Lecture 7: Greedy Algorithms

http://www.cs.virginia.edu/cs216

Menu

La vita e incerta - mangia il dolce per primo.

"Life is uncertain. Eat dessert first."
Ernestine Ulmer

Greed is Good?

- Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations* (1776)
  "It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interests."
- Invisible hand: individuals acting on personal greed produces (nearly) globally optimal results

Greedy Algorithms

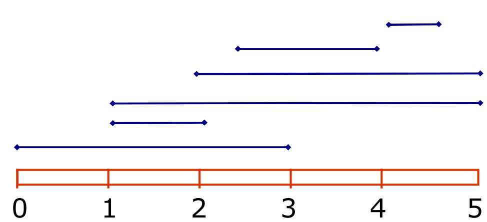
- Make the locally best choice "myopically" at each step
  - Need to figure out what "dessert" is
- Hope it leads to a globally good solution
  - Sometimes, can prove it leads to an optimal solution
  - Other times (like phylogeny), non-optimal, but usually okay if you get lucky

Interval Scheduling Problem

- Input: R, a set of n resource requests:
  $\{<s_0, f_0>, <s_1, f_1>, \ldots, <s_{n-1}, f_{n-1}>\}$
- Output: a subset S of R with no overlapping requests ($s_i > s_j < f_j$ for any $<s_i, f_i>, <s_j, f_j> \in S$) such that
  $|S| \geq |T|$ for any $T \subseteq R$ with no overlapping requests

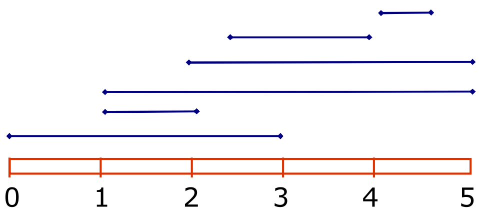
Example

R = {<0, 3>, <1, 2>, <1, 5>, <2, 5>, <2.5, 4>, <4, 4.5> }



Solution

$R = \{<0, 3>, <1, 2>, <1, 5>, <2, 5>, <2.5, 4>, <4, 4.5>\}$



Brute Force Algorithm

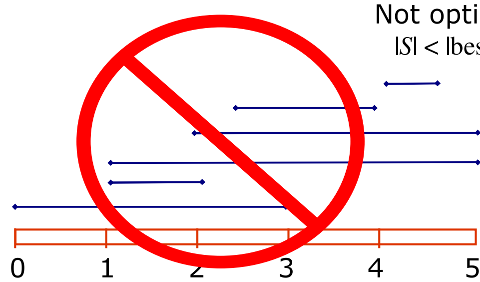
- Try all possible subsets
  - Filter out ones with overlapping intervals
  - Pick the largest subset
- Running time
  - How many subsets? 2^n
  - Constant work for each subset
  - $\Theta(2^n)$

Greedy Approaches

- Need to pick best subset by making myopic decisions, one element at a time
- Many possible criteria for making myopic decision
  - Earliest starting time?
  - Latest ending time?
  - Shortest?

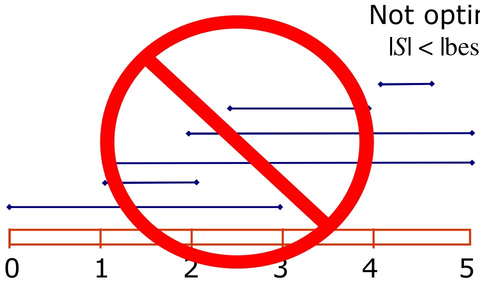
Greedy Approach: Earliest Starting

Not optimal:
$|S| < |best| = 3$



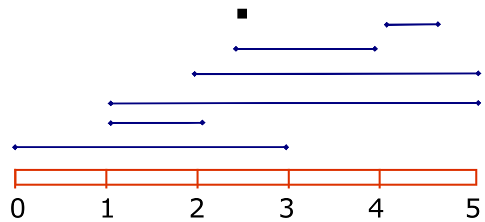
Greedy Approach: Earliest Finishing

Not optimal:
$|S| < |best| = 3$



Greedy Approach: Shortest Length

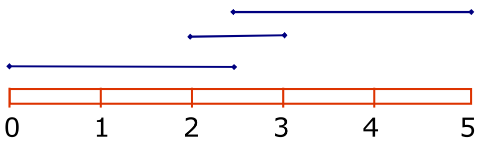
?



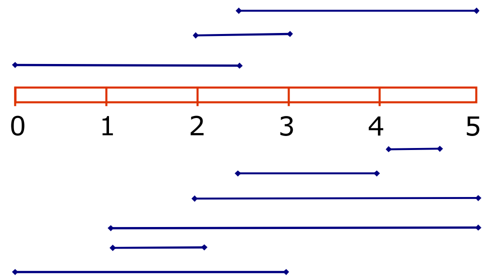
Greedy Approach: Shortest Length

R = {<0, 2.5>, <2, 3>, <2.5, 5>}

Not optimal:
$|S| < |\text{best}| = 2$



Greedy Approach: Pick Earliest Finishing Time



Greedy Algorithm: Running Time Analysis

- Straightforward implementation:
  - Search to find earliest finishing: $O(n)$
  - Eliminate matching elements: $O(n)$
  - Repeat (up to n times): $O(n^2)$
- Smarter implementation:
  - Sort by finishing time: $O(n \log n)$
  - Go through list, selecting if non-overlapped: $O(n)$
  - Running time $\in O(n \log n)$

Correctness?

