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CS 575

Design and Analysis of Computer Algorithms

Professor Michal Cutler

Lecture 22

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This class

- The theory of NP-completeness
  - Tractable and intractable problems
  - The classes P, NP and NPC

Classifying problems

- Classify problems as tractable or intractable.
- Problem is *tractable* if there **exists at least one** polynomial bound algorithm that solves it.
- An algorithm is *polynomial bound* if its worst case growth rate can be bound by a polynomial $p(n)$ in the size n of the problem

$$p(n) = a_n n^k + \ldots + a_1 n + a_0 \text{ where } k \text{ is a constant}$$

- Note: $O(\lg n) \subset O(n)$, $O(n \lg n) \subset O(n^2)$

Intractable problems

- Problem is *intractable* if it is not tractable.
  - **All** algorithms that solve the problem are not polynomial bound.
  - They have a worst case growth rate $f(n)$ which cannot be bound by a polynomial p(n) in the size n of the problem.
- For intractable problems the bounds are:

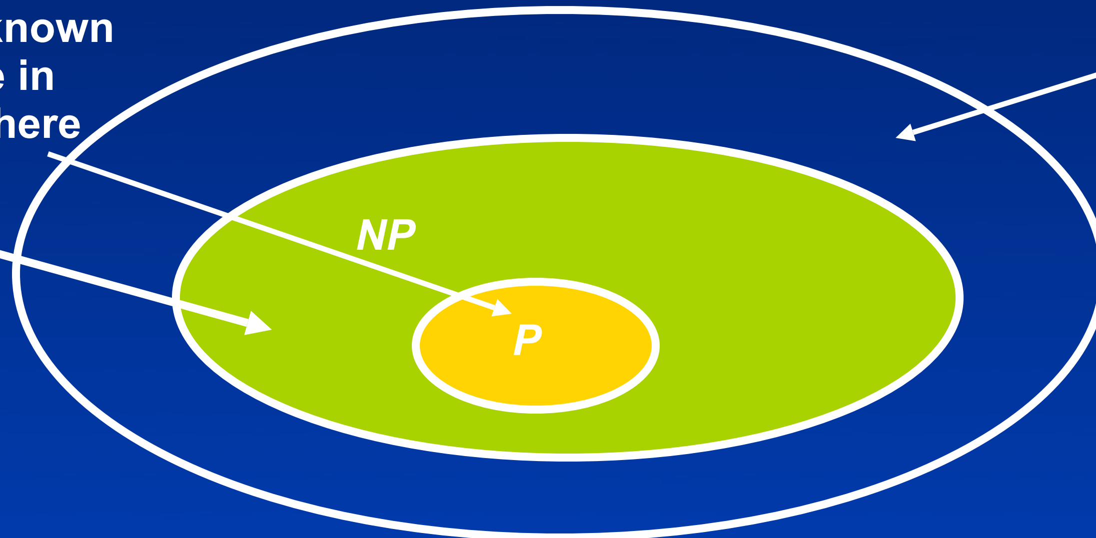
$$f(n) = c^n \text{, or } n^{\log n}\text{, etc.}$$

Why is this classification useful?

- If problem is intractable, no point in trying to find an efficient algorithm
- All algorithms will be too slow for **large inputs**.

Intractable problems

- Turing showed some problems are so hard that no algorithm can solve them (undecidable): Halting problem
- Other researchers showed some decidable problems from automata, mathematical logic, etc. are intractable: Presburger Arithmetic



Hard practical problems

- There are many practical problems for which no one has yet found a polynomial bound algorithm.
- Examples: traveling salesperson, 0/1 knapsack, graph coloring, bin packing etc.
- Most design automation problems such as testing and routing.
- Many networks, database and graph problems.

The theory of NP completeness

- The theory of NP-completeness enables showing that these problems are at least as hard as *NP-complete* problems
- Practical implication of knowing problem is NP-complete is that it is **probably** intractable (whether it is or not has not been proved yet)
- So any algorithm that solves it will probably be very slow for large inputs

We need to discuss

- Decision problems
- Converting optimization problems into decision problems
- The relationship between an optimization problem and its decision version
- The class P
- Verification algorithms
- The class NP
- The concept of polynomial transformations
- The class of NP-complete problems

