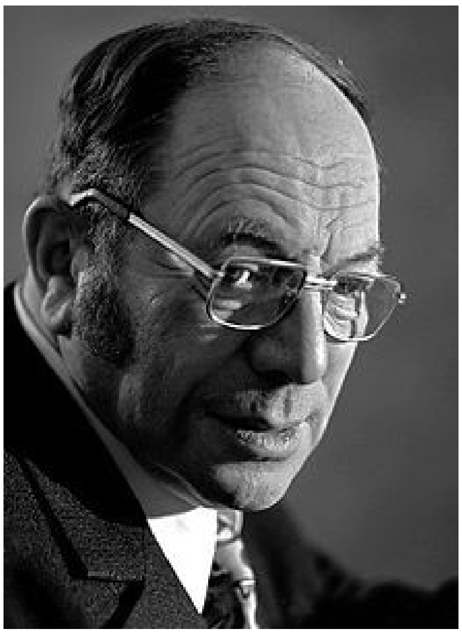
CS6501: Great Works in Computer Science 11 April 2013
Wei Wang

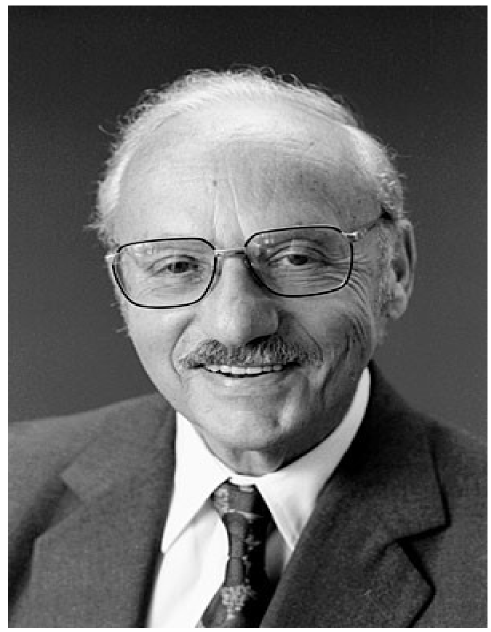
The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints

George B. Dantzig, Alex Orden, Philip Wolfe

1953



Leonid Kantorovich

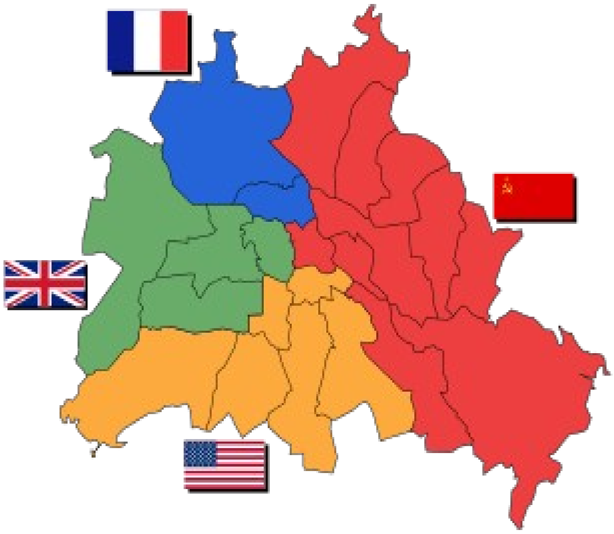


George B. Dantzig

"I want to emphasize again that the greater part of the problems of which I shall speak, relating to the organization and planning of production, are connected specifically with the Soviet system of economy and in the majority of cases do not arise in the economy of a capitalist society."

– Leonid Kantorovich

1. The Berlin Blockade



Road and water accesses to western Berlin were blocked on 18 June 1948.

Supplies required:
646 tons of flour and wheat, 125 tons of cereal, 64 tons of fat, 109 tons of meat and fish, 180 tons of dehydrated potatoes, 180 tons of sugar, 11 tons of coffee, 19 tons of powdered milk, 5 tons of whole milk for children, 3 tons of fresh yeast for baking, 144 tons of dehydrated vegetables, 38 tons of salt and 10 tons of cheese, 3,475 tons of coal and gasoline. Total 5000 tons per day.

Berlin Airlift:
225 airplanes
277,804 flights
2,325,809 tons of food and supplies
Planes departing every three minutes
On 21 April 1949 the tonnage of supplies flown into the city exceeded that previously brought by rail.

Blockade was lifted on 12 May 1949.

Linear programming was used to optimize the transportation. One of the first linear programming problems solved by hand using the simplex method.

