Practice 06

This lab refers to the following two definitions. The first we've seen before:

Definition 1 (Kleene Star) *Given any set A, the set A^* contains the empty sequence () and every sequence than can be created by appending an element of A to a different element of A^*.*

The second is often left undefined, relying on an intuitive sense of what it means to be finite. However, having a few rules might help:

Definition 2 (Finite) *Any integer you can write (and finish writing) in base-10 is finite. The sum or product of two finite numbers is finite. Any number whose absolute value is smaller than some finite number is finite.*

problem 1 *Proof by Induction*

Complete the following proof by induction of the theorem "every string in $\{\texttt{a}\}^*$ has finite length."

Proof. We use induction. The induction hypothesis is that a string has finite length.

Base case: λ ______________ has finite length, because

its length is 0, and 0 is finite.

Inductive step: Assume that a string s ______ has finite length. Then string of "a"s one longer than s ______ must have finite length because

it contains one more a than did s. Adding 1 (a finite number) to the length keeps the length finite.

(Note we could also have defined t as "the only element of $\{\texttt{a}\}^{|s|}$" or as "$(s,\texttt{a})$".)

It follows by induction that every strings in $\{\texttt{a}\}^*$ has finite length. □

Want more practice? Try proving that every element of $\{\texttt{a}, \texttt{b}\}^*$ is finite; that for every $x \in \mathbb{N}$ the sum of the elements in $\{y \mid y \in \mathbb{N} \wedge y < x\}$ is finite; that for every $x \in \mathbb{N}$ the sum of the elements in $\{y \mid y \in \mathbb{N} \wedge y < x\}$ is no greater than x^2; and all the practice problems in §5.1–5.3 (i.e., 5.1 through 5.29).

PROBLEM 2 *Proof by Contradiction*

Complete the following proof by contradiction of the theorem "there is no longest string in $\{\texttt{a}\}^*$."

Proof. Assume there was a longest string, $s \in \{\texttt{a}\}^*$, where $\nexists q \in \{\texttt{a}\}^* \,.\, q$ is longer than s.
Consider the sequence t defined as

a sequence of "a"s one "a" longer than s

By construction, t is longer than s.
We know that $t \in \{\texttt{a}\}^*$ because

it is constructed by appending an element of $\{\texttt{a}\}$ to s and s is a member of $\{\texttt{a}\}^*$.

But that contradicts $\nexists q \in \{\texttt{a}\}^* \,.\, q$ is longer than s.
Because

assuming there was a longest string led to a contradiction,

there is no longest string in $\{\texttt{a}\}^*$. □

Want more practice? Try proving that there is no longest string of $\{\texttt{a}, \texttt{b}\}^*$; that 2 is the only even prime number; that a subset cannot be larger than its superset; and most practice problems in §2 (i.e., 2.1 through 2.13) and problem 4.25 and 4.26.