Ian Davey
Challenge Problem Proof

First, we prove that the grammar does not generate any string in L_{ww}.

Proof by contradiction. Assume that the rules do produce strings in L_{ww}.

Without loss of generality, assume we use $X \rightarrow ZXZ$ *m* times before using $X \rightarrow 0$ and use $Y \rightarrow ZYZ$ *n* times before using $Y \rightarrow 1$, where *m* and *n* are any non-negative integer.

Case 1: The first production used is $S_{Even} \rightarrow XY$.

We end up with $Z^m 0 Z^m Z^n 1 Z^n$ where each Z has yet to generate a terminal. This can be written as $Z^m 0 Z^{n+m} 1 Z^n$, or $Z^m 0 Z^n Z^m 1 Z^n$.

For the string to be split into two identical parts w, each w must have equal length, which in this case would be $m + n + 1$.

For each w to be equal, each Z^m term must generate identical sequences (we'll call this sequence *a*). Each Z^n term must also generate an identical sequences (we'll call this sequence *b*). Each *w* can now be represented as $a\gamma b$, where γ is a single character. γ must be the terminal finally derived from both X and Y.

The only single character that can be derived from X is 0, so when derived from X, $\gamma = 0$. Likewise, when derived from Y, $\gamma = 1$. Therefore, the first *w* would be $a0b$ and the second *w* would be $a1b$.

However, each *w* is supposed to be identical. Hence, there is a contradiction.

Case 2: The first production used is the $S_{Even} \rightarrow XY$ derivation.

The proof is nearly identical to Case 1, except for swaping X and Y.

The two cases cover all possible derivations, and both lead to contradictions. Hence, the assumption is invalid and the rules cannot derive an element of L_{ww}.

Now, we prove that the grammar does generate all even-length strings in the complement of L_{ww}.

Proof-by-induction on the length of the strings.

We prove the grammar will derive

$$\{0, 1\}^m 0 \{0, 1\}^{m+n} 1 \{0, 1\}^n \text{ and } \{0, 1\}^m 0 \{0, 1\}^{m+n} 1 \{0, 1\}^n$$

which covers all possible strings in the complement of L_{ww}.

Each string s in the complement of L_{ww} has the length $|s| = 2(m + n + 1)$. Therefore, there should be $2^{2(m + n + 1)}$ possibilities for any string of length $|s|$ in Σ^*. Since X and Y cannot be the same (two characters in each string), this reduces the number of possibilities to $2^{2(m + n + 1)} - 2$.

Basis: The smallest possible strings in the complement of L_{ww} are 10 and 01. Both can be derived using the grammar: $S \to XY \to 01$ and $S \to YX \to 10$. In the basis step, $m = n = 0$.

Induction:
Let there be $m + 1$ Zs derived on either side of the first X or Y. This will make the length of the string $2[(m + 1) + n + 1]$.

Thus there are $2^{[(m + 1) + n + 1]}$ possibilities for any string of this length in Σ^*. X and Y still cannot derive to the same terminal, reducing the number of possibilities to $2^{2[(m + 1) + n + 1]} - 2$.

Now let there be $n + 1$ Zs derived on either side of the first X or Y. This will make the length of the string $2[(m + (n + 1) + 1]$. Thus there are $2^{[m + (n + 1) + 1]}$ possibilities for any string of this length in Σ^*. X and Y still cannot derive to the same terminal, reducing the number of possibilities to $2^{2[m + (n + 1) + 1]} - 2$.

Now let there be m + 1 *and* n + 1 Zs derived. This will make the length of the string $2[(m + 1) + (n + 1) + 1]$. Thus there are $2^{[(m+1)+(n+1)+1]}$ possibilities for any string of this length in Σ^*. X and Y still cannot derive to the same terminal, reducing the number of possibilities to $2^{2[(m + 1) + (n + 1) + 1]} - 2$.

The only possibilities eliminated for every set of strings with a common length are those which would put the strings in L_{ww}. Therefore, this grammar does indeed describe the complement of L_{ww}.