2. The linear programming problems

(1) Variables:

Three types of planes;
Number of planes of type 1 for food X_{1f} , for coal X_{1c}
Number of planes of type 2 for food X_{2f} , for coal X_{2c}
Number of planes of type 3 for food X_{3f} , for coal X_{3c}

(2) Constraints:

More than 1500 tons of food $100\times X_{1f}+200\times X_{2f}+150\times X_{3f}\geq 1500$
More than 3500 tons of coal $100\times X_{1c}+200\times X_{2c}+150\times X_{3c}\geq 3500$
No more than 10 planes of type 1 $X_{1f}+X_{1c}\leq 10$
No more than 22 planes of type 2 $X_{2f}+X_{2c}\leq 22$
No more than 10 planes of type 3 $X_{3f}+X_{3c}\leq 10$
All planes numbers has to be greater than or equal to 0 $X_{1f}, X_{2f}, X_{3f}, X_{1c}, X_{2c}, X_{3c}\geq 0$

(3) Object function (cost to minimize):

$$1000\times X_{1f}+1000\times X_{1c}+2000\times X_{2f}+2000\times X_{2c}+1200\times X_{3f}+1200\times X_{3c}$$

(4) In summary:

- A set of variables, must be greater than or equal to 0
- A set of linear inequality constrains
- An object function to optimize (maximize or minimize)

3. Solving the linear programming problem and simplex algorithm

(1) Basic idea:

- Optimal solutions are from the corner points of the solution space. Therefore, we just go over each corner point and find the optimal one.
- Simplex algorithm starts with one corner point, and the follows the edges to reach other corner points.

(2) Standard form:

- Cost function for maximization:
  $3x_1+x_2+2x_3$
- Constrains are always less than or equal to a constant:
  $x_1+x_2+3x_3\leq 30$
  $2x_1+2x_2+5x_3\leq 24$
  $4x_1+x_2+2x_3\leq 36$
- All variables are greater than or equal to 0:
  $x_{1,} x_{2,} x_3\geq 0$
- All linear programming problems can be converted to standard form

(3) Slack form:

- Cost function for maximization:

$z=3x_1+x_2+2x_3$

- Constrains are always less than or equal to a constant:
  $x_4=30-x_1+x_2+3x_3$
  $x_5=24-2x_1+2x_2+5x_3$
  $x_6=36-4x_1+x_2+2x_3$
- All variables are greater than or equal to 0:
  $x_{1,}x_{2,}x_{3,}x_{4,}x_{5,}x_6\geq 0$
- Equivalent to standard form

(4) Goal of simplex algorithm:
Switching the variables on both sides of the constrains, and transform the cost function under the following form is achieved:
$z=v-c_i x_i-c_j x_j-c_k x_k$

(5) Pivot operation:
First switch x_1 and x_6 using the last constraint:

$$x_1=9-\frac{x_2}{4}-\frac{x_3}{2}-\frac{x_6}{4}$$

Put in this new equation and replacing all x_1 's in other equations, we have a new slack form,

$$z=27+\frac{x_2}{4}+\frac{x_3}{2}-\frac{3x_6}{4}$$
$$x_1=9-\frac{x_2}{4}-\frac{x_3}{2}-\frac{x_6}{4}$$
$$x_4=21-\frac{3x_2}{4}-\frac{5x_3}{2}+\frac{x_6}{4}$$
$$x_5=6-\frac{3x_2}{2}-4x_3+\frac{x_6}{2}$$

(6) Another pivot:
Switch x_3 and x_5 using the last constraint, we have a new slack form:

$$z=\frac{111}{4}+\frac{x_2}{16}-\frac{x_5}{8}-\frac{11x_6}{16}$$
$$x_1=\frac{33}{4}-\frac{x_2}{16}+\frac{x_5}{8}-\frac{5x_6}{16}$$
$$x_3=\frac{3}{2}-\frac{3x_2}{8}-\frac{x_5}{4}+\frac{x_6}{8}$$
$$x_4=\frac{69}{4}+\frac{3x_2}{16}+\frac{5x_5}{8}-\frac{x_6}{16}$$

(7) Last piovt:
Switch x_2 and x_3 using the last constraint, we have the last slack form:

$$z=28-\frac{x_3}{6}-\frac{x_5}{6}-\frac{2x_6}{3}$$
$$x_1=8+\frac{x_3}{6}+\frac{x_5}{6}-\frac{x_6}{3}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$

Clearly, we have reach the optimal, which is 28 with values of the variables as (8,4,0,18,0,0).

4. Analysis and other issues of simplex algorithm and linear programming:

(1) Result found by simplex is turly optimal:
Duality by John Von Neumann

(2) Terminations:

- May not terminate, can loop among slack forms.
- To avoid loop, follow Bland's rule:
  - Choose the lowest-numbered (i.e., leftmost) variable with a positive cost.
  - Choose the constraints that mostly limits the value of the leaving variable; if there are multiple constraints with the same limits, choose the one that lowest-numbered.

