Soundness and Completeness of Axiomatic Semantics
One-Slide Summary

• A system of axiomatic semantics is **sound** if everything we can prove is also true. if $\vdash \{ A \} c \{ B \}$ then $\models \{ A \} c \{ B \}$

• We prove this by **simultaneous induction** on the structure of the operational semantics derivation and the axiomatic semantics proof.

• A system of axiomatic semantics is **complete** if we can prove all true things. if $\models \{ A \} c \{ B \}$ then $\vdash \{ A \} c \{ B \}$

• Our system is **relatively complete** (= just as complete as the underlying logic). We use **weakest preconditions** to reason about soundness. **Verification conditions** are preconditions that are easy to **compute**.
Soundness of Axiomatic Semantics

• Formal statement of **soundness**:
  \[
  \text{if } \not\vdash \{ A \} \subset \{ B \} \text{ then } \vdash \{ A \} \subset \{ B \}
  \]
  or, equivalently
  For all \( \sigma \), if \( \sigma \models A \)
  and \( \text{Op :: } <c, \sigma> \Downarrow \sigma' \)
  and \( \text{Pr :: } \not\vdash \{ A \} \subset \{ B \} \)
  then \( \sigma' \models B \)

• “Op” === “Opsem Derivation”
• “Pr” === “Axiomatic Proof”

How shall we prove this, oh class?
Not Easily!

- By induction on the structure of $c$?
  - No, problems with while and rule of consequence
- By induction on the structure of Op?
  - No, problems with while
- By induction on the structure of Pr?
  - No, problems with consequence
- By simultaneous induction on the structure of Op and Pr
  - Yes! New Technique!
Simultaneous Induction

• Consider two structures Op and Pr
  - Assume that $x < y$ iff $x$ is a substructure of $y$

• Define the ordering
  $$(o, p) \prec (o', p') \text{ iff }$$
  
  $o < o' \text{ or } o = o' \text{ and } p < p'$

  - Called lexicographic (dictionary) ordering

• This $\prec$ is a well-founded order and leads to simultaneous induction

• If $o < o'$ then $h$ can actually be larger than $h'$!

• It can even be unrelated to $h'$!
Soundness of Axiomatic Semantics

- Formal statement of soundness:
  \[ \text{If } \vdash \{ A \} \subseteq \{ B \} \text{ then } \models \{ A \} \subseteq \{ B \} \]
  or, equivalently
  For all \( \sigma \), if \( \sigma \vdash A \)
  \[ \text{and Op :: } \langle c, \sigma \rangle \downarrow \sigma' \]
  \[ \text{and Pr :: } \vdash \{ A \} \subseteq \{ B \} \]
  \[ \text{then } \sigma' \vdash B \]

- “Op” = “Opsem Derivation”
- “Pr” = “Axiomatic Proof”
Simultaneous Induction

- Consider two structures $O_p$ and $P_r$
  - Assume that $x < y$ iff $x$ is a substructure of $y$
- Define the ordering $(o, p) < (o', p')$ iff
  - $o < o'$ or $o = o'$ and $p < p'$
  - Called lexicographic (dictionary) ordering
- This $<$ is a well founded order and leads to simultaneous induction
- If $o < o'$ then $p$ can actually be larger than $p'$!
- It can even be unrelated to $p'$!
Soundness of the While Rule
(Indiana Proof and the Slide of Doom)

• Case: last rule used in Pr : ⊢ {A} c {B} was the while rule:

\[
\text{Pr}_1 :: \vdash \{A \land b\} \ c \ \{A\} \\
\vdash \{A\} \ \text{while b do c} \ \{A \land \neg \ b\}
\]

• Two possible rules for the root of Op (by inversion)
  - We’ll only do the complicated case:

\[
\begin{align*}
\text{Op}_1 :: \langle b, \sigma \rangle \Downarrow \text{true} \\
\text{Op}_2 :: \langle c, \sigma \rangle \Downarrow \sigma' \\
\text{Op}_3 :: \langle \text{while b do c}, \sigma' \rangle \Downarrow \sigma''
\end{align*}
\]

<while b do c, σ > \Downarrow \sigma''

Assume that \( \sigma \models A \)

To show that \( \sigma'' \models A \land \neg b \)

• By soundness of booleans and Op_1 we get \( \sigma \models b \)
  - Hence \( \sigma \models A \land b \)

• By IH on Pr_1 and Op_2 we get \( \sigma' \models A \)

• By IH on Pr and Op_3 we get \( \sigma''' \models A \land \neg b \), q.e.d. (tricky!)
Soundness of the While Rule

