3D Pie Slicing problem
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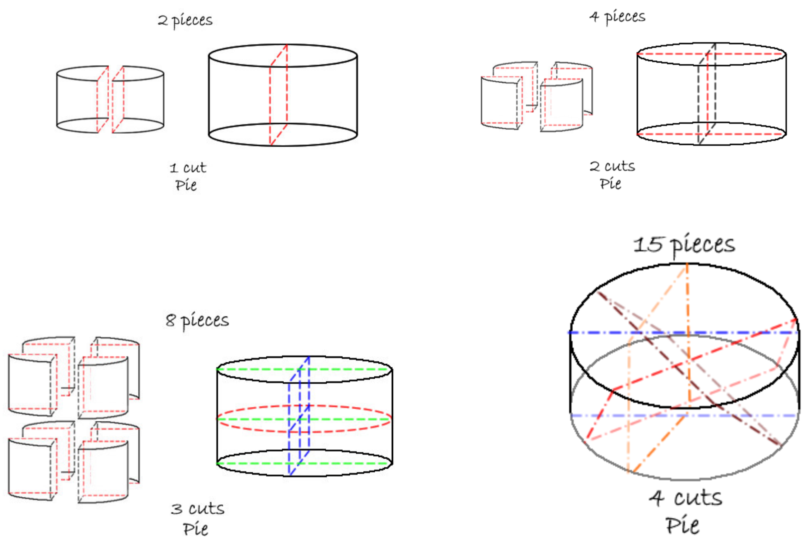
For Problem Set 1, Problem 6 asked what the maximum number of pieces that can be produced using "n cuts that are straight lines all the way across the pie". Liuyi cleverly noticed that more pieces can be produced if the cuts are not perpendicular to the pie, but instead slide it in three dimensions. Here is his proof that by angling cuts, it is possible to obtain $(n^3+5n+6)/6$ pieces using n cuts.



The four pictures above illustrate some solutions of "3D Pie Slicing" problem. (Note: I have spent couple hours but I am not able to decompose the 15 pieces with 4 cuts with nice vision. Here is one exhausting method you could try with scratch paper: choose one cutting-plane and split the pie into two pieces, find the intersecting line of any two planes and then split the pieces again. Keep doing this until you cannot split.)

Theorem: The maximum number of pieces that can be produced with n cuts is $(n^3+5n+6)/6$.

Proof: Proof by induction on the number of cuts.
Basis: For $n = 0$, $(n^3+5n+6)/6 = 6/6 = 1$ piece.

Induction:
Assume k cuts will give us maximum $(k^3+5k+6)/6$ pieces.
We can view each cut C_i as a plane. The $k+1^{st}$ cut, C_{k+1}, could intersect all the previous k cuts ($C_1, C_2, \dots, C_k$). It will form k intersecting lines in the plane created by C_{k+1} (each

two intersecting planes generate an intersecting line).

Note that k lines could cut a plane into maximum $(k^2+k+2)/2$ pieces. C_{k+1} could be divided into a maximum $(k^2+k+2)/2$ pieces. Each piece of C_{k+1} could cut the space that contains it into two parts, increasing the total number of pieces by one.

So $(k^2+k+2)/2$ pieces of the plane will create $(k^2+k+2)/2$ new pie pieces from the $k+1^{st}$ cut. Using the induction hypotheses, the total number of pieces is $(k^3+5k+6)/6 + (k^2+k+2)/2 = ((k+1)^3+5(k+1)+6)/6$. Replacing $n = k + 1$, we get $(n^3 + 5n+6)/6$, thus proving the induction step.