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Challenge Solution

Given:

$$\begin{aligned} S_{Even} &\rightarrow XY \mid YX \\ X &\rightarrow ZXZ \mid 0 \\ Y &\rightarrow ZYZ \mid 1 \\ Z &\rightarrow 0 \mid 1 \end{aligned}$$

Prove:

(1) The grammar does not produce any strings not in L_{ww} complement.
(2) The grammar can produce all even length strings in L_{ww} complement.

<u>Part (1)</u>

- ✓ From the class slides, we have already proved that the grammar cannot produce ε.

By definition, for the string to not be in L_{ww} complement, it must be an even-length string that may be divided into two parts *ww* such that the first half of is equivalent to the second half.

Using proof by contradiction:

Assumption: S_{Even} can produce a string that is not in L_{ww} complement.

The assumption suggests that we may split the final resultant string into two equal halves such that the first half is equivalent to the second. There are two cases for S_{Even} that we must consider:

First Case: All possible strings under the condition $S_{Even} \rightarrow XY$

The generated string must be even in length since:

$X \Rightarrow ZXZ, ZZXZZ, ZZZXZZZ..., Z^nXZ^n$, where $n \geq 0$ is an integer (odd length)
$X \Rightarrow 0$ (odd length)

$Y \Rightarrow ZYZ, ZZYZZ, ZZZYZZZ..., Z^nYZ^n$, where $n \geq 0$ is an integer (odd length)
$Y \Rightarrow 1$ (odd length)

Therefore, their combination (XY) must be even in length.

We notice that from the right sides of X and Y, they can never be the same value. Thus, if we can prove that X and Y are on separate corresponding halves of the string, where if X is the n^{th} element of the first half, while Y is the n^{th} element of the second, we may arrive at the contradiction we seek.

Using Induction:

Basis: $S_{Even} \Rightarrow XY$, $X \Rightarrow 0$, and $Y \Rightarrow 1$.

Here, the generated string is 01. As we can see, when split the string in half, X and Y are on separate corresponding halves of the string, where X is the 1^{st} element of the first half, while Y is the 1^{st} element of the second. This leads to a contradiction to our original assumption.

Inductive Step: $X \Rightarrow ZXZ$ or $Y \Rightarrow ZYZ$

Since we are covering all even length strings, we may inductively increase the length of the string by two via $X \Rightarrow ZXZ$ or $Y \Rightarrow ZYZ$. Looking at the transitions, we notice that since Z is the only variable that maybe added in between X and Y. Therefore, adding a Z after X is the same as adding a Z before Y and vise versa. As a result, we may model the final string generated by the grammar with the equation:

$$S_{Even} = Z^a\, X\, Z^{a+b}\, Y\, Z^b$$
$$S_{Even} = Z^a\, X\, Z^{b+a}\, Y\, Z^b$$

where a = the number of $X \Rightarrow ZXZ$ transitions taken
b = the number of $Y \Rightarrow ZYZ$ transitions taken

We see that when the string is split into $Z^a\, X\, Z^b$ and $Z^a\, Y\, Z^b$, although it is possible for all of the Z's to be equivalent, X will always be different from Y since X must lead eventually to 1 and Y to 0. Therefore, we reach a contradiction to our original assumption.

Second Case: All possible strings under the condition $S_{Even} \rightarrow YX$

We realize this is very similar to the first case and we can use the same argument to reach a contradiction. Using the same argument, we arrive at the final equation:

$$S_{Even} = Z^b\, Y\, Z^{a+b}\, X\, Z^a$$

From the equation above, we can use the same argument to reach a contradiction from our original assumption. Therefore, we see that a contradiction cannot be avoided, and thus our original assumption was false.

Part (2)

We have already shown in part 1 that: the grammar only produces even length strings and that it can produce some strings of any even length, and we have shown that when split into two equal halves; X and Y are on separate halves of the string. To complete our proof, we must show that given an even length, we may generate all possible strings of that length that are in L_{ww} complement.

The grammar includes the productions: $Z \rightarrow 0|1$. This means that Z can be any value in the alphabet of the language.

Let's then prove that for any given string generated by the grammar, we may split the string into two equal halves where X and Y appears as any n^{th} element of each half counting from left to right. This ensures that we may always generate a string provided the location of X and Y in their respective halves.

Consider the case for $S_{Even} \rightarrow XY$:

$X \Rightarrow ZXZ$ is equivalent to $X \Rightarrow ZX$; $Y \Rightarrow ZY$.
$Y \Rightarrow ZYZ$ is equivalent to $X \Rightarrow XZ$; $Y \Rightarrow YZ$.

This suggests that if we want to have X and Y appear as the first element for example, we would have only $Y \Rightarrow ZYZ$ transitions for fulfill the length of the string requirement. If we wanted them to appear as the second element, we would have one $X \Rightarrow ZXZ$ transition and the rest $Y \Rightarrow ZYZ$ transitions to fulfill the requirement. In fact, the general equation would be:

For a string of length s, if we want X and Y to be the n^{th} element after the string was divided into two equal length substrings, we would do the following number of transition rules:

$n-1 \quad X \Rightarrow ZXZ$ -and- $\frac{1}{2}s-n \quad Y \Rightarrow ZYZ$

A similar argument may be made for the $S_{Even} \rightarrow YX$ case, where the transition rules would be:

$n-1 \quad Y \Rightarrow ZYZ$ -and- $\frac{1}{2}s-n \quad X \Rightarrow ZXZ$

This assures that we can produce even length strings where X and Y can be anywhere within the individual substrings after it is divided equally.

Because Z fulfills the entire alphabet, S_{Even} may be XY or YX, and we can generate strings that are any even length that fulfills the language, allowing X and Y be at any location completes the proof by exhaustion.