University of Virginia, Department of Computer Science
cs150: Computer Science — Spring 2007

Problem Set 4: Creating Colossi

Out: 12 February 2007
Due: 19 February 2007 (beginning of class)

Collaboration Policy - Read Carefully

This problem set is intended to help you prepare for Exam 1. You may work on it by yourself or with other students, but you will need to do the exam on your own.

Regardless of whether you work alone or with a partner, you are encouraged to discuss this assignment with other students in the class and ask and provide help in useful ways. You may consult any outside resources you wish including books, papers, web sites and people. If you use resources other than the class materials, indicate what you used along with your answer.

You have only one week to complete this problem set. We recommend you **start early** and **take advantage of the staffed lab hours**.

Purpose

- Get practice with some of the material that will be on Exam 1 including: recursive definitions, procedures, lists, and measuring work.
- Learn about cryptography and the first problems that were solved by computers.

Download: Download *ps4.zip* to your machine and unzip it into your home directory `J:\cs150\ps4`.

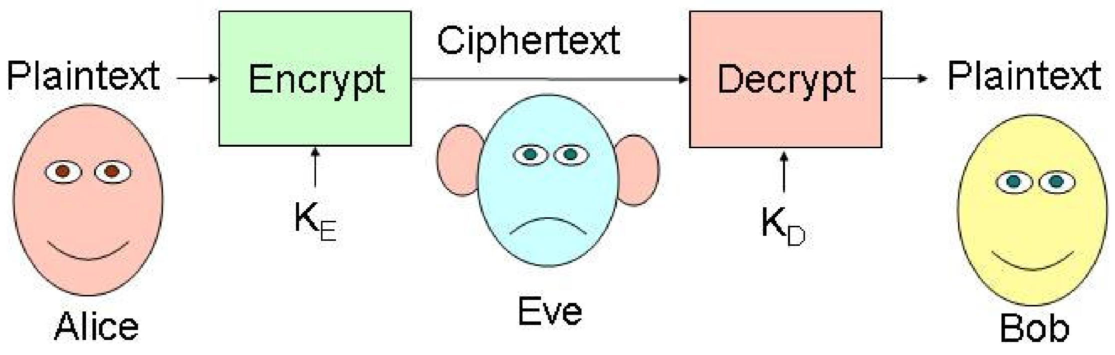
This file contains:

- *ps4.scm* — A template for your answers. You should do the problem set by editing this file.
- *lorenz.scm* — Some helpful definitions for this problem set.

Background

Cryptography means *secret writing*. The goal of much cryptography is for one person to send a message to another person over a channel that an adversary may be eavesdropping on without the eavesdropper understanding the message. We do this by having functions that *encrypt* and *decrypt* messages. The encrypt function takes a plaintext message and produces a ciphertext message. Encrypt scrambles and alters the letters of the plaintext message so that an eavesdropper will not be able to understand the message. The decrypt function takes a ciphertext message and produces the corresponding plaintext message. Encryption works as intended if the only person who can perform the decrypt function is the intended recipient of the message.

Since making up good encrypt and decrypt functions and keeping them secret is hard, most cryptosystems are designed to be secure even if the encryption and decryption algorithms are revealed. The security relies on a key which is kept secret and known only to the sender and receiver. The key alters the encryption and decryption algorithm in some way that would be hard for someone who doesn't know the key to figure out. If the sender and receiver use the same secret key we call it a *symmetric* cipher. If the sender and receiver can use different keys we call it an *asymmetric* cipher.



`Ciphertext = ((encrypt` K_E`) Plaintext))`

`Plaintext = ((encrypt` K_D`) Ciphertext))`

If $K_E = K_D$ it is *symmetric* encryption

If K_E is different from K_D it is *asymmetric* encryption

In this problem set, you will explore a symmetric cipher based on the Lorenz Cipher that was used by the German Army High Command to send some of the most important and secret messages during World War II. The Lorenz Cipher was broken by British Cryptographers at Bletchley Park. Arguably the first electronic programmable computer, *Colossus*, was designed and built by Tommy Flowers, a Post Office engineer working at Bletchley Park during the war.