Decision Problems

Decision Problems

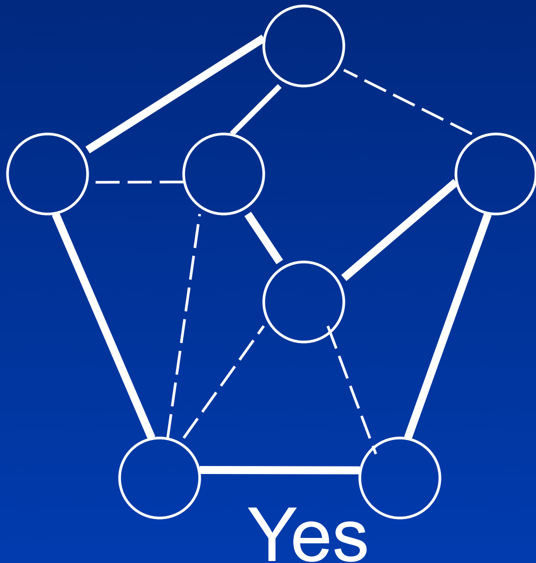
- A *decision* problem answers *yes or no* for a given input

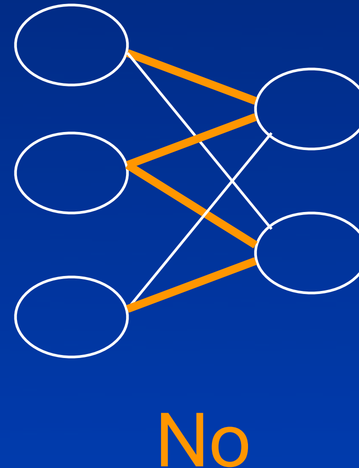
Decision Problems

- A *decision* problem answers *yes or no* for a given input
- Examples:
  - Given a graph *G* Is there a path from *s* to *t* of length at most *k*?
  - Does graph *G* contain a Hamiltonian cycle?
  - Given a graph *G* is it bipartite?
  - Given a bipartite graph does it contain a perfect match?
  - Can G be colored with at most k colors?
  - Given a flow network with supply and demand

A decision problem: HAMILTONIAN-CYCLE

- A *Hamiltonian cycle* of a graph *G* is a cycle that includes each vertex of the graph exactly once.
- Problem: Given a graph G, does G have a Hamiltonian cycle?





Converting to decision problems

- Optimization problems can be converted to decision problems (typically) by adding a bound B on the value to optimize, and asking the question:
  - Is there a solution whose value is at most B? (for a minimization problem)
  - Is there a solution whose value is at least B? (for a maximization problem)

An optimization problem: traveling salesman

- Given:
  - A finite set $C=\{c_1,\ldots,c_m\}$ of cities,
  - A distance function $d(c_i, c_j)$ of nonnegative numbers.
- Find the length of the **minimum** distance tour which includes every city exactly once

A decision problem for traveling salesman (TS)

- Given a finite set $C=\{c_1,\ldots,c_m\}$ of cities, a distance function $d(c_i, c_j)$ of nonnegative numbers and a bound B
- Is there a tour of all the cities (in which each city is visited exactly once) with total length **at most B**?
- There is no known polynomial bound algorithm for TS.

Relationship between optimization and decision version

- Have a solution to the optimization problem:
  - Find the optimal solution
  - Compare optimal solution to the bound B and answer “yes” or “no”.
- Example: Does G contain path from s to t of length <=B?

  D← Dijkstra(s) //length of shortest path from s to t
  return (D[t]<=B)

- **Tractable optimization problem → tractable decision problem**

Relationship between optimization and decision version

- **If the decision problem is "hard" → optimization problem is also "hard"**
  - Why? If the optimization was easy then the decision problem is easy.

The class P

The class P

- P is the class of decision problems that have polynomial bounded algorithms

The class P

- P is the class of decision problems that have polynomial bounded algorithms
- Problem in P
  - Given a weighted graph G, is there a spanning tree of weight at most B?
  - Is graph G a tree?
  - Is graph G acyclic?
  - Is G connected?
  - Can G be colored with two colors?
  - Is array A sorted?

Encoding and Time Complexity

- Run time is a function of problem size.
- An encoding is used to represent a problem instance.
- The size of a problem instance is the number of characters in a (reasonable) encoding of its input
- **The chosen encoding affects the time complexity of an algorithm!**

Effect of encoding on run time

- Assume: input to an algorithm is an integer *k*, and the algorithm's complexity is $\Theta(k)$
- A unary encoding of input k has size = k.
- A binary encoding of k has size = $\lfloor \lg k \rfloor + 1$.
- Unary encoding would make the algorithm linear
- Binary encoding shows the algorithm to be exponential (size <= lg k +1, $k >= 2^{size-1}$)
- We prefer to consider this algorithm exponential!

What is a reasonable encoding?

