UVa - cs302: Theory of Computation Spring 2008

Exam 1

Due: 28 February 2008

Your Name:

UVa Email Id:

Honor Policy. For this exam, you must **work alone**. You may consult the single page of notes you brought, but may not look at any other materials or aid or accept aid from other students.

Directions. Answer all five questions, including all sub-parts. You may use the backs of pages for your scratch work, but we will only grade answers that are written in the answer boxes, or that are found following clearly marked arrows from these boxes. The box for each answer is designed to be big enough to easily fit a full credit, correct answer. If you feel like you need more space to write your answer, then either your answer is incorrect, inelegant, or you are providing more detail than needed for full credit.

Question	Target	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1: Definitions. (20) For each question, provide a correct, clear, precise, and concise answer from the perspective of a theoretical computer scientist.

a. What is a *language*?

b. What is a *computer*?

c. What is a *problem*?

d. What does it mean for a machine to *recognize a language*?

Problem 2: Language Classification. (20)

For each described language, your answer should identify the simplest machine that can recognize the given language. The choices for machines (from simplest to least simple) are:

(A) Deterministic Finite Automaton with no cycles
(B) Deterministic Finite Automaton
(C) Deterministic Push-Down Automaton
(D) Nondeterministic Push-Down Automaton
(E) None of the Above

Your answer should include one of the five letters to identify the machine, and a brief justification. For these questions, you are *not* required to actually construct the machine or provide a proof for a full credit answer; a correct intuitive explanation is sufficient.

a. $\{w | w$ descibes the position of a chess board in which White can win $\}$

b. The language of all strings generated by the grammar: $S \rightarrow 0S \mid 1S \mid \epsilon$.

c. $\{0^i 1^{2i} 0^{3i} | i \geq 0\}$

d. The language of all strings in $\{a, b\}^*$ that do not include any substring of the form $a^i b^i$ where $i \geq 1$.

Problem 3: Problematic Proofs. (20)

Each of these "proofs" claims to prove a conjecture that is false. For each proof, identify the *first step* that is wrong, and briefly and clearly explain why.

a. **False Theorem:** The language

$$L = \{w | w \in \{0,1\}^* \text{ and the number of 1s in } w \text{ is odd } \}$$

is non-regular.

Claimed Proof. We use proof-by-contradiction to show L is not regular.

1. Assume L is regular.

2. Since L is regular, the pumping lemma should be satisfied.

3. Choose $s = 1$, which satisfies the requirements for the pumping lemma since it is in L (it has an odd number of 1s).

4. The only way to satisfy the pumping lemma requirements for dividing $s = xyz$ is to make $x = \epsilon$, $y = 1$, and $z = \epsilon$ since the pumping lemma requires $|y| > 0$.

5. But, this division leads to a contradiction of the pumping lemma, since for $i = 2$, $xy^i z = 11$ which is not in L (it has an even number of 1s).

6. Since the pumping lemma is not satisfied, the assumption must be invalid.

7. Thus, L is not regular.

b. **False Theorem:** All regular languages include the empty string.

Proof by induction. We induce on the size of the language.

1. *Basis*: A language of size 1 includes the empty string: the language $\{\epsilon\}$ is a regular language and it includes the empty string.
2. *Induction*: We assume that all regular languages of size n include the empty string, and show that all regular languages of size $n + 1$ include the empty string.

2a. Suppose L is a language of size $n + 1$.
2b. For some language M of size n, L = $M \cup w$ where w is some string not in M.
2c. By the induction hypothesis, M includes the empty string, so $L = M \cup w$ also includes the empty string.

c. **False Theorem:** If A and B are context-free languages, then $A \cap B$ is a context-free language.

Proof by construction.

1. Since A and B are context-free languages, there exist deterministic pushdown automata $P_A = (Q_A, \Sigma, \Gamma, \delta_A, q_0A, F_A)$ and $P_B = (Q_B, \Sigma, \Gamma, \delta_B, q_0B, F_B)$ that recognize A and B respectively.

2. We can construct the deterministic pushdown automaton $P_{AB} = (Q_{AB}, \Sigma, \Gamma_{AB}, \delta_{AB}, q_{0AB}, F_{AB})$ that recognizes the intersection of A and B by simulating both P_A and P_B simultaneously:

2a. $Q_{AB} = Q_A \times Q_B$
2b. $\Gamma_{AB} = \Gamma_A \times \Gamma_B$
2c. $\delta_{AB}((q_a, q_b), \sigma, (\gamma_a, \gamma_b)) = ((r_a, r_b), (y_a, y_b))$
where $\delta_A(q_a, \sigma, \gamma_a) = (r_a, y_a)$ and $\delta_B(q_b, \sigma, \gamma_b) = (r_b, y_b)$
2d. $q_{0AB} = (q_0A, q_0B)$
2e. $F_{AB} = \{(q_a, q_b) | q_a \in F_A \wedge q_b \in F_B\}$

3. Since P_{AB} accepts only if both P_A and P_B would accept, the language recognized by P_{AB} is the intersection of the languages recognized by P_A and P_B.

Problem 4: Proofs. (20) For each part, write a clear, concise, and convincing proof. You may use the pumping lemma for regular languages, pumping lemma for context-free languages, as well as the closure properties for regular and context-free languages.

a. Prove that the language $\{0^i 1^j | i, j \geq 0 \wedge i \neq j\}$ is not regular.

b. Prove that the language $\{a^i b^j a^j b^i\}$ is context-free.

c. Prove that the language $\left\{0^n 1^{n^2}\right\}$ (the number of 1s is the square of the number of 0s) is not context-free.

Problem 5: Closure. (20) Define $H(L)$ as the set of even-length strings in L. That is,

$$H(L) = \{w | w \in L \text{ and } |w| = 2k \text{ for some } k \geq 0\}$$

a. If L is a regular language, is $H(L)$ a regular language? (State clearly "Yes" or "No", and support your answer with a convincing proof.)

b. [Bonus] If L is a context-free language, is $H(L)$ a context-free language? (State clearly "Yes" or "No", and support your answer with a convincing proof.)

End of Exam

Enjoy your break!