Proof by Raghu Rajkumar

Theorem: The following grammar produces the language $\{w \mid |w| \text{ is even} \wedge w \notin zz\}$

$S_{Even} \rightarrow XY \mid YX$
$X \rightarrow ZXZ \mid 0$
$Y \rightarrow ZYZ \mid 1$
$Z \rightarrow 0 \mid 1$

String indexing formulae, notations and ideas used in proof:

1. $\forall$ String s, Odd($|s|$) $\Rightarrow$ The middle character of s is indexed by $(|s|+1)/2$
2. u[i] refers to the i^{th} character of u, starting at index 1.
3. Two characters of $s = uv$, $|u| = |v|$ are corresponding
   $\Leftrightarrow s[i], s[j]$ such that $j - i = |s|/2$
   $\Leftrightarrow s[i], s[j]$ such that $s[j] = v[i]$

Lemma 1: X and Y produce only odd length strings.

Let s be any string such that Even($|s|$)

Case 1 (Basis): $|s| = 0$.

There is no rule $X \rightarrow \varepsilon$ or $Y \rightarrow \varepsilon$. Hence, the string s can not be produced.

Case 2 (Induction): $|s| > 0$.

The only rules that lead to the production of strings greater than length 1 are $X \rightarrow ZXZ$ and $Y \rightarrow ZYZ$. Hence, $s = ZtZ$, where t is an even-length string of length $|s - 2|$. Repeating this process recursively leads to Case 1, where s^n, $|s^n| = 0$, cannot be produced by any rule.

Therefore, in either case, the string s cannot be produced by the grammar.

$\Rightarrow$ All strings produced by X and Y are of odd length.

To prove that the grammar recognizes $\{w \mid |w| \text{ is even} \wedge w \notin zz\}$ involves proving two parts:

1. Odd length strings are not produced by the grammar. This is proved in Part 1.
2. No string in L^{ww} is produced by the grammar, and all even strings *not* in L^{ww} are produced by the grammar. These are proved separately in subparts 2a and 2b.

Part 1: Odd length strings are not produced by the grammar.

Let $X \rightarrow x$ and and $Y \rightarrow y$.

The only possible productions for S_{Even} are $S_{Even} \rightarrow XY$ and $S_{Even} \rightarrow YX$.

Hence, the length of the resulting string is $|x + y|$

Odd(x) $\wedge$ Odd(y) By *Lemma* 1

$\Rightarrow$ Even($|x + y|$)

Therefore, only even length strings are produced by the grammar.

Part 2a: All even length strings not in L^{ww} are produced by the grammar.

Let $s = uv$ such that $|u| = |v| = n$ (say) $\wedge u \neq v \Leftrightarrow s \notin L^{ww}$

$\Rightarrow \exists\, i$: N | $u[i] \neq v[i]$.

Let $p = 2i - 1$, $q = n - p$

Then $s = \Sigma^{(p-1)/2}1\Sigma^{(p-1)/2}\Sigma^{(q-1)/2}0\Sigma^{(q-1)/2}$ or $s = \Sigma^{(p-1)/2}0\Sigma^{(p-1)/2}\Sigma^{(q-1)/2}1\Sigma^{(q-1)/2}$, where the corresponding i^{th} characters of u and v are different. The first p characters of the string match the pattern for $X \to Z^{(p-1)/2}0Z^{(p-1)/2}$ and the next q characters match the pattern for $Y \to Z^{(q-1)/2}1Z^{(q-1)/2}$ in the first case, and vice versa for the second case.

Hence, $S_{Even} \to XY|YX \to^* s$.

$\Rightarrow \forall$ String s, Even(s) $\wedge\ s \notin L^{ww} \Rightarrow S_{Even} \to^* s$

Part 2b: No even length string in L^{ww} is produced by the grammar.

Let $s \in L^{ww}$. Assume that the grammar produces s, i.e., $S_{Even} \to s$.

Case 1: $S_{Even} \to XY \to^* s$.

$X = Z^k0Z^k \wedge Y = Z^l1Z^l$

$\Rightarrow XY \to^* s = Z^k0Z^k Z^l1Z^l = xy$

Let $p = 2k + 1$, $q = 2l + 1$;

$|s| = n = p + q$;

Midpoint of $x = m_x = (p+1)/2 = k+1$.

Midpoint of $y = m_y = (q+1)/2 = l+1$.

$s[m_x] = s[k+1] = 0$, $s[m_y] = s[p+l+1] = 0$

$m_y - m_x = p+l-k = k+l+1 = (2k+2l+2)/2 = n/2 = |s|/2$

$\Rightarrow$ The characters at m_x and m_y are corresponding characters.

Since $s \in L^{ww}$, these corresponding characters should be equal. But we have proved that $s[m_x] = 0 \neq 1 = s[m_y]$. This is a contradiction. Our assumption that the grammar produces s is incorrect.

$\Rightarrow S_{Even}$ does not derive s

Case 2: $S_{Even} \to YX \to^* s$.

The proof for this is the same as for Case 1, with X and Y swapped.

Hence, $\forall$ String s, $s \in L^{ww} \Rightarrow S_{Even}$ does not derive s

Combining parts 1 & 2, the grammar accepts only even length strings $\notin L^{ww}$. ∈