Ten Collosi were built in 1943 and 1944, and used to break some of the most important German messages during World War II. Messages broken by Colossus were crucial to the D-Day invasion since the allies were able to learn that their campaign to deceive Hitler about where the attack would come was succeeding and knew where German troops were positioned.

Bletchley Park (Summer 2004)

Measuring Cost

Before answering question 1, you should have read Chapter 6 of the course book. You can skip ahead to question 2 and return to this question later.

For each of the following subquestions, you will be given two functions, *f* and *g* that take a single parameter *n* (which must be a non-negative integer) and asked to determine which of $O(g(n))$, $\Omega(g(n))$, and $\Theta(g(n))$ are true for $f(n)$. It is possible that more than one holds. In addition to identifying the properties that hold, you should justify your answer by:

- For each property that is true, select *c* and n_0 values and show why the necessary inequality holds for all $n > n_0$.
- For each property that is false, explain why no such *c* and n_0 values exists. One way to do this is to show how for any selected *c* value, you can find infinitely many *n* values that fail to satisfy the necessary property.

For example, if *f* is n^2 and *g* is n^3 you would argue:

- n^2 **is** $O(n^3)$ since if we pick $c = 1$ and $n_0 = 1$ then $c * n^2 < n^3$ for all $n > 1$.
- n^2 **is not** $\Omega(n^3)$ since for any c value we know $c * n^2$ is not $> n_3$ for all $n > c$. This is the case since $c * n^2$ is less than n^3 if c is less than n. For any finite value c, there are infinitely many n values greater than c that fail to satisfy the needed property.
- n^2 **is not** $\Theta(n^3)$ since n^2 is not $\Omega(n^3)$.

Question 1: For each f and g pair below, argue convincingly whether or not f is (1) $O(g)$, (2) $\Omega(g)$, and (3) $\Theta(g)$ as explained above. For all questions, assume n is a non-negative integer.

a. f: $n + 3$, g: n
b. f: $n^2 + n$, g: n^2
c. f: 2^n, g: 3^n
d. f: 2^n, g: n^n
e. f: the federal debt n years from today, g: the US population n years from today (this one requires a more informal argument)

The Exclusive Or (XOR)

The exclusive-or (*xor*, sometimes pronounced "zor") function is one of the most useful functions for cryptography. The *xor* function evaluates to true if exactly one of the inputs is true. In the English language people often use "or" when they mean to say "xor" since English does not have a word meaning precisely xor. For example, take the statement *"This Tuesday night we can work on our CS150 problem set* **or** *watch the exciting finale of "Big Brother 6"*." Since only one can actually take place (assuming they occur at the same time), we should actually say **xor**. On the other hand, the statement "*working on our CS150 problem set* **or** *watching the exciting finale of "Big Brother 6"* is a worthwhile activity" is true if either one or both of these activities is worthwhile (and in the opinion of the CS150 staff, at least, that makes it a true statement). The statement, "*working on our CS150 problem set* **xor** *watching the exciting finale of "Big Brother 6"* is a worthwhile activity" is also likely to be true.

In cryptography, it is usually easier to deal with `1` and `0` instead of true and false. For this problem set, use `1` to represent true and `0` to represent false. We will call each `0` or `1` a *bit*, since this is the smallest unit of information.

Question 2: Define the `xor` function that takes two bits as parameters and evaluates to `1` if exactly one of the parameters is `1` and evaluates to `0` otherwise.

Your `xor` function should produce these evaluations:

```
> (xor 0 0)    > (xor 1 0)
0              1
> (xor 0 1)    > (xor 1 1)
1              0
```

The xor function has several properties that make it useful in cryptography:

- *invertibility* — `(xor (xor a b) b)` always evaluates to `a`, no matter what `b` is. This means if we encrypt a message by xor-ing it with a key, we can decrypt the message by using xor again with the same key!
- *perfect secrecy* — if `a` is a message bit and `r` is a perfectly random bit (fifty percent chance it is true and fifty percent chance it is false), then `(xor a r)` is equally likely to be true or false regardless of the message bit *a*.

