Type Systems For:
Exceptions,
Continuations,
and Recursive Types

I think the short attention span of television is great.

As far as I'm concerned, if something is so complicated that you can't explain it in 10 seconds, then it's probably not worth knowing anyway.

My time is valuable. I can't go thinking about one subject for minutes on end. I'm a busy man.

Who's been sitting here for three hours.

...at six thoughts a minute.
One-Slide Summary

- **Exceptions** are like non-local gotos; they are used to propagate errors. We will use contextual semantics to model them.

- **Continuations** allow you to take a snapshot of the current execution and store it for later use. They are often used for threads or backtracking. We will use contextual semantics to model them.

- **Recursive types** describe recursive data structures. $F_1$ with them, $F_1^\mu$, is as powerful as the untyped lambda calculus.
Exceptions

- A mechanism that allows **non-local control flow**
  - Useful for implementing the **propagation of errors to caller**
- Exceptions ensure* that errors are not ignored
  - Compare with the **manual error handling** in C
- Languages with exceptions:
  - C++, ML, Modula-3, Java, C#, ...
- We assume that there is a special type **exn** of exceptions
  - exn could be int to model error codes
  - In Java or C++, exn is a special object types

* Supposedly.
Modeling Exceptions

• Syntax

\[ e ::= \ldots \mid \text{raise } e \mid \text{try } e_1 \text{ handle } x \Rightarrow e_2 \]
\[ \tau ::= \ldots \mid \text{exn} \]

• We ignore here how exception values are created
  - In examples we will use integers as exception values
• The handler binds \( x \) in \( e_2 \) to the actual exception value
• The “raise” expression never returns to the immediately enclosing context
  - \( 1 + \text{raise } 2 \) is well-typed
  - if (\( \text{raise } 2 \)) then 1 else 2 is also well-typed
  - \( \text{(raise } 2) \text{ 5 } \) is also well-typed
  - What should be the type of \( \text{raise} \)?
Example with Exceptions

• A (strange) factorial function
  let f = \( x: \text{int} \). \( \text{res}: \text{int}. \) if \( x = 0 \) then
  raise res
  else
  f (x - 1) (res \* x)
  in try f 5 1 handle x ⇒ x

• The function returns in one step from the recursion

• The top-level handler catches the exception and turns it into a regular result
Typing Exceptions

• New typing rules

\[ \Gamma \vdash e : \text{exn} \]
\[ \Gamma \vdash \text{raise } e : \tau \]

\[ \Gamma \vdash e_1 : \tau \quad \Gamma, x : \text{exn} \vdash e_2 : \tau \]
\[ \Gamma \vdash \text{try } e_1 \text{ handle } x \implies e_2 : \tau \]

• A raise expression has an *arbitrary type*
  • This is a clear sign that the expression does not return to its evaluation context

• The type of the body of try and of the handler must match
  • Just like for conditionals
Dynamics of Exceptions

- The result of evaluation can be an **uncaught exception**
  - Evaluation answers: \( a ::= v \mid \text{uncaught } v \)
  - “uncaught \( v \)” has an *arbitrary type*

- Raising an exception has global effects

- It is convenient to use **contextual semantics**
  - Exceptions **propagate** through some contexts but not through others
  - We **distinguish** the handling contexts that intercept exceptions (this will be new)
Contexts for Exceptions

- **Contexts**
  
  \[ H ::= \bullet \mid H \ e \mid v \ H \mid \text{raise} \ H \mid \text{try} \ H \ \text{handle} \ x \Rightarrow e \]

- **Propagating contexts**
  
  - Contexts that propagate exceptions to their own enclosing contexts
  
  \[ P ::= \bullet \mid P \ e \mid v \ P \mid \text{raise} \ P \]

- **(New) Decomposition theorem**
  
  - If \( e \) is not a value and \( e \) is well-typed then it can be decomposed in exactly one of the following ways:
    
    - \( H[(\lambda x: \tau. \ e) \ v] \) (normal lambda calculus)
    - \( H[\text{try} \ v \ \text{handle} \ x \Rightarrow e] \) (handle it or not)
    - \( H[\text{try} \ P[\text{raise} \ v] \ \text{handle} \ x \Rightarrow e] \) (propagate!)
    - \( P[\text{raise} \ v] \) (uncaught exception)
Contextual Semantics for Exceptions

• Small-step reduction rules

\[ H[(\lambda x: \tau. \ e) \ v] \rightarrow H[[v/x] \ e] \]
\[ H[\text{try } v \ \text{handle } x \Rightarrow e] \rightarrow H[v] \]
\[ H[\text{try } P[\text{raise } v] \ \text{handle } x \Rightarrow e] \rightarrow H[[v/x] e] \]
\[ P[\text{raise } v] \rightarrow \text{uncaught } v \]

• The handler is ignored if the body of try completes normally

• A raised exception propagates (in one step) to the closest enclosing handler or to the top of the program
Exceptional Commentary

- The addition of exceptions preserves type soundness
- Exceptions are like non-local goto
- However, they cannot be used to implement recursion
  - Thus we still cannot write (well-typed) non-terminating programs
- There are a number of ways to implement exceptions (e.g., “zero-cost” exceptions)
Continuations

• Some languages have a mechanism for taking a snapshot of the execution and storing it for later use
  - Later the execution can be reinstated from the snapshot
  - Useful for implementing threads, for example
  - Examples: Scheme, LISP, ML, C (yes, really!)

