Lambda Calculus

I read this library book you got me.

What did you think of it?

It really made me see things differently. It's given me a lot to think about.

I'm glad you enjoyed it.

It's complicating my life. Don't get me any more.
Plan

• Introduce lambda calculus
  - Syntax
  - Substitution
  - Operational Semantics (... with contexts!)
  - Evaluations strategies
  - Equality

• Later:
  - Relationship to programming languages
  - Study of types and type systems
Lambda Background

- Developed in 1930’s by Alonzo Church
- Subsequently studied by many people
  - Still studied today!
- Considered the “testbed” for procedural and functional languages
  - Simple
  - Powerful
  - Easy to extend with new features of interest
  - Lambda:PL :: Turning Machine:Complexity
  - Somewhat like a crowbar ...

“Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus.”

(Landin ’66)
Lambda Syntax

• The $\lambda$-calculus has 3 kinds of expressions (terms)
  
  $e ::= x$  Variables
  
  | $\lambda x. e$  Functions (abstractions)
  
  | $e_1 e_2$  Application

• $\lambda x. e$ is a one-argument anonymous function with body $e$

• $e_1 e_2$ is a function application

• Application associates to the left

  $x y z = (x y) z$

• Abstraction extends far to the right

  $\lambda x. x \lambda y. x y z = (\lambda x. (x [\lambda y. {(x y) z}]))$
Why Should I Care?

- A language with 3 expressions? Woof!
- Li and Zdancewic. *Downgrading policies and relaxed noninterference*. POPL ’05
  - Just one example of a recent PL/security paper

4. LOCAL DOWNGRADING POLICIES

4.1 Label Definition

Definition 4.1.1 (The policy language). In *Figure 1*.

<table>
<thead>
<tr>
<th>Syntax</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Types</td>
<td>( \tau ::= \text{int} \mid \tau \to \tau )</td>
</tr>
<tr>
<td>Constants</td>
<td>( c ::= c_i )</td>
</tr>
<tr>
<td>Operators</td>
<td>( \oplus ::= +, -, ., \ldots )</td>
</tr>
<tr>
<td>Terms</td>
<td>( m ::= \lambda x: \tau.\ m \mid m \ m \mid x \mid c \mid m \oplus m )</td>
</tr>
<tr>
<td>Policies</td>
<td>( n ::= \lambda x: \text{int.\ } m )</td>
</tr>
<tr>
<td>Labels</td>
<td>( l ::= {n_1, \ldots, n_k} \quad (k \geq 1) )</td>
</tr>
</tbody>
</table>

*Figure 1: \( L_{\text{local}} \) Label Syntax*

The core of the policy language is a variant of the simply-typed \( \lambda \)-calculus with a base type, binary operators and constants. A *dowgrading policy* is a \( \lambda \)-term that specifies how an integer can be dowgraded: when this \( \lambda \)-term is applied to the annotated integer, the result becomes public. A labeled integer consists of the preceding policies, specifying

\[
\begin{align*}
\Gamma \vdash m : \tau & \quad \text{Q-REFL} \\
\Gamma \vdash m \equiv m : \tau & \\
\Gamma \vdash m_1 \equiv m_2 : \tau & \quad \text{Q-SYMM} \\
\Gamma \vdash m_2 \equiv m_1 : \tau & \\
\Gamma \vdash m_1 \equiv m_2 : \tau & \quad \text{Q-TRANS} \\
\Gamma \vdash m_2 \equiv m_3 : \tau & \quad \Gamma \vdash m_3 \equiv m_3 : \tau \\
\Gamma \vdash m_1 \equiv m_3 : \tau & \\
\Gamma, x : \tau_1 \vdash m_1 \equiv m_2 : \tau_2 & \quad \text{Q-ABS} \\
\Gamma \vdash \lambda x : \tau_1.\ m_1 \equiv \lambda x : \tau_1.\ m_2 : \tau_1 \to \tau_2 & \\
\Gamma \vdash m_1 \equiv m_2 : \tau_1 \to \tau_2 & \quad \text{Q-APP} \\
\Gamma \vdash m_3 \equiv m_4 : \tau_1 & \\
\Gamma \vdash m_1 \ m_3 \equiv m_2 \ m_4 : \tau_2 & \\
\Gamma \vdash m_1 \ m_3 \equiv m_2 \ m_4 : \tau_2 & \\
\Gamma \vdash m_1 \oplus m_3 \equiv m_2 \oplus m_4 : \text{int} & \quad \text{Q-BinOP} \\
\Gamma \vdash m_1 \oplus m_3 \equiv m_2 \oplus m_4 : \text{int} & \\
\Gamma \vdash m_1 \oplus m_3 \equiv m_2 \oplus m_4 : \text{int} &
\end{align*}
\]
Lambda Celebrity Representative

