Practice 08

Theorem 1 $\forall n \in \mathbb{N} \, . \, \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$

PROBLEM 1 *Proof by Induction*

Prove the above theorem using induction.
Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{i=0}^{0} i = 0$ and $\frac{(0)(1)}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. Adding $n+1$ to both sides, we have $n + 1 + \sum_{i=0}^{n} i = n + 1 + \frac{(n)(n+1)}{2}$; the left-had side is equivalent to $\sum_{i=0}^{n+1} i$ by the definition of summation; the right-hand side can be rearranged using algebra to get $\frac{2(n+1)+(n)(n+1)}{2} = \frac{(2+n)(n+1)}{2} = \frac{(n+1)((n+1)+1)}{2}$; this means that $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$, or in other words that the theorem holds for $n+1$.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$.

□

PROBLEM 2 *Proof by Contradiction*

Prove the above theorem using contradiction and the well-ordering principle.
Proof.

We proceed by contradiction, with case analysis.

Assume that the theorem is false; that is, $\exists n \in \mathbb{N} \,.\, \sum_{i=0}^{n} i \neq \frac{(n)(n+1)}{2}$. By the well-ordering principle there must be a smallest such n; call that smallest n where the theorem is false n_0.
Because $n_0 \in \mathbb{N}$, either $n_0 = 0$ or $n_0 > 0$. We consider each case separately.

Case $n_0 = 0$ This means that $\sum_{i=0}^{0} i \neq \frac{(n)(n+1)}{2}$. But both sides are equal to 0, so they must be equal to each other, which contradicts our assumption.

Case $n_0 > 0$ This means that $\sum_{i=0}^{n_0} i \neq \frac{(n_0)(n_0+1)}{2}$. Subtracting n_0 from both sides, we get $\sum_{i=0}^{n_0-1} i \neq \frac{(n_0)(n_0+1)}{2} - n_0$ the right-hand side of which simplifies as $\frac{(n_0)(n_0+1) - 2n_0}{2} = \frac{(n_0)(n_0-1)}{2}$. But that means the theorem is also false for $n_0 - 1$, which contradicts our assumption that n_0 was the smallest such n.

Because both cases led to a contradiction, the assumption always leads to a contradiction, which means the assumption must be false. Hence, the theorem must be true for all $n \in \mathbb{N}$.

□

You might consider grading your own work on the following rubric:

Inductive Proof

- ☐ Identifies induction as proof structure
- ☐ Labels base case and inductive step
- ☐ Base case is smallest allowable n
- ☐ Base case is shown to hold via algebra
- ☐ Inductive case assumes theorem holds for n and considers $n+1$
- ☐ Inductive case reduces $n+1$ to n via algebra
- ☐ Proof ends by stating some form of "by induction, holds for all n"

Proof by Contradiction

- ☐ Identifies proof by contradiction as proof structure
- ☐ Assumes the theorem is false
- ☐ Either assumes it is false for some n, or recognizes that $\neg\forall \equiv \exists\neg$
- ☐ Uses well-ordering principle (considers smallest such n)
- Shows that n can't be the smallest such n because
    - ☐ Showing that true for n implies true for $n-1$
    - ☐ Either showing that there is always an $n-1$, or that the ns that do not have an $n-1$ also meet the theorem
- ☐ State explicitly that assuming not-theorem led to contradiction (noting it did so in all cases if case analysis used)
- ☐ Proof ends with some form of "by contradiction, theorem true"

You might also try doing the same two proof types with other summation formulae, such as

$$\begin{aligned}
\sum_{i=0}^{n} i^2 &= \frac{(n+1)(2n+1)(n)}{6} \\
\sum_{i=1}^{n+1} i^2 &= \frac{(n+2)(2n+3)(n+1)}{6} \\
\sum_{i=2}^{n+2} i^2 &= \frac{(n+3)(2n+5)(n+2)}{6} \\
6\sum_{i=0}^{n} i^3 - i &= \binom{n+2}{4} \\
\sum_{x=0}^{n} \frac{x^2-1}{x+1} &= \frac{(n+1)(n-1)}{2} \\
\sum_{x=0}^{n} x^3 - x^2 &= \frac{(n+1)(3n+2)(n)(n-1)}{12} \\
\sum_{i=0}^{n} 3i^2 + 2i &= \frac{(2n+3)(n+1)(n)}{2} \\
\sum_{i=n}^{\infty} \frac{1}{2^i} &= \frac{2}{2^n}
\end{aligned}$$

Note: at least one of the above formulae is false. In the process of proving it you should find the normal methods not working, revealing the non-truth.