The second property means that if a message is encrypted by converting the message into a sequence of bits, and xor-ing each bit in the message with a perfectly random, secret bit known only to the sender and receiver, then we can send a message with perfect secrecy! This is known as a *one-time pad*.

Fortunately for the allies, the Nazis didn't have a way of generating and distributing perfectly random bit sequences. Instead, they generated non-random sequences of bits using rotors. The Soviet Union also used a one-time pad cipher following WWII. Because their key bits were not truly random, and because they sometimes reused keys, the NSA was able to decrypt many important KGB messages. Details of this were released in 1995, and are available at the NSA's web site. We will look at how the Lorenz cipher used `xor` to encrypt messages soon. First, we consider how to turn messages into sequences of bits.

The Baudot Code

The code used by the Nazis was the Baudot code. It translates letters and other common characters into a 5 bit code. With five bits, we have $2^5 = 32$ possible values, so this is enough to give each letter in the alphabet a different code and have a few codes left over for spaces and other symbols. Modern computers typically used either 7-bit ASCII encodings to have $2^7 = 128$ possible characters so that lowercase and additional punctuation characters can be represented, or 16-bit Unicode encodings that are able to represent $2^{16} = 65536$ different characters to support languages like Chinese that have many different characters.

Table 1 shows the letter mappings for the Baudot code. For example, the letter `H` is represented by `10100` while `I` is `00110`. We can put letters together just by concatenating their encodings. For example, the string "HI" is represented in Baudot as `10100 00110`. There are some values in the Baudot code that are awkward to print: carriage return, line feed, letter shift, figure shift and *error*. For our purposes we will use printable characters unused in the Baudot code to represent those values so we can print encrypted messages as normal strings. Table 1 shows the replacement values in parenthesis.

A	00011	H	10100	O	11000	V	11110	*space*	00100
B	11001	I	00110	P	10110	W	10011	*carriage return* (,)	01000
C	01110	J	01011	Q	10111	X	11101	*line feed* (-)	00010
D	01001	K	01111	R	01010	Y	10101	*letter shift* (.)	11111
E	00001	L	10010	S	00101	Z	10001	*figure shift* (!)	11011
F	01101	M	11100	T	10000			*error* (*)	00000
G	11010	N	01100	U	00111				

Table 1. Baudot Code mappings.

We can use lists of ones and zeros to represent Baudot codes. `H` is represented as `(list 1 0 1 0 0)`. A string is represented as a list of these lists: "HI" is
`(list (list 1 0 1 0 0) (list 0 0 1 1 0))`.

We have provided these two functions in *lorenz.scm*:

- `(char-to-baudot c)` — evaluates to the baudot code, represented as a list of five bits, corresponding to the character `c`.
- `(baudot-to-char code)` — evaluates to the character associated with the Baudot code represented by the list of five bits passed as the `code` parameter.

Characters are represented in MzScheme using `#\`*`character`*. For example, `#\H` is the character `H` and `#\space` is the space character.

You can convert a string to a list of characters using `string->list`, and turn a list of characters into a string using `list->string`:

```
> (string->list "HI")
(#\H #\I)
> (list->string (list #\H #\I))
"HI"
```

Question 3:
a. Write a function `string-to-baudot` that takes a string and transforms it into a list of Baudot codes.
b. Write the inverse function, `baudot-to-string`, that takes a list of lists of baudot codes and transforms it back into a string.
c. Describe the complexity of your `baudot-to-string` procedure using Θ notation. Make sure to explain carefully what any variable you use means.

Your functions should be inverses. Hence, you can test your code by evaluating `baudot-to-string` composed with `string-to-baudot`. For example,

```
(baudot-to-string (string-to-baudot "HELLO"))
```

should evaluate to `"HELLO"`.