• Milton Friedman?
• Morgan Freeman?
• C. S. Friedman?
Gordon Freeman

- Best-selling PC FPS to date...
Examples of Lambda Expressions

• The identity function:
  \[ I = \text{def} \lambda x. x \]

• A function that, given an argument \( y \), discards it and yields the identity function:
  \[ \lambda y. (\lambda x. x) \]

• A function that, given an function \( f \), invokes it on the identity function:
  \[ \lambda f. f (\lambda x. x) \]

“There goes our grant money.”
Scope of Variables

As in all languages with variables, it is important to discuss the notion of scope

- The **scope** of an identifier is the portion of a program where the identifier is accessible

An abstraction $\lambda x. E$ **binds** variable $x$ in $E$

- $x$ is the newly introduced variable
- $E$ is the scope of $x$ (unless $x$ is shadowed)

- We say $x$ is **bound** in $\lambda x. E$
- Just like formal function arguments are bound in the function body
Free and Bound Variables

A variable is said to be **free** in E if it has occurrences that are not bound in E.

We can define the free variables of an expression E recursively as follows:

- $\text{Free}(x) = \{x\}$
- $\text{Free}(E_1 E_2) = \text{Free}(E_1) \cup \text{Free}(E_2)$
- $\text{Free}(\lambda x. E) = \text{Free}(E) - \{x\}$

Example: $\text{Free}(\lambda x. x (\lambda y. x y z)) = \{z\}$

Free variables are (implicitly or explicitly) declared outside the expression.
Free Your Mind!

• Just as in any language with statically-nested scoping we have to worry about variable shadowing:
  - An occurrence of a variable might refer to different things in different contexts

• Example in IMP with locals:
  \[
  \text{let } x = 5 \text{ in } x + (\text{let } x = 9 \text{ in } x) + x
  \]

• In \(\lambda\)-calculus:
  \[
  \lambda x. \ x \ (\lambda x. \ x) \ x
  \]
Renaming Bound Variables

- $\lambda$-terms that can be obtained from one another by renaming bound variables are considered identical
- This is called $\alpha$-equivalence
- Renaming bound vars is called $\alpha$-renaming
- Ex: $\lambda x. x$ is identical to $\lambda y. y$ and to $\lambda z. z$
- Intuition:
  - By changing the name of a formal argument and all of its occurrences in the function body, the behavior of the function does not change
  - In $\lambda$-calculus such functions are considered identical
Make It Easy On Yourself

• Convention: we will always try to rename bound variables so that they are all unique
  - e.g., write $\lambda x. x (\lambda y.y) x$ instead of $\lambda x. x (\lambda x.x) x$

• This makes it easy to see the scope of bindings and also prevents confusion!
Substitution

- The substitution of F for x in E (written [F/x]E)
  - Step 1. Rename bound variables in E and F so they are unique
  - Step 2. Perform the textual substitution of f for X in E
- Called capture-avoiding substitution
- Example: \([y \ (\lambda x. \ x) \ / \ x] \ \lambda y. \ (\lambda x. \ x) \ y \ x\)
  - After renaming: \([y \ (\lambda x. \ x) \ / \ x] \ \lambda z. \ (\lambda u. \ u) \ z \ x\)
  - After substitution: \(\lambda z. \ (\lambda u. \ u) \ z \ (y \ (\lambda x. \ x))\)
- If we are not careful with scopes we might get:
  \(\lambda y. \ (\lambda x. \ x) \ y \ (y \ (\lambda x. \ x))\) ← wrong!
The De Bruijn Notation