- Note that in the last use of IH the derivation $\text{Pr}$ did not decrease
- But $\text{Op}_3$ was a sub-derivation of $\text{Op}$
- See Winskel, Chapter 6.5, for a soundness proof with denotational semantics
Completeness of Axiomatic Semantics

- If \( \models \{A\} \subseteq \{B\} \) can we always derive \( \vdash \{A\} \subseteq \{B\} \)?
- If so, axiomatic semantics is \textbf{complete}
- If not then there are valid properties of programs that we cannot verify with Hoare rules :-(
- \textbf{Good news:} for our language the Hoare triples are complete
- \textbf{Bad news:} only if the underlying logic is complete (whenever \( \models A \) we also have \( \vdash A \))
- this is called \textbf{relative completeness}
Examples, General Plan

• OK, so:
  \[ \vdash \{ x < 5 \land z = 2 \} y := x + 2 \{ y < 7 \} \]

• Can we prove it?
  \[ ? \vdash ? \{ x < 5 \land z = 2 \} y := x + 2 \{ y < 7 \} \]

• Well, we could easily prove:
  \[ \vdash \{ x+2 < 7 \} y := x + 2 \{ y < 7 \} \]

• And we know ...
  \[ \vdash x < 5 \land z = 2 \Rightarrow x+2 < 7 \]

• Shouldn’t those two proofs be enough?
Proof Idea

- Dijkstra’s idea: To verify that \( \{ A \} \preceq \{ B \} \)
  a) Find out all predicates \( A' \) such that \( \models \{ A' \} \preceq \{ B \} \)
  - call this set \( \text{Pre}(c, B) \) (Pre = “pre-conditions”)
  b) Verify for one \( A' \in \text{Pre}(c, B) \) that \( A \Rightarrow A' \)

- Assertions can be ordered:
  
  false \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] true

  strong \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] \[\Rightarrow\] weak

  weakest precondition: \( \text{WP}(c, B) \)

- Thus: compute \( \text{WP}(c, B) \) and prove \( A \Rightarrow \text{WP}(c, B) \)
Proof Idea (Cont.)

• **Completeness** of axiomatic semantics:
  
  If \( \vdash \{ A \} \subset \{ B \} \) then \( \vdash \{ A \} \subset \{ B \} \)

• Assuming that we can compute \( \text{wp}(c, B) \) with the following properties:
  
  • \( \text{wp} \) is a precondition (according to the Hoare rules)
    
    \( \vdash \{ \text{wp}(c, B) \} \subset \{ B \} \)
  
  • \( \text{wp} \) is (truly) the weakest precondition
    
    If \( \vdash \{ A \} \subset \{ B \} \) then \( \vdash A \Rightarrow \text{wp}(c, B) \)

    \( \vdash A \Rightarrow \text{wp}(c, B) \)

    \( \vdash \{ \text{wp}(c, B) \} \subset \{ B \} \)

    \( \vdash \{ A \} \subset \{ B \} \)

• We also need that whenever \( \vdash A \) then \( \vdash A ! \)
Complete the following Garrison Keillor catchphrase: "And that's the news from Lake Wobegon, where all the women are strong, ..."
Q: Movie Music (422 / 842)
• This 1986 song by Queen is the theme for the Highlander movie and television series.
Q: Bonus

• Despite having physically appeared in only about ten movies, this Indian singer has received the *Bharat Ratna* (India's highest civilian honor) and holds the Guinness Book of World Records entry for “most recordings” (30,000 songs by 1987). At one point the Pakistani prime minister said the he would “gladly exchange [her] for Kashmir”. She is the sister of Asha Bhosle and specializes in “playback” or “voiceover” movie music.
Q: Advertising (797 / 842)

• Identify the company associated with two of the following four advertising slogans or symbols.
  - "Reach out and touch someone."
  - "It just keeps going and going and going."
  - "Now You're Playing With Power!"
  - "We bring good things to life."
Axiomatic Semantics: Preconditions

My students drew me into another political argument.

Lately, political debates bother me. They just show how good smart people are at rationalizing.

Eh; it happens.

The world is so complicated - the more I learn, the less clear anything gets. There are too many ideas and arguments to pick and choose from. How can I trust myself to know the truth about anything?

And if everything I know is so shaky, what on earth am I doing teaching?

I guess you just do your best. No one can impart perfect universal truths to their students.

*Ahem*

... except math teachers.