The Lorenz Cipher

Lorenz Cipher Machine

The Lorenz cipher was an encryption algorithm developed by the Germans during World War II. It was used primarily for communications between high commanders. The original Lorenz machine consisted of 12 wheels, each one having 23 to 61 unique positions. Each position of a wheel represented either a one or a zero.

The first 5 wheels were called the *K* wheels. Each bit of the Baudot representation of a letter was xor-ed with the value showing on the respective wheel. The same process was repeated with the next 5 wheels, named the *S* wheels. The resulting value represented the encrypted letter. After each message letter the *K* wheels turn one rotation. The movement of the *S* wheels was determined by the positions of the final two wheels, called the *M* wheels.

Like most ciphers, the Lorenz machine also required a key. The key was the starting position of each of the 12 wheels. To decipher the message you simply need to start the wheels with the same position as was used to encrypt and enter the ciphertext. There were 16,033,955,073,056,318,658 possible starting positions. This made the Nazis very confident that without knowing the key (starting positions of the wheels), no one would be able to break messages encrypted using the Lorenz machine. (But, their confidence was misplaced.)

Since the full Lorenz cipher would be too hard for this problem set, we will implement a simplified version. You should be suitably amazed that the allied cryptographers in 1943 were able to build a computer to solve a problem that is still hard for us to solve today! (Of course, they did have more that a week to solve it, and more serious motivation than we can provide in this course.)

Our Lorenz machine will use 11 wheels, each with only 5 positions. The first five wheels will be the *K* wheels and the second five the *S* wheels. Each of these will only have a single starting position for all 5 — that is, unlike the real Lorenz machine, for this problem set we will assume all five *K* wheels start in the same position and all five *S* wheels start in the same position.

The final wheel will act as the *M* wheel. After each letter all the *K* wheels and the *M* wheel should rotate. If the *M* wheel shows a `1` the *S* wheels should also rotate, but if the M wheel shows a `0` the *S* wheels do not rotate.

We have provided 3 lists that represent the wheels. The first is called `K-wheels` and is a list of lists, each of the inner lists containing the 5 settings. The definition is: `(define K-wheels (list (list 1 1 0 1 0) (list 0 1 0 0 1) (list 1 0 0 1 0) (list 1 1 1 0 1) (list 1 0 0 0 1)))`.

There is a similar list called `S-wheels` to represent the *S* wheels of our simulated machine. The final list represents the *M* wheels and is just a single list. The definition is `(define M-wheel (list 0 0 1 0 1))`.

Question 4: To rotate our wheels, we will take the number at the front of the list and move it to the back. The first number in the list will represent the current position of the wheel.

a. Define a function `rotate-wheel` that takes one of the wheels as a parameter. It should return the wheel rotated once. For example, `(rotate-wheel (list 1 0 0 1 0))` should evaluate to `(0 0 1 0 1)`. Although all the wheels in our simulated Lorenz cipher machine have five bits, your `rotate-wheel` procedure should work for any length list.

b. Define a function `rotate-wheel-by` that takes a wheel as the first parameter and a number as the second. The function should return the wheel list rotated by the number passed in. For example, `(rotate-wheel-by (list 1 0 0 1 0) 2)` should evaluate to `(0 1 0 1 0)` and `(rotate-wheel-by` *`wheel`* `5)` should evaluate to *`wheel`*.

Question 5:
a. Define a function `rotate-wheel-list` that takes a list of wheels (like `K-wheels`) as a parameter and evaluates to a list of wheels where each of the wheels in the parameter list of wheels has rotated once. For example, `(rotate-wheel-list K-wheels)` should evaluate to `((1 0 1 0 1) (1 0 0 1 0) (0 0 1 0 1) (1 1 0 1 1) (0 0 0 1 1))`.

b. Define a function `rotate-wheel-list-by` that takes a list of wheels and a number as parameters, and evaluates to a list where each of the wheels in the parameter list of wheels has rotated the number parameter times. For example, `(rotate-wheel-list-by K-wheels 5)` should evaluate to the same list as K-wheels.

c. Describe the complexity of your `rotate-wheel-list-by` procedure using Θ notation. Be sure to explain what all variables you use mean.