• An alternative syntax that avoids naming of bound variables (and the subsequent confusions)
• The **De Bruijn index** of a variable *occurrence* is that number of lambda that separate the occurrence from its binding lambda in the abstract syntax tree
• The **De Bruijn notation** replaces names of occurrences with their De Bruijn indices
• Examples:
  - $\lambda\ x.\ x$ $\lambda.\ 0$
  - $\lambda\ x.\ \lambda\ x.\ x$ $\lambda.\ \lambda.\ 0$
  - $\lambda\ x.\ \lambda\ y.\ y$ $\lambda.\ \lambda.\ 0$
  - $(\lambda\ x.\ x\ x)\ (\lambda\ z.\ z\ z)$ $(\lambda.\ 0\ 0)\ (\lambda.\ 0\ 0)$
  - $\lambda\ x.\ (\lambda\ x.\ \lambda\ y.\ x)\ x$ $\lambda.\ (\lambda.\ \lambda.\ 1)\ 0$

Identical terms have identical representations!
Combinators

- A $\lambda$-term without free variables is **closed** or a **combinator**
- Some interesting combinators:
  
  $I = \lambda x. x$
  
  $K = \lambda x. \lambda y. x$
  
  $S = \lambda f. \lambda g. \lambda x. f x (g x)$
  
  $D = \lambda x. x x$
  
  $Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$

- Theorem: any closed term is equivalent to one written with just $S$, $K$ and $I$
  
  - Example: $D =^\beta S I I$
  
  - (we’ll discuss this form of equivalence later)
Q: Music (241 / 842)

• Name the singer and his crossover 1982 album that holds (as of 2005) the record of being the best-selling album of all-original material in the US (26 times platinum, 37 weeks as Billboard #1). Much of that success was the result of the singer's use of the MTV music video.
Q: Movies (262 / 842)

- Name two of the three rules given for the pet Gizmo in the 1984 movie Gremlins.
This 1993 Mel Brooks parody film features "The Man in Black" as "Kevin Costner" and also stars Patrick Stewart as King Richard. It includes the exchange: "And why would the people listen to you? / Because, unlike some other Robin Hoods, I can speak with an English accent."
Q: General (452 / 842)

- Name any 3 of the 22 letters in the Hebrew alphabet.
This 1965 Wham-O toy is an extremely elastic sphere made of a rubber polymer with a high coefficient of restitution. When dropped from shoulder level onto a hard surface it rebounds to about 90% of its original height.
A: Games (489 / 842)

- **Super Ball**
  - *Trivia: When Lamar Hunt saw his daughter playing with a Super Ball, it inspired him to name the new AFL-NFL World Championship Game the Super Bowl.*
Informal Semantics

• We consider only closed terms
• The evaluation of

\[(\lambda x. e) f\]

- Binds \(x\) to \(f\)
- Evaluates \(e\) with the new binding
- Yields the result of this evaluation

• Like a function call, or like “let \(x = f\) in \(e\)”
• Example:

\[(\lambda f. f (f e)) g\] evaluates to \(g (g e)\)
Operational Semantics

• Many operational semantics for the \( \lambda \)-calculus
• All are based on the equation
  \[
  (\lambda x. e) f =_\beta [f/x]e
  \]
  usually read from left to right
• This is called the **\( \beta \)-rule** and the evaluation step a **\( \beta \)-reduction**
• The subterm \( (\lambda x. e) f \) is a **\( \beta \)-redex**
• We write \( e \to^{\beta}_1 g \) to say that \( e \) \( \beta \)-reduces to \( g \) in one step
• We write \( e \to^{\beta}_* g \) to say that \( e \) \( \beta \)-reduces to \( g \) in 0 or more steps
  - Remind you of the small-step opsem term rewriting?
Examples of Evaluation

• The identity function:
  \[(\lambda \ x. \ x) \ E \rightarrow [E / x] \ x = E\]

• Another example with the identity:
  \[(\lambda \ f. \ f (\lambda \ x. \ x)) (\lambda \ x. \ x) \rightarrow\]
  \[\[\lambda \ x. \ x / f\] f (\lambda \ x. \ x)) =\]
  \[\[\lambda \ x. \ x / f\] f (\lambda \ y. \ y)) =\]
  \[(\lambda \ x. \ x) (\lambda \ y. \ y) \rightarrow\]
  \[\[\lambda \ y. \ y / x\] x = \lambda \ y. \ y\]