- Numbers should not have unary representation
- Binary, octal, decimal, ect, are reasonable
- Representation should not have “padding” to artificially increase problem size

Is the algorithm polynomial bound?

- The sieve algorithm for finding whether a number *n* is prime divides it by 2, 3, …, $\sqrt{n}$
- It is not polynomial bound

$\sqrt{n}$

- Notice that there is no difference in complexity if the encoding is in binary or in decimal.

Other definitions of P

- The class P can be defined as a class of formal languages:
  - P = { L | there exists an polynomial bounded algorithm A that decides L}
  - P = { L | L is accepted by a polynomial time algorithm}
- We can prove the second definition given the first

The goal of verification algorithms

- The goal of a verification algorithm for a problem is to verify a 1 answer for a given problem instance (i.e., if the answer is 1 the verification algorithm verifies this answer)

- The inputs to the verification algorithm are:
  - the original input (problem instance) and
  - a *certificate* (some representation of a solution if it exists)

Verification Algorithms

- A *verification algorithm*, takes a problem instance x and *answers* 1, if there **exists** a certificate y such that the answer for x with certificate y is 1

Polynomial bound verification algorithms

- Given a decision problem *d*.

- A verification algorithm for *d* is *polynomial bound*
  if given an *input x* to *d*, there exists a "short" certificate *y*,
  such that ($|y|=O(|x|^c)$) where c is a constant,
  and a polynomial bound algorithm A(x, y) that verifies an answer 1 for d with input x

Note: |y| is the size of the certificate, |x| is the size of the input

A verification algorithm for TS

- Given a problem instance x for TS and a certificate *y*
  - Check that *y* is indeed a cycle that includes every vertex exactly once
  - Compute the length of the cycle
  - Verify that the length of the cycle is at most B
- Is the size of y polynomial in the size of x?
- Is the verification algorithm polynomial?

The class NP

- NP is the class of decision problems for which there is a polynomial bounded verification algorithm
- It can be shown that:
  - all decision problems in P, and
  - decision problems such as traveling salesman, knapsack, bin pack, are also in NP

The relation between P and NP

- $P \subseteq NP$

- If a decision problem is solvable in polynomial time, a polynomial time verification algorithm can easily be designed that *ignores the certificate* and answers “yes” for all inputs with the answer “yes”.

The relation between P and NP

- **PATH**: Does G (non-negative weights) contain path from *s* to *t* of length <=B?

```
verPath(G, s, certificate)
    D← Dijkstra(s)
    if (D[t]<=B)
        return "yes"
```

The relation between P and NP (cont)

- It is not known whether P = NP.
- Problems in P can be *solved* "quickly"
- Problems in NP can be *verified* "quickly".
- It is probably easier to verify a solution than to solve a problem.
- Many researchers believe that P and NP are not the same class.

Reducibility

- We will study the relative hardness of languages through polynomial-time reducibility.
- A problem Q can be reduced to another problem Q' if any instance of Q can be rephrased as an instance of Q', such that the solution to the instance of Q' provides a solution to the instance of Q.

Example

- The problem of solving linear equations reduces to the problem of solving quadratic equations.
- Given an instance (a, b) to solve the linear equations problem $ax + b = 0$, we can create an instance (0, a, b) to solve the quadratic equations problem $0x^2+ax+b=0$, whose solution provides a solution to our original problem.

Polynomial reductions

- **Motivation**: The definition of NP-completeness uses the notion of *polynomial reductions* of one problem π_1 to another problem π_2, written as