Now that we can rotate our wheels, let's figure out how to implement Lorenz encryption using our *K* and *S* wheels. Since both sets of wheels are doing the same thing, we should be able to write one procedure that will work with either the *K* wheels or the *S* wheels.

Question 6: Define a function `wheel-encrypt` that takes a Baudot-encoded letter (a list of 5 bits) and a list of wheels as parameters. The function should xor each bit of the Baudot list with the first value of its respective wheel and return the resulting list. For example, `(wheel-encrypt (list 0 0 0 1 1) K-wheels)` should produce `(1 0 1 0 0)`. Your `wheel-encrypt` procedure should be $\Theta(n)$ where n is the length of the list passed as the first parameter to `wheel-encrypt`. Argue convincingly why your `wheel-encrypt` procedure is $\Theta(n)$.

We now have all the procedures we need to implement our simplified Lorenz machine. A quick review of how the machine should work:

- Each of the five bits of the Baudot representation of a letter is xored with the current position of the respective *K* wheels.
- The resulting five bits from the previous step are xored with the current position of the respective *S* wheels to produce the ciphertext.
- The *K* wheels are rotated one position.
- The *S* wheels are rotated one position only if the current position of the *M* wheel is 1.
- The *M* wheel is rotated one position.
- The process is repeated for each letter in the message.

Question 7:
a. Define a function called `do-lorenz` that takes a list of Baudot values, the *K* wheels, *S* wheels and *M* wheel. The function should encrypt the first Baudot code with the *K* and *S* wheels, then recursively encrypt the rest of the Baudot codes with the wheels rotated. The function should return the encrypted values in the form of a list of Baudot values.

b. Define a function `lorenz` that takes four parameters: a string and three integers. The integers represent the starting positions of the three wheels, respectively. The function should call `do-lorenz` with the string converted to Baudot and the wheels rotated to the correct starting positions. The function should return the ciphertext in the form of a string.

You should now be able to encrypt strings using the simplified Lorenz cipher. To test it, call your `lorenz` function with a string and offsets of your choice to produce ciphertext. Since our encryption and decryption functions are the same, if you evaluate `lorenz` again using the ciphertext and the same offsets you should get your original message back.

Cracking the Code

The first Lorenz-encrypted messages were intercepted by the British in early 1940. The intercepts were sent to Bletchley Park, the highly secret British base set up specifically to break enemy codes. The code-breakers at Bletchley Park had little success with the Lorenz Cipher until the Germans made a major mistake in late 1941. A German operator had nearly finished sending a long message using a Lorenz machine when the receiver radioed back to tell him that the message had not been received correctly. The operator then reset his machine back to the same starting position and began sending the message again. But the operator, probably frustrated at having to resend the message, abbreviated some of the words he had typed out completely the first time. This led to two nearly identical messages encrypted using the same starting positions.

The messages were sent to John Tiltman at Bletchley Park. Tiltman was able to discern both messages and determine the generated key. The messages were then passed on to Bill Tutte who, after two months of work, figured out the complete structure of the Lorenz machine only from knowing the key it generated. The British were then able to break the Lorenz codes, but much of the work needed to be done by hand, which took a number of weeks to complete. By the time the messages were decrypted they were mostly useless.

The problem was given to Tommy Flowers, an electronics engineer from the Royal Post Office. Flowers designed and built a device called Colossus that worked primarily with electronic valves. The Colossus was the first electronic programmable computer. It was able to decrypt the Lorenz messages in a matter of hours, a huge improvement from the previous methods. The British built ten more Colossi and were able to decrypt an enormous amount of messages sent between Hitler and his high commanders. The British kept Colossus secret until the 1970s. After the war, eight of the Colossi were quickly destroyed and the remaining two were destroyed in 1960 and all drawings were burnt. The details of the breaking of the Lorenz Cipher were kept secret until 2000, but are now available at
http://www.codesandciphers.org.uk/documents/newman/newmix.htm.

