Lecture 18: Important Undecidable Problems
(and what to do about them)

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Menu

- What does undecidability mean for problems real people (not just CS theorists) care about?
- What do you do when your application's requirements require "solving" an undecidable problem?

Undecidability + Rice + Church-Turing

Language and Problems: any problem can be restated are a language recognition problem

Undecidability: undecidable languages that cannot be decided by ***any*** Turing Machine

Rice's Theorem: ***all*** nontrivial properties about the language of a TM are undecidable

Church-Turing Thesis: ***any*** mechanical computation can be done by some TM

Conclusion: any nontrivial property about general mechanical computations cannot be decided!

Program Halting Problem

- **Input:** a program P in some programming language
- **Output: true** if P terminates; **false** if P runs forever.

Examples

halts("2+2") **True**

```
halts("def f(n):
        if n==0: return 1
        else: return n * f(n-1)
     f(10)")
```
True

```
halts("def f(n):
        if n==0: return 1
        else: return n * f(n-1)
     f(10.5)")
```
False

Tougher Example

```
halts("
  def isPerfectNumber(n): # n is perfect if factors sum to n
      divs = findDivisors(n)
      return n == sum(divs)
  i = 3
  while not isPerfectNumber (i): i = i + 2
  print i")
```

Unknown

Note: it is unknown where an odd perfect number exists. (Numbers up to 10^{300} have been tried without finding one yet.)

If you had **halts**, you could prove or disprove nearly every open mathematical problem!

- Does an odd perfect number exist?
- Reimann hypothesis: The real part of any non-trivial zero of the Riemann zeta function is ½.
- Goldbach conjecture: Every number > 2 is the sum of three primes (including 1).
- Poincaré conjecture: Every simply connected closed three-manifold is homeomorphic to the three-sphere.
- ...

This suggests it is unlikely halts exists...but doesn't prove it (yet).

More Convincing Non-Existence Proof

```
def paradox():
  if halts("paradox()"):
     while True: print "You lose"
  else:
     return "You lose"
```

If **halts**("paradox()") is **True**: paradox() loops forever
If **halts**("paradox()") is **False**: paradox() halts

Neither option makes sense, so halts must not exist!

Recall from Lecture 16...

Assume there exists some TM H that **decides** A_{TM}.

Define D ($\langle M \rangle$) = Construct a TM that:
Outputs the **opposite** of the result of simulating H on input $\langle M, \langle M \rangle\rangle$

If D **accepts** $\langle D \rangle$:
$H(D, \langle D \rangle)$ **accepts** and $D(\langle D \rangle)$ **rejects**
If D **rejects** $\langle D \rangle$,
$H(D, \langle D \rangle)$ **rejects** and $D(\langle D \rangle)$ **accepts**

Whatever D does, it must do the opposite, so there is a contraction!

Alternate Proof: Reduction

- A Python procedure that solves halts must not exist, since if it did we could:
  - Write a TM simulator in Python:

    ```
    def simulateTM(M,w):
       # simulates M on input w
    ```

  - Determine if a TM M halts on w using halts:

    halts("simulateTM(M,w)")
- But, we know $HALT_{TM}$ is undecidable. Hence, **halts** for Python must not exist.

Does this work for Java?

Universal Programming Language

- Definition: a programming language that can describe every algorithm.
- Equivalently: a programming language that can simulate every Turing Machine.
- Equivalently: a programming language in which you can implement a Universal Turing Machine.

Which of these are Universal Programming Languages?

BASIC
C++
Python
Fortran
x86
Java
SQL
COBOL
HTML
PLAN
Scheme
C#
PostScript
JavaScript
Ruby

Proofs

- BASIC, C, C++, C#, Fortran, Java, JavaScript, PostScript, Python, Ruby, Scheme, etc.:
  - Proof: implement a TM simulator in the PL
- HTML is not universal:
  - Proof: show some algorithm that cannot be implemented in HTML
  - Easy choice: an infinite loop
- PLAN (Packet Language for Active Networks):
  - Designed to be non-universal: resource-constrained language

Why is it *impossible* for a programming language to be both ***universal*** and ***resource-constrained***?

