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**Challenge Problem Solution**

**Theorem:** $S_{\text{Even}}$, as defined below, generates all even strings that are not in $L_{\text{ww}}$.

(1) $S_{\text{Even}} \rightarrow XY \mid YX$

(2) $X \rightarrow ZXZ \mid \mathbf{0}$

(3) $Y \rightarrow ZYZ \mid \mathbf{1}$

(4) $Z \rightarrow \mathbf{0} \mid \mathbf{1}$

**Lemma 1.** $S_{\text{Even}}$ can generate all the strings length equal to $2n$ ($n > 0$, $n$ is integer) in the following two forms:

$$Z^j\mathbf{0}Z^jZ^k\mathbf{1}Z^k \text{ and } Z^j\mathbf{1}Z^jZ^k\mathbf{0}Z^k$$

where $j$, $k$ are integers, $j \geq 0$, $k \geq 0$, and $2j+2k+2 = 2(j+k+1) = 2n$. This follows from the grammar rules. The first form corresponds to derivations of the form:

$$S_{\text{Even}} \rightarrow XY \rightarrow ZXZY \rightarrow \ldots \rightarrow Z^jXZ^jY \rightarrow Z^j\mathbf{0}Z^jY \rightarrow Z^j\mathbf{0}Z^jZYZ \rightarrow \ldots \rightarrow Z^j\mathbf{0}Z^jZ^kYZ^k \rightarrow Z^j\mathbf{0}Z^jZ^k\mathbf{1}Z^k$$

($j$ uses of rule 2a) (by rule 2b) ($k$ uses of rule 3a) (using rule 3b)

The second form corresponds similarly to derivations that start with $S_{\text{Even}} \rightarrow YX$.

Since each $Z$ produces either 0 or 1, we can equivalently write the two forms above by replacing $Z^jZ^k$ with $Z^kZ^j$as: $Z^j\mathbf{0}Z^kZ^j\mathbf{1}Z^k$ and $Z^j\mathbf{1}Z^kZ^j\mathbf{0}Z^k$

**Lemma 2.** We know the grammar could only generate strings in the following format $Z^j\mathbf{0}Z^jZ^k\mathbf{1}Z^k$ or $Z^j\mathbf{1}Z^jZ^k\mathbf{0}Z^k$.

**Proof.** With only $X$ productions $X \rightarrow ZXZ \mid \mathbf{0}$ we can only generate strings in the following format by repeating using the grammar. As there is no other production for X, we can conclude that $X = Z^j0Z^j$

Similarly, with only $Y$ productions $Y \rightarrow ZYZ \mid \mathbf{1}$ we can only generate strings in the following format $Y = Z^k1Z^k$ by repeating using the grammar. As there is no other rule for $Y$, we can conclude that $Y = Z^k1Z^k$

Combining both parts, with grammar $S_{\text{Even}} \rightarrow XY \mid YX$ we know that all strings produced by $S_{\text{Even}}$ could only be the following format $Z^j0Z^jZ^k1Z^k$ or $Z^j1Z^jZ^k0Z^k$.

**Proof:**

To prove that $S_{\text{Even}}$ generates $L_{\wedge\text{ww}}$, the language of all even-length strings that are *not* composed of two matching halves, we observe that for any string $s \in L_{\wedge\text{ww}}$ there must be some position $i$ in the string where the value of $s[i]$ is different from the value of $s[|s|/2 + i]$. We show that $S_{\text{Even}}$

can generate all such strings, and generates no strings without a mismatch at some position. There are only two symbols in the alphabet, so these two cases cover all possibilities: $s[i]=\mathbf{0}$ and $s[i]=\mathbf{1}$ where $0 \leq i < |s|/2$.

**Case 1.** $s[i] = \mathbf{0}$.

First, we show the grammar produces all of the even-length strings in $L_{\text{^ww}}$ where there is some $i$, $0 \leq i < |s|/2$, such that $s[i] = \mathbf{0}$ and $s[|s|/2 + i] = \mathbf{1}$. All even length strings $t$ of length $2n$ can be divided into two equal length strings of length $n$: $t = t_1t_2$. Since we are covering the case where $s[i] = 0$, and $s[|s|/2 + i] = 1$, all strings have the form $t = Z^i\mathbf{0}Z^kZ^i\mathbf{1}Z^k$ where $n = i + k + 1$. From the lemma, $S_{\text{Even}}$ generates all such strings by substituting $i$ and $j$.

Now, we show that the grammar does not produce any strings in $L_{\text{ww}}$. We prove by contradiction: if $t$ is in $L_{\text{ww}}$, then $t_1$ must equal to $t_2$. We know the grammar could only generate strings in the following format $Z^j\mathbf{0}Z^jZ^k\mathbf{1}Z^k$ or $Z^j\mathbf{1}Z^jZ^k\mathbf{0}Z^k$ (from Lemma 2). If we separate the string into two equal length strings $t=t_1t_2$, we can only separate it as $t_1= Z^j\mathbf{0}Z^k$ and $t_2= Z^j\mathbf{1}Z^k$, then, in $t_1$, there must be j symbols in front of the 0; same in $t_2$, there are j symbols in font of the 1. As there will be different alphabet appear on the same position, therefore $t_1$ always NOT equal to $t_2$. Thus, we have a contradiction.

**Case 2.** $s[i] = \mathbf{1}$. This follows identically to case 1, except using the second form corresponding to the $S_{\text{Even}} \rightarrow YX$ rule.