• Consider the expression: $e_1 + e_2$ in a context $C$
  - How to express a snapshot of the execution right after evaluating $e_1$ but before evaluating $e_2$ and the rest of $C$?
  - Idea: as a context $C_1 = C [\bullet + e_2 ]$
    - Alternatively, as $\lambda x_1. C [x_1 + e_2 ]$
  - When we finish evaluating $e_1$ to $v_1$, we fill the context and continue with $C[v_1 + e_2 ]$
  - But the $C_1$ continuation is still available and we can continue several times, with different replacements for $e_1$
Continuation Uses in “Real Life”

• You’re walking and come to a fork in the road
• You save a continuation “right” for going right
• But you go left (with the “right” continuation in hand)
• You encounter Bender. Bender coerces you into joining his computer dating service.
• You save a continuation “bad-date” for going on the date.
• You decide to invoke the “right” continuation
• So, you go right (no evil date obligation, but with the “bad-date” continuation in hand)
• A train hits you!
• On your last breath, you invoke the “bad-date” continuation
Continuations

• Syntax:
  
  e ::= callcc k in e  |  throw e₁ e₂
  
  τ ::= ...  |  τ cont

• τ cont - the type of a continuation that expects a τ

• callcc k in e - sets k to the current context of the execution and then evaluates expression e
  - when e terminates, the whole callcc terminates
  - e can invoke the saved continuation (many times even)
  - when e invokes k it is as if “callcc k in e” returns
  - k is bound in e

• throw e₁ e₂ - evaluates e₁ to a continuation, e₂ to a value and invokes the continuation with the value of e₂ (just wait, we’ll explain it!)
Example with Continuations

• Example: another strange factorial

\[
\text{callcc } k \text{ in }
\]
\[
\text{let } f = \lambda x: \text{int}. \lambda \text{res: int}. \begin{cases} 
\text{if } x = 0 \text{ then throw } k \text{ res} \\
\text{else } f(x - 1)(x * \text{res})
\end{cases}
\]
\[
in f 5 1
\]

• First we save the current context
  - This is the top-level context
  - A throw to k of value v means “pretend the whole callcc evaluates to v”

• This simulates exceptions
• Continuations are \textit{strictly more powerful} than exceptions
  - The destination is not tied to the call stack
Q: Movies (364 / 842)

• According to Vizzini in the movie *The Princess Bride*, what are two classic blunders?
148. Except living with others our whole life, we are both alone, solitary.

211. It was an uncomfortable silence. It was as if they were both as ease with each other.

270. He probably does know me but he where's a mask, so illogically he could be a number of people that I know.

426. Though her grades proved otherwise, Maebly wasn't an idiot.

313. Hermione Granger was head girl, and to everyone's surprise, so was Draco Malfoy.
Computer Science

• This American Turing-award winner is known as the DARPA program manager in charge of funding groups developing TCP/IP. He funded and founded ICANN and the Internet Society. He helped develop the first commercial email system connected to the internet.
This 1953 dystopian novel by Ray Bradbury has censorship as a major theme. The main character, Guy Montag, is a fireman.
Static Semantics of Continuations

\[ \Gamma, k : \tau \text{ cont} \vdash e : \tau \]

\[ \Gamma \vdash \text{callcc} \ k \ \text{in} \ e : \tau \]

\[ \Gamma \vdash e_1 : \tau \text{ cont} \quad \Gamma \vdash e_2 : \tau \]

\[ \Gamma \vdash \text{throw} \ e_1 \ e_2 : \tau' \]

• Note that the result of callcc is of type \( \tau \)
  “callcc \( k \) in \( e \)” returns in two possible situations
  - \( e \) \textit{throws} to \( k \) a value of type \( \tau \), or
  - \( e \) \textit{terminates normally} with a value of type \( \tau \)

• Note that throw has any type \( \tau' \)
  - Since it \textit{never returns} to its enclosing context
Dynamic Semantics of Continuations

- Use contextual semantics (wow, again!)
  - Contexts are now manipulated directly
  - Contexts are values of type $\tau$ cont

Contexts

$$H ::= \bullet \mid H\ e \mid v\ H \mid \text{throw}\ H_1\ e_2 \mid \text{throw}\ v_1\ H_2$$

