Set Theory

1 Set Theory Exercise — Problem

This exercise is meant to help you refresh your knowledge of set theory and functions. Let \( X \) and \( Y \) be sets. Let \( \mathcal{P}(X) \) denote the powerset of \( X \) (the set of all subsets of \( X \)). Show that there is a 1-1 correspondence (i.e., a bijection) between the sets \( A \) and \( B \), where \( A = X \to \mathcal{P}(Y) \) and \( B = \mathcal{P}(X \times Y) \). Note that \( A \) is a set of functions and \( B \) is a (or can be viewed as a) set of relations. This correspondence will allow us to use functional notation for certain sets in class. This is Exercise 1.4 from page 8 of Winskel’s book.

2 Set Theory Exercise — Solution 1 (Injective + Surjective)

Let us construct a function \( f : (X \to \mathcal{P}(Y)) \to \mathcal{P}(X \times Y) \). We choose \( f \) as follows:

\[ f(a) = \{ (x, y) \mid y \in a(x) \} \]

A function \( f \) is injective (or one-to-one) if for all \( a_1, a_2 \in A \), if \( f(a_1) = f(a_2) \) then \( a_1 = a_2 \). Let \( a_1 \) and \( a_2 \) be arbitrary elements of \( A \), and assume \( f(a_1) = f(a_2) \). Then, by definition of \( f \):

\[ \{ (x, y) \mid y \in a_1(x) \} = \{ (x, y) \mid y \in a_2(x) \} \]

By the axiom of extensionality in Set Theory, two sets are equal if they have exactly the same elements. Applied to the two sets above, we find that for any \( (x, y) \), whenever \( y \in a_1(x) \), we also have \( y \in a_2(x) \). Applying the axiom of extensionality to \( a_1(x) \) and \( a_2(x) \), we find that they must be equal sets (because for all \( y \) they either both contain that same \( y \) or both do not contain that same \( y \)). So for any \( x \), \( a_1(x) = a_2(x) \). Thus by the definition of function, \( a_1 \) and \( a_2 \) are equal functions (they agree on all arguments). Thus \( f \) is injective.

A function \( f : A \to B \) is surjective (or onto) if, for every \( b \in B \) there is an \( a \in A \) with \( f(a) = b \). To demonstrate this, let \( b \) be an arbitrary element of \( B \). So \( b \in \mathcal{P}(X \times Y) \) (by definition of \( B \), above). So every element of \( b \) is of the form \( (x, y) \) with \( x \in X \) and \( y \in Y \). We now construct an \( a \) such that \( f(a) = b \). By definition of \( f \), \( f(a) = \{ (x, y) \mid y \in a(x) \} \). So we pick our function \( a \) by letting \( a(x) = \{ y \mid (x, y) \in b \} \). By substitution, \( f(a) = \{ (x, y) \mid y \in \{ y \mid (x, y) \in b \} \} \), which simplifies to \( f(a) = \{ (x, y) \mid (x, y) \in b \} \). Since \( f(a) \) is the set of elements that are exactly those elements found in \( b \), by the axiom of extensionality, \( f(a) = b \). So the function \( f \) is surjective.

Since \( f \) is injective and surjective, it is also bijective (i.e., invertible). Since there exists an invertible function \( f : A \to B \), there is a 1-1 correspondence between \( A \) and \( B \). QED.

3 Set Theory Exercise — Solution 2 (Explicit Inverse)

In this alternate solution, we’ll construct an invertible \( f \) by explicitly showing its inverse. Let \( f \) be as in the previous solution:

\[ f(a) = \{ (x, y) \mid y \in a(x) \} \]

We introduce a second function, \( g \), that we will show to be the inverse of \( f \). Since \( g : B \to A \) and \( A \) is a set of functions, every \( g(b) \) will be a function. We define \( g \) as follows:

\[ (g(b))(x) = \{ y \mid (x, y) \in b \} \]

An optional presentation of the same \( g \) in the style of the lambda calculus is:

\[ g(b) = \lambda x. \{ y \mid (x, y) \in b \} \]

However, we have not yet introduced the lambda calculus in class. Do not worry if you are not familiar with it. In either case, \( g(b) \) returns a function. When that function is presented with the argument \( x \), it returns the set \( \{ y \mid (x, y) \in b \} \).
By the definition of invertible, to show that $f$ and $g$ are inverses, we show that that $f$ composed with $g$ is the identity function: $g(f(a)) = a$. Let $a$ be an arbitrary element of $A$. So $a$ is a function mapping $X$ to $\mathcal{P}(Y)$. To show that $g(f(a)) = a$, since they’re both functions, we’ll show that they behave the same way on all inputs: $(g(f(a))(x) = a(x)$. Now we expand $g(f(a))(x)$ by definition of $f(a)$:

$$g(f(a))(x) = g(\{(x, y) \mid y \in a(x)\})(x)$$

Now we expand by definition of $(g(b))(x)$:

$$g(f(a))(x) = \{y \mid (x, y) \in \{(x, y) \mid y \in a(x)\}\}$$

By simplification we have:

$$g(f(a))(x) = \{y \mid y \in a(x)\}$$

By the axiom of extensionality, the set of all elements found in $a(x)$ is exactly $a(x)$ itself, so we have:

$$g(f(a))(x) = a(x)$$

So we’re done: $f$ composed with $g$ is the identity function, so $f$ and $g$ are inverses, so $f : A \to B$ is invertible, so there is a 1-to-1 correspondence between $A$ and $B$. QED.