Thank you.
Weakest Preconditions

• Define $wp(c, B)$ inductively on $c$, following the Hoare rules:

$\begin{align*}
wp(c_1; c_2, B) &= wp(c_1, wp(c_2, B)) \\
wp(x := e, B) &= [e/x]B \\
wp(\text{if } E \text{ then } c_1 \text{ else } c_2, B) &= \{ E \Rightarrow A_1 \land \neg E \Rightarrow A_2 \} \text{ if } E \text{ then } c_1 \text{ else } c_2 \{ B \} \\
wp(\text{if } E \text{ then } c_1 \text{ else } c_2, B) &= E \Rightarrow wp(c_1, B) \land \neg E \Rightarrow wp(c_2, B)
\end{align*}$
Weakest Preconditions for Loops

• We start from the unwinding equivalence
  \[
  \text{while } b \text{ do } c = \text{if } b \text{ then } c; \text{ while } b \text{ do } c \text{ else skip}
  \]

• Let \( w = \text{while } b \text{ do } c \) and \( W = \text{wp}(w, B) \)

• We have that
  \[
  W = b \Rightarrow \text{wp}(c, W) \land \neg b \Rightarrow B
  \]

• But this is a recursive equation!
  - We know how to solve these using domain theory

• But we need a domain for assertions
A Partial Order for Assertions

• Which assertion contains the least information?
  - “true” - does not say anything about the state

• What is an appropriate information ordering?
  \[
  A \subseteq A' \quad \text{iff} \quad \models A' \Rightarrow A
  \]

• Is this partial order complete?
  - Take a chain \( A_1 \subseteq A_2 \subseteq \ldots \)
  - Let \( \land A_i \) be the infinite conjunction of \( A_i \)
    \[
    \sigma \models \land A_i \quad \text{iff} \quad \text{for all } i \text{ we have that } \sigma \models A_i
    \]
  - I assert that \( \land A_i \) is the least upper bound

• Can \( \land A_i \) be expressed in our language of assertions?
  - In many cases: yes (see Winskel), we’ll assume yes for now
Weakest Precondition for WHILE

- Use the fixed-point theorem
  \[ F(A) = b \Rightarrow wp(c, A) \land \neg b \Rightarrow B \]
  - (Where did this come from? Two slides back!)
  - I assert that F is both monotonic and continuous

- The least-fixed point (= the weakest fixed point) is
  \[ wp(w, B) = \bigwedge F^i(true) \]

- Notice that unlike for denotational semantics of IMP we are not working on a flat domain!
Weakest Preconditions (Cont.)

- Define a family of wp’s
  - \(wp_k(\text{while } e \text{ do } c, B)\) = weakest precondition on which the loop terminates in \(B\) if it terminates in \(k\) or fewer iterations

\[
wp_0 = \neg E \Rightarrow B
\]

\[
wp_1 = E \Rightarrow wp(c, wp_0) \land \neg E \Rightarrow B
\]

... 

- \(wp(\text{while } e \text{ do } c, B) = \bigwedge_{k \geq 0} wp_k = \text{lub} \{wp_k \mid k \geq 0\}\)

- See Necula document on the web page for the proof of completeness with weakest preconditions

- Weakest preconditions are
  - Impossible to compute (in general)
  - Can we find something easier to compute yet sufficient?
Not Quite Weakest Preconditions

• Recall what we are trying to do:

\[
\begin{align*}
\text{false} & \Rightarrow \\
\text{Pre}(s, B) & \\
\text{true} & \\
\end{align*}
\]

- strong
- weakest
- weak

A
verification
precondition: \( WP(c, B) \)
condition: \( VC(c, B) \)

• Construct a verification condition: \( VC(c, B) \)
  - Our loops will be annotated with loop invariants!
  - \( VC \) is guaranteed to be stronger than \( WP \)
  - But still weaker than \( A \): \( A \Rightarrow VC(c, B) \Rightarrow WP(c, B) \)
Groundwork

• Factor out the hard work
  - Loop invariants
  - Function specifications (pre- and post-conditions)

• Assume programs are annotated with such specs
  - Good software engineering practice anyway
  - Requiring annotations = Kiss of Death?

• New form of while that includes a loop invariant:

  \[ \text{while}_{\text{Inv}} b \text{ do } c \]

  - Invariant formula Inv must hold every time before b is evaluated

• A process for computing VC(annotated_command, post_condition) is called VCGen
Verification Condition Generation

- Mostly follows the definition of the wp function:

\[
\begin{align*}
\text{VC}(\text{skip}, B) &= B \\
\text{VC}(c_1; c_2, B) &= \text{VC}(c_1, \text{VC}(c_2, B)) \\
\text{VC}(\text{if } b \text{ then } c_1 \text{ else } c_2, B) &= b \Rightarrow \text{VC}(c_1, B) \land \neg b \Rightarrow \text{VC}(c_2, B) \\
\text{VC}(x := e, B) &= [e/x] B \\
\text{VC}(\text{let } x = e \text{ in } c, B) &= [e/x] \text{VC}(c, B) \\
\text{VC}(\text{while}_\text{inv} b \text{ do } c, B) &= ?
\end{align*}
\]
VCGen for WHILE

\[ VC(\text{while} \_ \text{Inv} \ e \ \text{do} \ c, \ B) = \]
\[ \text{Inv} \land (\forall x_1...x_n. \ \text{Inv} \Rightarrow (e \Rightarrow VC(c, \text{Inv}) \land \neg e \Rightarrow B)) \]

