1. Problem 1.

- a). A point on the side of a cylinder is uniquely located by height $h$ and angle $\phi$ (see fig.1). To sample points in a area uniform manner, choose $h = HU_1 = U_1$ (height=1) and $\phi = 2\pi U_2$ where $U_1$ and $U_2$ are uniformly distributed random variables. Corresponding xyz coordinates are $x = R \cos \phi = \cos \phi$ (radius=1), $y = \sin \phi$ and $z = h$.

- b). A point on the side of a cone is uniquely located by height $h$ and angle $\phi$ (see fig.1). Since the surface area of a unit cone is $\pi \sqrt{2}$, the differential probability is $\frac{dh}{2 \pi \sqrt{2}}$. Suppose the distribution is symmetric w.r.t. $\theta = \frac{\pi}{2}$, the differential probability must satisfy $\rho(h)\rho(\phi) dh d\phi = \frac{dA}{\pi \sqrt{2}} = \frac{1}{\pi \sqrt{2}} \sqrt{2} dh (1 - h) d\phi$, so we have $\rho(h)\rho(\phi) = \frac{1 - h}{\pi}$. Integrate the equation w.r.t. $h$ and $\phi$ respectively, consider the fact that $\int_0^1 \rho(h) dh = 1$ and $\int_0^{2\pi} \rho(\phi) d\phi = 1$, we have $\rho(h) = 2(1 - h)$ and $\rho(\phi) = \frac{1}{2\pi}$. So the C.D.F. is $P(h) = 2h - h^2$ and $P(\phi) = \frac{\phi}{2\pi}$. So choose $h = 1 - \sqrt{U_1}$ and $\phi = 2\pi U_2$ to sample points.

- c). A point on a unit sphere is uniquely located by the two angles $\theta$ and $\phi$. Using the same derivation of b), we get $\rho(\theta)\rho(\phi) = \frac{\sin \theta}{4\pi}$, where $4\pi$ is the surfaces area of unit sphere. Integrate the eq with respect to $\theta$ and $\phi$ we get $\rho(\theta) = \frac{\sin \theta}{2\pi}$ and $\rho = \frac{1}{2\pi}$, and then $P(\theta) = \frac{1 - \cos \theta}{2}$, $P(\phi) = \frac{\phi}{2\pi}$. So we choose $\theta = \arccos(1 - 2U_1)$, $\phi = 2\pi U_2$ to sample points.

![Figure 1: Three figures for problem 1.](image)

2. Problem 2. Suppose the normalized micro-facet distribution is $\rho(\alpha) = c D(\alpha)$ where $c$ is constant. The integral $\int_\omega D(\alpha) \cos \alpha d\omega = \int_\theta \int_\phi e^{-\cos^2 \theta/\cos^2 \beta} \cos \theta \sin \theta d\theta d\phi = \pi \cos^2 \beta (1 - e^{-\cos^2 \beta})$. (The integration turns out to be very easy to work out since you can substitute integral variables several times: $\cos \theta \to x \to x^2$). The normalization constant $c$ is the reciprocal of the integration. Obviously the distribution is symmetric w.r.t. $\phi$, so we sample $\phi = 2\pi U_2$, now $\rho(\theta) = 2\pi c e^{-\cos^2 \theta/\cos^2 \beta} \cos \theta \sin \theta$, hence $P(\theta) = \int_\theta^\Theta \rho(\theta) = \pi c \cos^2 \beta (e^{-\cos^2 \theta/\cos^2 \beta} - e^{-1/\cos^2 \beta})$, and after cleaning up this mess, we get $\theta$ is chosen according to $(e^{-\cos^2 \theta/\cos^2 \beta} - e^{-1/\cos^2 \beta})/(1 - e^{-1/\cos^2 \beta}) = U_1$ (Just compute the reverse function). As stated before, $\phi$ is chosen according to $\phi = 2\pi U_2$.

3. Problem 3. The normalization constant $c$ is equal to the reciprocal of the integration $\int_\omega D(\alpha) \cos \alpha d\omega = \frac{2\pi}{7}$. Still since the distribution is symmetric w.r.t. $\phi$, we work out $P(\theta) = 1 - \cos^7 \theta$, hence to correctly sample points, we choose $\theta = \arccos \sqrt[7]{U_1}$ and $\phi = 2\pi U_2$.  

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