Our simplified cipher will be much easier to break. Since there are only 5 starting positions for the *K* wheels, 5 for the *S* wheels, and 5 for the *M* wheel, there are only 125 different keys. This is such a small number that we can simply try every possibility and look at the results to find the original message. Breaking a cipher by trying all possible key values is called a *brute-force attack*.

Question 8: Define a procedure that takes a ciphertext and evaluates `lorenz` on the ciphertext for all 125 possible starting positions. It may be helpful to use the function `printf` that will print out a string to the interactions window.

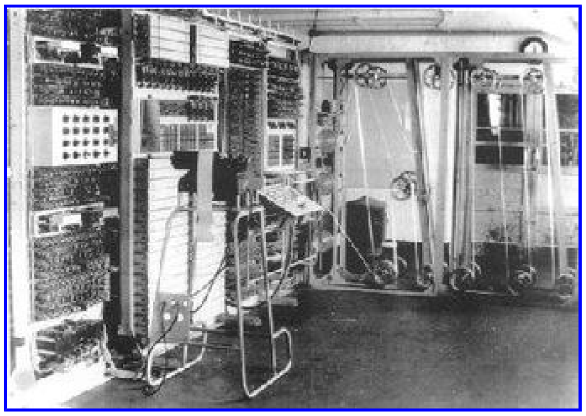
To test your procedure, try decrypting the ciphertext defined as `ciphertext` in *ps4.scm*. If your procedure works correctly, one of the messages generated will look like sensible English.

Question 9:

a. The actual Lorenz cipher operated similarly to the one in this problem set except instead of only having 5 positions in each wheel, each wheel had many more positions (up to 61). Suppose *n* is the number of possible positions for each Lorenz cipher wheel, and the procedure used to cryptanalyze the cipher is the same as your code (except extended as necessary to handle the extra wheel positions). Describe how the amount of work needed to break a Lorenz-encrypted message grows as a function of the number of wheel positions, *n*, using Θ notation. Assume the message length and number of wheels are fixed.

b. Suppose instead that it was possible to add *S* and *K* wheels to the Lorenz cipher, and *w* represents the number of *S* and *K* wheels (for example, for the cipher machine in the problem set, *w* is 5). Describe how the amount of work needed to break a Lorenz-encrypted message grows as a function of *w*, using Θ notation. Assume the message length and number of wheel positions are fixed.

c. If the Nazis had learned of Bletchley Park's success in breaking the Lorenz cipher during the war, what changes that could be done without building and redistributing completly new cipher machines, would be most likely to prevent further successful cryptanalysis?


Colossus (Original, 1943)

Colossus (Rebuilt, 2004)

Avail yourself of these means to communicate to us at seasonable intervals
a copy of your journal, notes & observations of every kind,
putting into cipher whatever might do injury if betrayed.

Thomas Jefferson's instructions to Captain Lewis for the Expedition to the Pacific.
The complete text is available from
http://www.th-jefferson.org/html/instructions.html.

It is regretted that it is not possible to give an adequate idea of the fascination of a Colossus at work: its sheer bulk and apparent complexity; the fantastic speed of thin paper tape round the glittering pulleys; the childish pleasure of not-not, span, print main heading and other gadgets; the wizardry of purely mechanical decoding letter by letter (one novice thought she was being hoaxed); the uncanny action of the typewriter in printing the correct scores without and beyond human aid; the stepping of display; periods of eager expectation culminating in the sudden appearance of the longed-for score; the strange rhythms characterizing every type of run; the stately break-in, the erratic short run, the regularity of wheel-breaking, the stolid rectangle interrupted by the wild leaps of the carriage-return, the frantic chatter of a motor run, the ludicrous frenzy of hosts of bogus scores.

D. Mitchie, J. Good, G. Timms. *General Report on Tunny*, 1945. (Released to the Public Record Office in 2000). *http://www.alanturing.net/tunny_report*