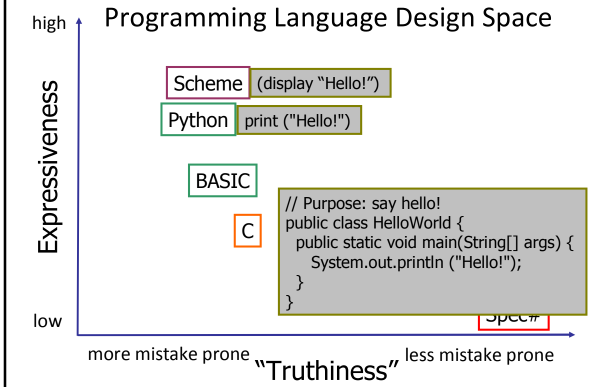
Resource-constrained means it is possible to determine an upper bound on the resources any program in the language can consume.

All universal programming language are ***equivalent*** in power: they can all simulate a TM, which can carry out any mechanical algorithm.

Why so many programming languages?

Proliferation of Universal PLs

- "Aesthetics"
  - Some people like :=, others prefer =.
  - Some people think whitespace shouldn't matter (e.g., Java), others think programs should be formatted like they mean (e.g., Python)
  - Some people like **goto**, others like **throw**.
- Expressiveness vs. "Truthiness"
  - How much you can say with a little code vs. how likely it is your code means what you think it does



Are any **Important** Problems Undecidable?

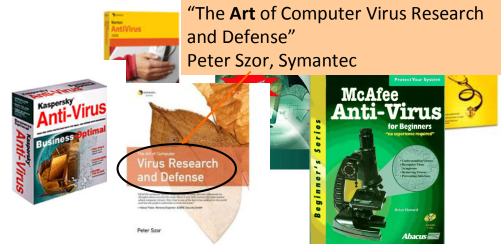
Virus Detection

- **Input:** a program *P*
- **Output: True** if executing *P* would cause a file on the host computer to be "infected"; **False** otherwise.

Rough Proof:
def halts(P):
  return isVirus(**"removePossibleInfections(P) infectFile()"**)

Therefore: Anti-Virus programs cannot exist!

"The **Art** of Computer Virus Research and Defense"
Peter Szor, Symantec



Vulnerability Detection

- **Input**: a program *P*
- **Output: True** if there is some input *w*, such that running *P* on *w* leads to a security compromise; **False** otherwise.

Rough Proof:
def haltsOnInput(P,w): # we know this doesn't exist either
  return isVulnerable(**"P(w) compromiseSecurity()"**)

Example: Morris Internet Worm (1988)

- *P* = fingerd
  - Program used to query user status (running on most Unix servers)
- isVulnerable(*P*)?

  Yes, for *w* = "nop^{400} pushl \$68732f pushl \$6e69622f movl sp,r10 pushl \$0 pushl \$0 pushl r10 pushl \$3 movl sp,ap chmk \$3b"
  - Worm infected several thousand computers (~10% of Internet in 1988)

Impossibility of Vulnerability Detection

"Solving" Undecidable Problems

- Undecidable means there is no program that
  1. **Always** gives the correct answer, and
  2. **Always** terminates
- Must give up one of these:
  - Giving up #2 is not acceptable in most cases
  - Must give up #1: none of the anti-virus or vulnerability detection products are always correct
- *Or* change the problem
  - e.g., detect file infections during an execution, make resilient execution environments, etc.

Actual isVirus Programs

- Sometimes give the wrong answer:
  - "False positive": say P is a virus when it isn't
  - "False negative": say P is safe when it is
- Database of known viruses: if P matches one of these, it is a virus
- Clever virus authors can make viruses that change each time they propagate
  - It is undecidable to determine if a given program is the same as a known virus
  - Emulate program for a limited number of steps; if it doesn't do anything bad, assume it is safe

Recap

- If you can simulate a Turing Machine with programming language PL, it is a *universal programming language*
- There is no algorithm for deciding halts for $P \in$ PL: if there was, we could decide A_{TM}.
- There is no way to determine in general if $P \in$ PL is a virus, or a program containing vulnerabilities, or any interesting property…
- We can build algorithms that get it right some of the time (and this can be valuable).

PS5 is due Tuesday