(3) Complexity:

- There are *n* original variable; *m* constraints yields *m* added variables
- Each slack form is basically choosing *m* variables to put in the left side of the equations. Therefore, there are $\binom{m+n}{m}$ ways to choose these variables.
- Given a fixed set of *m* variables on the left side, there is only one slack form.
- Therefore, there are at most $\binom{m+n}{m}$ pivot operations.
- Simplex algorithm is no polynomial time. However, linear programming has been shown to have strong polynomial time solution.
- In practice simplex algorithm is very fast for most linear programming problems.

(4) Finding the correct starting point:

- Because simplex follows the simplex algorithm follows the edges to find corner points, if the start point is not on the edge, simplex algorithm will break.
- 0-point is not always feasible.
  Maximize $2x_1 - x2$
  Subject to $2x_1 - x_2 \le 2$
  $x_1 - 5x_2 \le -4$
  $x_{1,} x_2 \ge 0$
- Auxiliary problem: Adding a new variable
  Maximize $-x_0$
  Subject to $2x_1 - x_2 - x_0 \le 2$
  $x_1 - 5x_2 - x_0 \le -4$

$x_{0,} x_{1,} x_2 \geq 0$

- Slack form of the auxiliary problem:
  $z = -x_0$
  $x_3 = 2 - 2x_1 - x_2 + x_0$
  $x_4 = -4 - x_1 + 5x_2 + x_0$
- Transform the slack from by switching x_0 and x_4 :
  $z = -4 - x_1 + 5x_2 - x_4$
  $x_0 = 4 + x_1 - 5x_2 + x_4$
  $x_3 = 6 - x_1 - 4x_2 + x_4$
- Solve this problem with simplex:
  $z = -x_0$
  $x_2 = \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5}$
  $x_3 = \frac{14}{5} + \frac{5x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5}$
  Max is 0
- Replace x_0 with 0, and put x_2 back into the original problem:
  $z = -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5}$
  $x_2 = \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5}$
  $x_3 = \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5}$
  And we have a feasible starting point to work on.

(5) Feasibility:
If the auxiliary problem maximum value is 0, then the original problem is feasible.
The original is first checked with its auxiliary problem, if the problem is feasible, it is then computed using simplex.

(6) Unbounded:
If a right-sided variable has only positive coefficients, then the problem is unbounded.
Maximize $z = 88 + 5x_1 - 7x_2 - 3x_3$
Subject to $x_4 = 8 + 7x_1 - 11x_2 - 2x_3$
$x_5 = 13 + 2x_1 - x_2 - 5x_3$
$x_{1,} x_{2,} x_{3,} x_{4,} x_5 \geq 0$

5. Translating other problems in to linear programming:

(1) Shortest path:
Given two vertices *s* and *t*, find the shortest path from *s* to *t*.
Maximize d_t
Subject to $d_v < d_u + w(u, v)$ for each edge $(u, v) \in E$
$d_s = 0$

6. Integer linear programming problem:

   (1) Min weight vertex cover:
   Given a set of vertices *V* and edges E and vertex weights W, find a set S with minimal weight, where S is a subset of V, and for every edge, at lest one of its vertices is in S.

   Minimize $\sum_{v \in V} w_v y_v$

   Subject to $y_u + y_v \geq 1 \; \forall e = (u, v) \in E$

   $y_v, y_u = 0, 1$

   (2) Min weight vertex cover is know to be NP-complete, and integer linear programming is NP-hard.

   (3) One can still solve in using Simplex in real space, and round the results up. For this particular problem, a rounded solution is smaller than 2 times of the real optimal solution.

   (4) Rounding may be not optimal, and may not even satisfy the constraints.

7. Impact of linear programming:

   (1) A handy algorithm for solving optimization problems. Many problems can be reduced into a linear programming problem, and be solved with simplex. There has been a flurry of work in 80s trying to approximate optimal solutions for NP-complete problems.

   (2) Widely used in industries and businesses for optimal resource allocations.

   - The only algorithm taught in business school.
   - Delta claims $100 million $ saving per year using LP.
   - A Nobel prize was awarded to Harry Markowitz for using LP to optimize portfolio profit.
   - A list of applications:
     - Agriculture. Diet problem.
     - Computer science. Compiler register allocation, data mining.
     - Electrical engineering. VLSI design, optimal clocking.
     - Energy. Blending petroleum products.
     - Economics. Equilibrium theory, two-person zero-sum games.
     - Environment. Water quality management.
     - Finance. Portfolio optimization.
     - Logistics. Supply-chain management.
     - Management. Hotel yield management.
     - Marketing. Direct mail advertising.
     - Manufacturing. Production line balancing, cutting stock.

- Medicine. Radioactive seed placement in cancer treatment.
- Operations research. Airline crew assignment, vehicle routing.
- Physics. Ground states of 3-D Ising spin glasses.
- Plasma physics. Optimal stellarator design.
- Telecommunication. Network design, Internet routing.