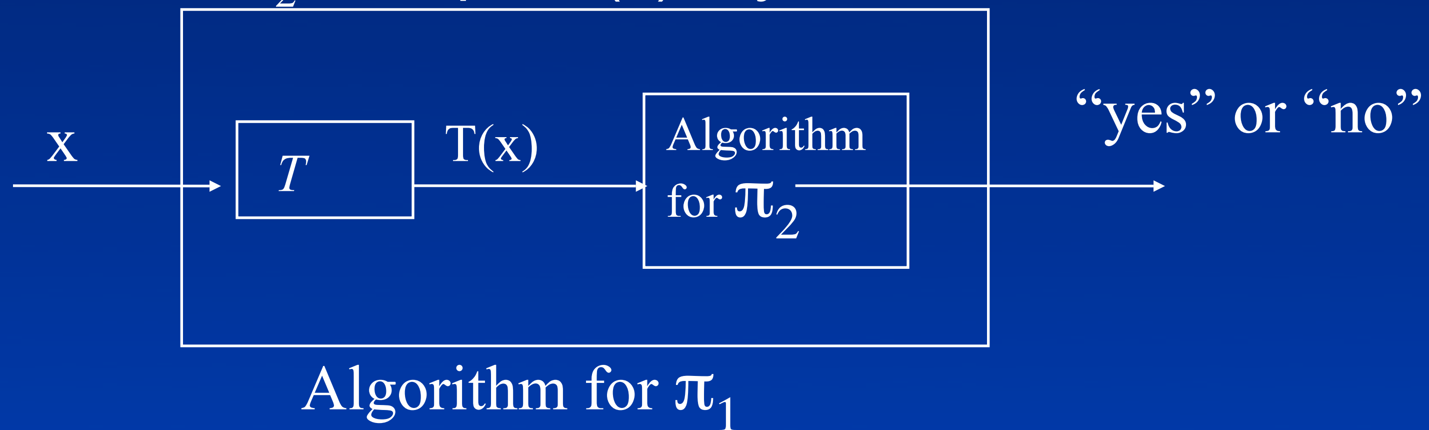
$$\pi_1 \propto \pi_2$$

- Let T be a function that converts any input x for decision problem π_1 into input T(x) for decision problem π_2

Polynomial reductions

T(x) is a *polynomial reduction* from π_1 to π_2 if:

1. T can be computed in polynomial bounded time
2. The answer to π_1 for input x is *yes* if and only if the answer to π_2 for input T(x) is *yes*.



Implications of polynomial reductions

- **Theorem**: If $\pi_1 \propto \pi_2$ and π_2 is in P, then π_1 is in P
- Proof outline: We need to sum:

1. The number of steps to convert x to T(x) and
2. The number of steps needed by algorithm A for π_2 to provide the answer for T(x).

- Then, we need to show that the sum is bound by a polynomial in the size n of x.

Proof

- Let $p(n) = O(n^c)$ be a polynomial bound on the computation of T
- Let $q(m) = O(m^{c'})$ be a polynomial bound on algorithm A for π_2

1. The conversion needs $O(n^c)$ steps where $|x|=n$.
2. The size of the problem generated by T(x) is at most p(n) characters (assuming that at each step the program for T writes a symbol).
3. So algorithm A with input of at most p(n) symbols does at most $q(p(n)) = O((n^c)^{c'})$ steps
4. The total work to transform x to T(x) and then apply A to get the correct answer, is at most p(n)+q(p(n)) = O

NP-completeness

- A language L (or problem corresponding to the language) is *NP-complete* if

1. $L \in NP$ and
2. $L' \propto L$ for every $L' \in NP$.

- The class of NP-complete problems (languages) is called NPC.

Theorem

1. If any problem in NPC is polynomial-time solvable, then P = NP.
2. If any problem in NP is not polynomial-time solvable, then all problems in NPC are not polynomial-time solvable.

Proof

1. Given: $L \in P$ and $L \in NPC$

   For any $L' \in NP$ $L' \propto L$ (definition of NPC).

   So by last theorem, $L' \in P$ and $P = NP$.

2. Given: $L \in NP$ and $L \notin P$

   Let $L' \in NPC$. Assume that $L' \in P$. Then $L \propto L'$ (definition of NPC) and thus by last theorem, $L \in P$.

   Contradiction. So $L' \notin P$ and all NPC problems are not polynomial time solvable

Proving NP completeness

- Cook showed CNF-SATISFIABILITY is NP-complete.
- To prove any other problem NP-complete, we then just have to reduce CNF- SATISFIABILITY to that problem and show that problem lies in the class NP.

Theorem: polynomial reducibility is transitive

If L1 $\propto$ L2 and L2 $\propto$ L3 then L1 $\propto$ L3

Proof outline:

- There is a polynomially computable function
  - that reduces all inputs $x \in$ L1 into an input T1(x) $\in$ L2
  - that reduces all inputs $y \in$ L2 into an input T2(y) $\in$ L3
- Applying T1 to x and T2 to T1(x) reduces any instance $x \in$ L1 to an instance T2(T1(x)) $\in$ L3, and this conversion requires a polynomial number of steps

Shortcut for NP-completeness Proofs

- To prove a language *L* is NP-complete:

Prove $L \in NP$.

Choose $L' \in NPC$, and show $L' \propto L$

- $L \in NP$. We will show that every $M \in NP$ satisfies $M \propto L$, and thus L is NP-complete
  - Let $M \in NP$. $M \propto L'$ (definition of NPC), and $L' \propto L$ (proved by us). So by transitivity $M \propto L$