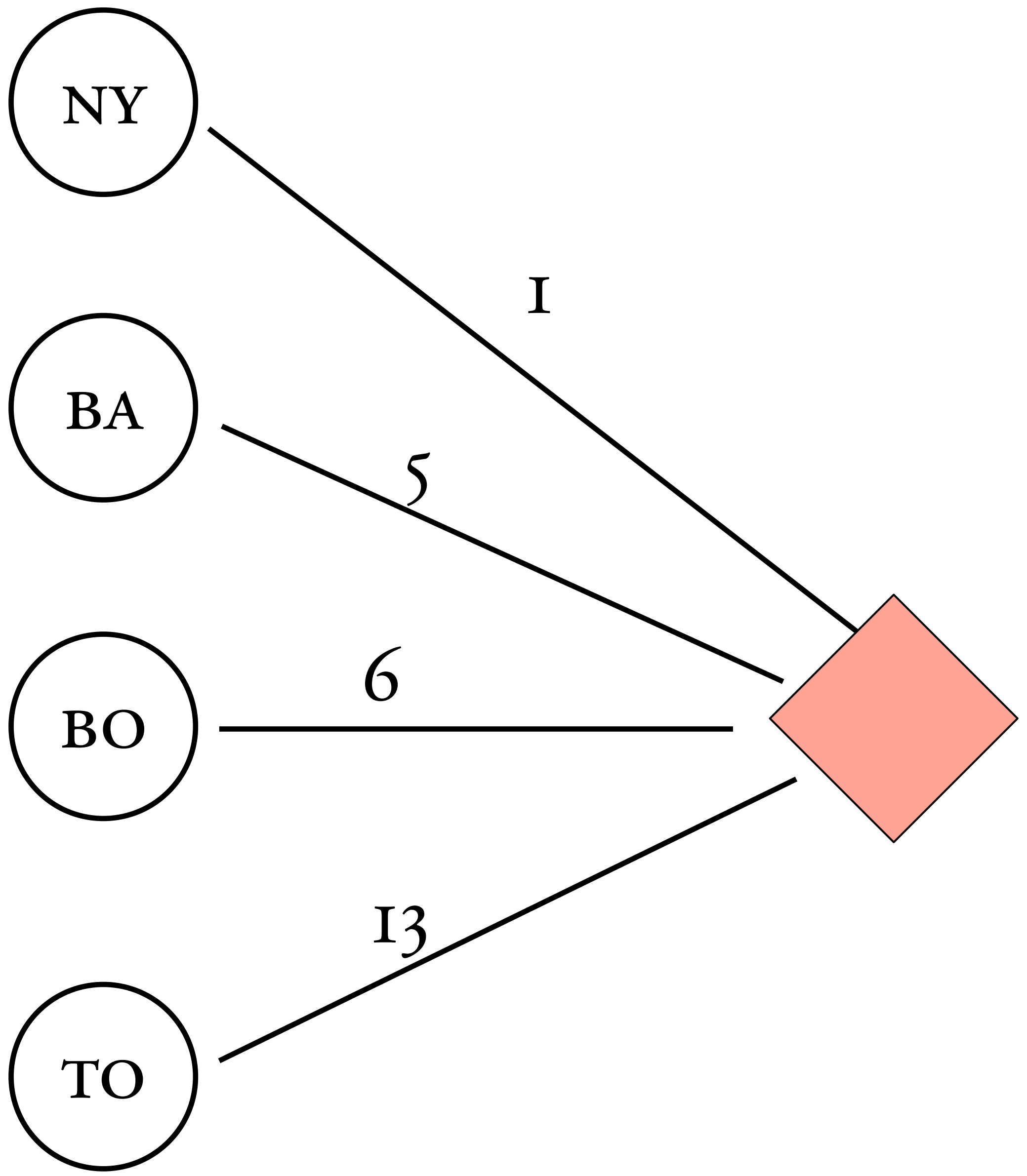

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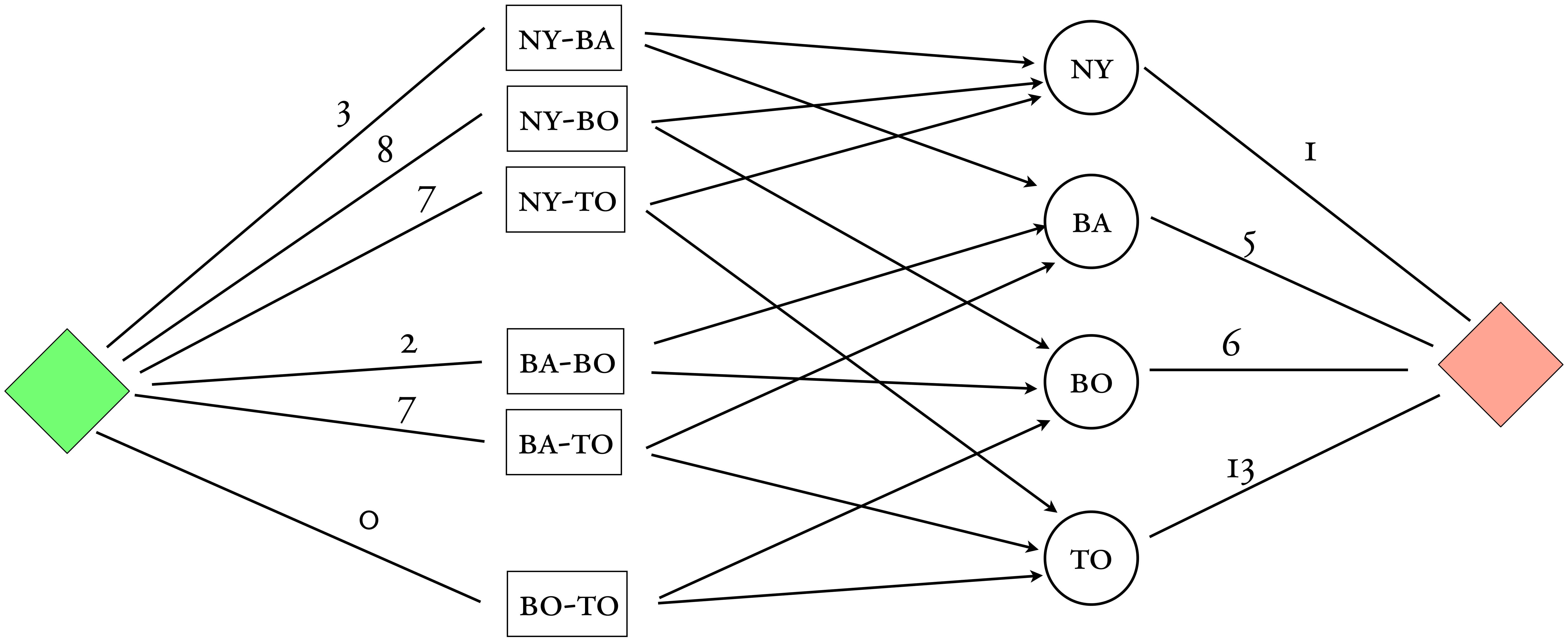

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