• A non-terminating evaluation:
  \[(\lambda \ x. \ xx) (\lambda \ y. \ yy) \rightarrow\]
  \[\[\lambda \ y. \ yy / x\] xx = (\lambda \ y. \ yy) (\lambda \ y. \ yy) \rightarrow \ldots\]

• Try \(T \ T\), where \(T = \lambda x. \ x \ x \ x\)
Evaluation and the Static Scope

• The definition of substitution guarantees that evaluation respects static scoping:

\[(\lambda x. (\lambda y. y x)) (y (\lambda x. x)) \rightarrow_\beta \lambda z. z (y (\lambda v. v))\]

(y remains free, i.e., defined externally)

• If we forget to rename the bound y:

\[(\lambda x. (\lambda y. y x)) (y (\lambda x. x)) \rightarrow_\beta^* \lambda y. y (y (\lambda v. v))\]

(y was free before but is bound now)
Another View of Reduction

• The application

\[ \lambda x. e \]

\[ e \]

\[ g \]

• Becomes:

\[ (\text{terms can grow substantially through } \beta\text{-reduction!}) \]

I am g!

Not one g, but three!
Normal Forms

• A term without redexes is in **normal form**
• A reduction sequence stops at a normal form

• If \( e \) is in normal form and \( e \rightarrow^* f \) then \( e \) is identical to \( f \)

• \( K = \lambda x. \lambda y. x \) is in normal form
• \( K I \) is *not* in normal form
Nondeterministic Evaluation

- We define a small-step reduction relation

\[ (\lambda x. e) f \rightarrow [f/x]e \]

\[ e_1 \rightarrow e_2 \]

\[ e_1 f \rightarrow e_2 f \]

\[ f_1 \rightarrow f_2 \]

\[ e f_1 \rightarrow e f_2 \]

\[ e \rightarrow f \]

\[ \lambda x. e \rightarrow \lambda x. f \]

- This is a non-deterministic semantics

- Note that we evaluate under \( \lambda \) (where?)
Lambda Calculus Contexts

- Define contexts with one hole
  - $H ::= \bullet \mid \lambda x. H \mid H e \mid e H$
- Write $H[e]$ to denote the filling of the hole in $H$ with the expression $e$
- Example:
  - $H = \lambda x. x \bullet \quad H[\lambda y. y] = \lambda x. x (\lambda y. y)$
- Filling the hole allows variable capture!
  - $H = \lambda x. x \bullet \quad H[x] = \lambda x. x x$
Contextual Opsem

- Contexts allow concise formulations of **congruence** rules (application of local reduction rules on subterms)
- Reduction occurs at a $\beta$-redex that can be anywhere inside the expression
- The latter rule is called a **congruence** or structural rule
- The above rules do not specify which redex must be reduced first
The Order of Evaluation

- In a \( \lambda \)-term there could be more than one instance of \((\lambda \ x. \ e) \ f\), as in:

\[
(\lambda \ y. \ (\lambda \ x. \ x) \ y) \ E
\]

- Could reduce the **inner** or **outer** \( \lambda \)
- Which one should we pick?

\[
(\lambda \ y. \ [y/x] \ x) \ E = (\lambda \ y. \ y) \ E \quad \quad [E/y] \ (\lambda \ x. \ x) \ y = (\lambda \ x. \ x) \ E
\]
The Diamond Property

- A relation $R$ has the **diamond property** if whenever $e \ R \ e_1$ and $e \ R \ e_2$ then there exists $e_3$ such that $e_1 \ R \ e_3$ and $e_2 \ R \ e_3$.