- \text{Inv} \text{ is the loop invariant (provided externally)}
- \(x_1, ..., x_n\) are all the variables modified in \(c\)
- The \(\forall\) is similar to the \(\forall\) in mathematical induction:
  \[ P(0) \land \forall n \in \mathbb{N}. \ P(n) \Rightarrow P(n+1) \]
Example VCGen Problem

• Let’s compute the VC of this program with respect to post-condition $x \neq 0$

\[
x = 0; \\
y = 2; \\
\text{while}_{x+y=2} y > 0 \text{ do} \\
\quad y := y - 1; \\
\quad x := x + 1
\]

First, what do we expect? What precondition do we need to ensure $x \neq 0$ after this?
Example of VC

• By the sequencing rule, first we do the while loop (call it w):

```plaintext
while_{x+y=2} y > 0 do
  y := y - 1;
  x := x + 1
```

• \( \text{VCGen}(w, x \neq 0) = x+y=2 \land \forall x,y. x+y=2 \Rightarrow (y>0 \Rightarrow VC(c, x+y=2) \land y\leq 0 \Rightarrow x \neq 0) \)

• \( \text{VCGen}(y:=y-1 ; x:=x+1, x+y=2) = (x+1) + (y-1) = 2 \)

• \( w \) Result: \( x+y=2 \land \forall x,y. x+y=2 \Rightarrow (y>0 \Rightarrow (x+1)+(y-1)=2 \land y\leq 0 \Rightarrow x \neq 0) \)
Example of VC (2)

• \( \text{VC}(w, x \neq 0) = x+y=2 \land \) 
  \[ \forall x, y. \ x+y=2 \Rightarrow \]
  \[ (y>0 \Rightarrow (x+1)+(y-1)=2 \land y\leq 0 \Rightarrow x \neq 0) \]

• \( \text{VC}(x := 0; y := 2 ; w, x \neq 0) = 0+2=2 \land \) 
  \[ \forall x, y. \ x+y=2 \Rightarrow \]
  \[ (y>0 \Rightarrow (x+1)+(y-1)=2 \land y\leq 0 \Rightarrow x \neq 0) \]

• So now we ask an automated theorem prover to prove it.
Thoreau, Thoreau, Thoreau

$ ./Simplify

> (AND (EQ (+ 0 2) 2)
   (FORALL (x y) (IMPLIES (EQ (+ x y) 2)
      (AND (IMPLIES (> y 0)
       (EQ (+ (+ x 1)(- y 1)) 2))
       (IMPLIES (<= y 0) (NEQ x 0))))))

1: Valid.

- Huzzah!
- Simplify is a non-trivial five megabytes
Can We Mess Up VCGen?

• The invariant is from the user (= the adversary, the untrusted code base)

• Let’s use a loop invariant that is too weak, like “true”.

• \( VC = true \land \forall x, y. true \Rightarrow \) 
  \( (y > 0 \Rightarrow true \land y \leq 0 \Rightarrow x \neq 0) \)

• Let’s use a loop invariant that is false, like “\( x \neq 0 \)”.

• \( VC = 0 \neq 0 \land \forall x, y. x \neq 0 \Rightarrow \) 
  \( (y > 0 \Rightarrow x + 1 \neq 0 \land y \leq 0 \Rightarrow x \neq 0) \)
Emerson, Emerson, Emerson

$ ./Simplify

> (AND TRUE
   (FORALL ( x y ) (IMPLIES TRUE
       (AND (IMPLIES (> y 0) TRUE)
            (IMPLIES (<= y 0) (NEQ x 0))))))

Counterexample: context:
   (AND
       (EQ x 0)
       (<= y 0)
   )

1: Invalid.

• OK, so we won’t be fooled.
Soundness of VCGen

• Simple form

\[ \vdash \{ \text{VC}(c, B) \} \ c \ \{ B \} \]

• Or equivalently that

\[ \vdash \text{VC}(c, B) \Rightarrow \wp(c, B) \]

• Proof is by induction on the structure of c
  - Try it!

• Soundness holds for any choice of invariant!

• Next: properties and extensions of VCs
Questions

• Homework?
• Project proposal?