Evaluation rules

- $H[(\lambda x. e)\ v] \rightarrow H[[v/x]\ e]$
- $H[\text{callcc}\ k\ in\ e] \rightarrow H[[H/k]\ e]$
- $H[\text{throw}\ H_1\ v_2] \rightarrow H_1[v_2]$

- callcc duplicates the current continuation

- Note that throw abandons its own context
Implementing Coroutines with Continuations

• Example:

let client = \k. let res = callcc k' in throw k k' in
  print (fst res);
  client (snd res)
- “client k” will invoke “k” to get an integer and a continuation for
  obtaining more integers (for now, assume the list & recursion work)

let getnext =
  \L. \k. if L = nil then raise 999
    else getnext (cdr L) (callcc k' in throw k (car L, k'))
- “getnext L k” will send to “k” the first element of L along with a
  continuation that can be used to get more elements of L

getnext [0;1;2;3;4;5] (callcc k in client k)
Continuation Comments

• In our semantics the continuation saves the entire context: program counter, local variables, call stack, and the heap!
• In actual implementations the heap is not saved!
• Saving the stack is done with various tricks, but it is expensive in general
• Few languages implement continuations
  - Because their presence complicates the whole compiler considerably
  - Unless you use a continuation-passing-style of compilation (more on this next)
Continuation Passing Style

• A style of compilation where evaluation of a function *never returns directly*: instead the function is *given a continuation to invoke with its result*.

• Instead of

\[
\text{f(int a) \{ return h(g(e)); \}}
\]

• we write

\[
\text{f(int a, cont k) \{ g(e, \lambda r. h(r, k)) \}}
\]

• Advantages:
  - interesting compilation scheme (supports callcc easily)
  - *no need for a stack*, can have multiple return addresses (e.g., for an error case)
  - fast and safe (non-preemptive) multithreading
Continuation Passing Style

- Let \( e ::= x \mid n \mid e_1 + e_2 \mid \text{if } e_1 \text{ then } e_2 \text{ else } e_3 \mid \lambda x. e \mid e_1 e_2 \)

- Define \( \text{cps}(e, k) \) as the code that computes \( e \) in CPS and passes the result to continuation \( k \)
  \[
  \begin{align*}
  \text{cps}(x, k) &= k \ x \\
  \text{cps}(n, k) &= k \ n \\
  \text{cps}(e_1 + e_2, k) &= \\
  &\quad \text{cps}(e_1, \ \lambda n_1. \text{cps}(e_2, \lambda n_2. k \ (n_1 + n_2))) \\
  \text{cps}(\lambda x. e, k) &= k \ (\lambda x \lambda k'. \text{cps}(e, k')) \\
  \text{cps}(e_1 \ e_2, k) &= \text{cps}(e_1, \ \lambda f_1. \text{cps}(e_2, \lambda v_2. f_1 \ v_2 \ k))
  \end{align*}
  \]

- Example: \( \text{cps } (h(g(5)), k) = g(5, \ \lambda x. h \ x \ k) \)
  - Notice the order of evaluation is made explicit
Recursive Types: Lists

- We want to define recursive data structures
- Example: lists
  - A list of elements of type $\tau$ (a $\tau$ list) is either empty or it is a pair of a $\tau$ and a $\tau$ list
  
  $$\tau\text{ list} = \text{unit} + (\tau \times \tau\text{ list})$$

  - This is a recursive equation. We take its solution to be the smallest set of values $L$ that satisfies the equation
  $$L = \{ * \} \cup (T \times L)$$
  where $T$ is the set of values of type $\tau$
  - Another interpretation is that the recursive equation is taken up-to (modulo) set isomorphism
Recursive Types

- We introduce a **recursive type constructor** \( \mu \) (mu):
  \[
  \mu t. \tau
  \]
  - The type variable \( t \) is **bound** in \( \tau \)
  - This stands for the solution to the equation
    \[
    t \simeq \tau \quad (t \text{ is isomorphic with } \tau)
    \]
  - Example: \( \tau \text{ list} = \mu t. (\text{unit} + \tau \times t) \)
  - This also allows “unnamed” recursive types

- We introduce syntactic (sugary) operations for the conversion between \( \mu t. \tau \) and \( [\mu t. \tau/t] \tau \)
  - e.g. between “\( \tau \text{ list} \)” and “unit + (\( \tau \times \tau \text{ list} \)”

\[
\begin{align*}
e & ::= \ldots \mid \text{fold}_{\mu t. \tau} e \mid \text{unfold}_{\mu t. \tau} e \\
\tau & ::= \ldots \mid t \mid \mu t. \tau
\end{align*}
\]
Example with Recursive Types