- $\rightarrow_\beta$ does *not* have the diamond property.
- $\rightarrow_\beta^*$ has the diamond property.
- Also called the **confluence property**.
A Diamond In The Rough

• Languages defined by non-deterministic sets of rules are common
  - Logic programming languages
  - Expert systems
  - Constraint satisfaction systems
    • And thus most pointer analyses ...
  - Dataflow systems
  - Makefiles

• It is useful to know whether such systems have the diamond property
(Beta) Equality

- Let $=_{\beta}$ be the reflexive, transitive and symmetric closure of $\rightarrow_{\beta}$

$$=_{\beta} \text{ is } (\rightarrow_{\beta} \cup \leftarrow_{\beta})^*$$

- That is, $e =_{\beta} f$ if $e$ converts to $f$ via a sequence of forward and backward $\rightarrow_{\beta}$
The Church-Rosser Theorem

- If $e_1 =_\beta e_2$ then there exists $e_3$ such that $e_1 \rightarrow_\beta^* e_3$ and $e_2 \rightarrow_\beta^* e_3$

- Proof (informal): apply the diamond property as many times as necessary
Corollaries

- If $e_1 = \beta e_2$ and $e_1$ and $e_2$ are normal forms then $e_1$ is identical to $e_2$
  - From C-R we have $\exists e_3. \ e_1 \rightarrow^* \beta e_3$ and $e_2 \rightarrow^* \beta e_3$
  - Since $e_1$ and $e_2$ are normal forms they are identical to $e_3$

- If $e \rightarrow^* \beta e_1$ and $e \rightarrow^* \beta e_2$ and $e_1$ and $e_2$ are normal forms then $e_1$ is identical to $e_2$
  - “All terms have a unique normal form.”
Evaluation Strategies

• Church-Rosser theorem says that independent of the reduction strategy we will find $\leq 1$ normal form
• But some reduction strategies might find 0
• $((\lambda y. y y) (\lambda y. y y)) \rightarrow (\lambda x. z) ((\lambda y. y y) (\lambda y. y y)) \rightarrow ...$
• $((\lambda y. y y) (\lambda y. y y)) \rightarrow z$
• There are three traditional strategies
  - normal order  (never used, always works)
  - call-by-name  (rarely used, cf. TeX)
  - call-by-value  (amazingly popular)
Civilization: Call By Value

• Normal Order
  - Evaluates the left-most redex not contained in another redex
  - If there is a normal form, this finds it
  - Not used in practice: requires partially evaluating function pointers and looking “inside” functions

• Call-By-Name ("lazy")
  - Don’t reduce under \( \lambda \), don’t evaluate a function argument (until you need to)
  - Does not always evaluate to a normal form

• Call-By-Value ("eager" or "strict")
  - Don’t reduce under \( \lambda \), do evaluate a function’s argument right away
  - Finds normal forms less often than the other two
Endgame

• This time: $\lambda$ syntax, semantics, reductions, equality, ...

• Next time: encodings, real programs, type systems, and all the fun stuff!

Wisely done, Mr. Freeman. I will see you up ahead.
Homework

• Read Leroy article, think about axiomatic
• Homework 5 Due Later
Tricksy On The Board Answer

- Is this rule unsound?

\[
\begin{array}{c}
\vdash \{ \text{A} \land p \} \ c_{\text{then}} \ \{ \text{B} \text{then} \} \\
\vdash \{ \text{A} \land \neg p \} \ c_{\text{else}} \ \{ \text{B} \text{else} \}
\end{array}
\]

\[
\vdash \{ \text{A} \} \text{ if } p \text{ then } c_{\text{then}} \text{ else } c_{\text{else}} \ \{ \text{B} \text{then} \lor \text{B} \text{else} \}
\]

- Nope: it’s our basic rule plus 2x consequence

\[
\begin{array}{c}
\vdash \{ \text{A} \land p \} \ c_1 \ \{ \text{B} \} \\
\vdash \{ \text{A} \land \neg p \} \ c_2 \ \{ \text{B} \}
\end{array}
\]

\[
\vdash \{ \text{A} \} \text{ if } p \text{ then } c_1 \text{ else } c_2 \ \{ \text{B} \}
\]

\[
\begin{array}{c}
\vdash \text{A’ } \Rightarrow \text{A} \\
\vdash \{ \text{A} \} \ c \ \{ \text{B} \} \\
\vdash \text{B } \Rightarrow \text{B’}
\end{array}
\]

\[
\vdash \{ \text{A’} \} \ c \ \{ \text{B’} \}
\]

- Note that \( \text{B}_{\text{then}} \Rightarrow \text{B}_{\text{then}} \lor \text{B}_{\text{else}} \)