- Lists
  \[ \tau \text{ list} = \mu t. (\text{unit} + \tau \times t) \]
  \[ \text{nil}_\tau = \text{fold}_{\tau \text{ list}} (\text{injl } *) \]
  \[ \text{cons}_\tau = \lambda x:\tau. \lambda L:\tau \text{ list. } \text{fold}_{\tau \text{ list}} \text{ injr } (x, L) \]

- A list length function
  \[ \text{length}_\tau = \lambda L:\tau \text{ list.} \]
  \[ \quad \text{case (unfold}_{\tau \text{ list}} L \text{ of } \text{ injl } x \Rightarrow 0 \]
  \[ \quad \quad \quad \text{ | injr } y \Rightarrow 1 + \text{length}_\tau (\text{snd } y) \]

- (At home ...) Verify that
  - \( \text{nil}_\tau : \tau \text{ list} \)
  - \( \text{cons}_\tau : \tau \rightarrow \tau \text{ list} \rightarrow \tau \text{ list} \)
  - \( \text{length}_\tau : \tau \text{ list} \rightarrow \text{int} \)
Type Rules for Recursive Types

\[ \Gamma \vdash e : \mu t.\tau \]

\[ \Gamma \vdash \text{unfold}_{\mu t.\tau} e : [\mu t.\tau / t] \tau \]

\[ \Gamma \vdash e : [\mu t.\tau / t] \tau \]

\[ \Gamma \vdash \text{fold}_{\mu t.\tau} e : \mu t.\tau \]

- The typing rules are **syntax directed**
- Often, for syntactic simplicity, the fold and unfold operators are **omitted**
  - This makes type checking somewhat harder
Dynamics of Recursive Types

- We add a new form of values

\[ \nu ::= \ldots \mid \text{fold}_{\mu \tau} \nu \]

- The purpose of fold is to ensure that the value has the recursive type and not its unfolding

- The evaluation rules:

\[
\begin{align*}
\text{fold}_{\mu \tau} e & \Downarrow \text{fold}_{\mu \tau} \nu \\
\text{unfold}_{\mu \tau} e & \Downarrow \nu
\end{align*}
\]

- The folding annotations are for type checking only
- They can be dropped after type checking
Recursive Types in ML

• The language ML uses a simple syntactic trick to avoid having to write the explicit fold and unfold.

• In ML recursive types are bundled with union types:
  
  \[ \text{type } t = C_1 \text{ of } \tau_1 \mid C_2 \text{ of } \tau_2 \mid \ldots \mid C_n \text{ of } \tau_n \]
  
  (* \( t \) can appear in \( \tau_i \) *)

  - e.g., “type intlist = Nil of unit \mid Cons of int * intlist”

• When the programmer writes \( \text{Cons } (5, l) \)
  
  - the compiler treats it as \( \text{fold}_{\text{intlist}} \text{ (injr } (5, l)) \)

• When the programmer writes
  
  - case \( e \) of Nil \( \Rightarrow \ldots \mid \text{Cons } (h, t) \Rightarrow \ldots \)
  
  the compiler treats it as

  - case unfold_{\text{intlist}} \text{ e of Nil } \Rightarrow \ldots \mid \text{Cons } (h,t) \Rightarrow \ldots
Encoding Call-by-Value
\( \lambda \)-calculus in \( F_1^\mu \)

- So far, \( F_1 \) was \textcolor{red}{so weak} that we could not encode non-terminating computations
  - Cannot encode recursion
  - Cannot write the \( \lambda x.x \ x \) (self-application)
- The addition of recursive types makes typed \( \lambda \)-calculus \textcolor{blue}{as expressive as untyped \( \lambda \)-calculus}!
- We could show a conversion algorithm from call-by-value untyped \( \lambda \)-calculus to call-by-value \( F_1^\mu \)
Untyped Programming in $F_{1}^{\mu}$

- We write $e$ for the conversion of the term $e$ to $F_{1}^{\mu}$
  - The type of $e$ is $V = \mu t. \ t \rightarrow t$
- The conversion rules
  $x \quad = x$
  $\lambda x.\ e \quad = \text{fold}_V (\lambda x:V.\ e)$
  $e_1 e_2 \quad = (\text{unfold}_V \ e_1) \ e_2$
- Verify that
  - $\vdash e : V$
  - $e \Downarrow v$ if and only if $e \Downarrow v$
- We can express non-terminating computation
  \[ D = (\text{unfold}_V (\text{fold}_V (\lambda x:V. (\text{unfold}_V x) x))) (\text{fold}_V (\lambda x:V. (\text{unfold}_V x) x))) \]
or, equivalently
  \[ D = (\lambda x:V. (\text{unfold}_V x) x) (\text{fold}_V (\lambda x:V. (\text{unfold}_V x) x))) \]
Homework

- Read Wikipedia articles
- Work on your projects!