- How to prove a greedy algorithm is ***non***optimal
  - Find a counterexample: some input where the greedy algorithm does not find the best solution
- How to prove a greedy algorithm is **optimal**
  - By induction: always best up to some size
  - By exchange argument: swapping any element in solution cannot improve result

Proof

- The greedy algorithm produces,

$$R = \{ r_0, \ldots, r_{k-1} \}$$

- Suppose there is a better subset,

$$Q = \{ q_0, \ldots, q_{k-1}, q_k \}$$

- Sort both by finishing time, so

$f_{r_i} < f_{r_j}$ for all $0 \le i < j < k$

$f_{q_i} < f_{q_j}$ for all $0 \le i < j < k+1$

Proof

$$R = \{ r_0, \ldots, r_{k-1} \}$$

$$Q = \{ q_0, \ldots, q_{k-1}, q_k \}$$

$f_{r_i} < f_{r_j}$ for all $0 \le i < j < k$

$f_{q_i} < f_{q_j}$ for all $0 \le i < j < k+1$

Strategy:

1. Prove by induction $f_{r_i} \le f_{q_j}$ for all $i < k$
2. Then, since $f_{r_{k-1}} \le f_{q_{k-1}}$ if q_k is valid, it would have also been added to R.

Induction Proof: $f_{ri} \le f_{qj}$

$R = \{ r_0, \ldots, r_{k\text{-}1} \}$ $Q = \{ q_0, \ldots, q_{k\text{-}1}, q_k \}$

- Basis: $f_{r0} \le f_{q0}$
  - Greedy algorithm choose r_0 as the element with the earliest finishing time
  - So, $f_{r0} \le f_j$ for all j
- Induction: $f_{ri\text{-}1} \le f_{qj\text{-}1} \Rightarrow f_{ri} \le f_{qj}$
  - Since $f_{r_{i\text{-}1}} \le f_{q_{j\text{-}1}}$ we know $s_{qi} \ge f_{r_{i\text{-}1}}$
  - So, greedy algorithm could choose q_i
  - If $f_{q_i} < f_{r_j}$, greedy algorithm would have chosen f_{qj} instead of f_{rj}

Knapsack Problems

- You have a collection of items, each has a value and weight
- How to optimally fill a knapsack with as many items as you can carry

Scheduling: weight = time, one deadline for all tasks
Budget allocation: weight = cost

General Knapsack Problem

- Input: a set of n items {<name_0, value_0, weight_0>, …, <$\text{name}_{n\text{-}1}$, $\text{value}_{n\text{-}1}$, $\text{weight}_{n\text{-}1}$>}, and maxweight
- Output: a subset of the input items such that the sum of the weights of all items in the output set is ≤ *maxweight* and there is no subset with weight sum ≤ *maxweight* with a greater value sum

Brute Force Knapsack

```
def knapsack (items, maxweight):
   best = {}
   bestvalue = 0
   for s in allPossibleSubsets (items):
      value = 0
      weight = 0
      for item in s:
         value += item.value
         weight += item.weight
      if weight <= maxweight:
         if value > bestvalue:
            best = s
            bestvalue = value
   return best
```

(Defining and analyzing this might be a good Exam 1 question)

Brute Force Knapsack Analysis

- How many subsets are there?

2^n

- How much work for each subset?

```
for item in s:
      value += item.value
      weight += item.weight
```

Average size of each subset is $n/2$ (there are as many subsets with size c and of size $n - c$)

Running time $\in \Theta(n2^n)$

Dynamic Programming

- Section this week: dynamic programming solution to the knapsack problem
- Running time in

$O(maxweight * n)$

Greedy Knapsack Algorithm

- Repeat until no more items fit:
  - Add the most valuable item that fits
- "Greedy": always picks the most valuable item that fits first

Greedy Knapsack Algorithm

```
def knapsack_greedy (items, maxweight):
  result = []
  weight = 0
  while True:
     # try to add the best item
     weightleft = maxweight - weight
     bestitem = None
     for item in items:
        if item.weight <= weightleft \
           and (bestitem == None \
                 or item.value > bestitem.value):
           bestitem = item
     if bestitem == None: break
     else:
        result.append (bestitem)
        weight += bestitem.weight
  return result
```

Running Time $\in \Theta(n^2)$

Is Greedy Algorithm Correct?

No.
Proof by counterexample:
Consider input
items = {<"gold", 100, 1 >,
<"platinum", 110, 3>
<"silver", 80, 2 >}
maxweight = 3
Greedy algorithm picks {<"platinum">} value = 110, but {<"gold">, "silver">} has weight <= 3 and value = 180

Same Weights Knapsack Problem

- Input: a set of n items $\{<\text{name}_0, \text{value}_0>, \ldots, <\text{name}_{n\text{-}1}, \text{value}_{n\text{-}1}>\}$, each having weight w, and maxweight
- Output: a subset of the input items such that the sum of the weights of all items in the output set is $\leq$ *maxweight* and there is no subset with weight sum $\leq$ *maxweight* with a greater value sum

Greedy Algorithm Correct

- It keeps adding items until maxweight would be exceeded, so the result contains k items where $kw <= maxweight$ and $(k+1)w > maxweight$
- Hence, cannot add any item (weight w) without removing another item
- But, any item with value > value of the lowest value item in result would have already been added by greedy algorithm

Subset Sum Problem

- Knapsack problem where $value = weight$
- Input: set of n positive integers, $\{w_0, \ldots, w_{n\text{-}1}\}$, maximum weight W
- Output: a subset S of the input set such that the sum of the elements of $S \leq W$ and there is no subset of the input set whose sum is greater than the sum of S and $\leq W$

Brute Force Subset Sum

```
def subsetsum (items, maxweight):
  best = {}
  for s in allPossibleSubsets (items):
     if sum (s) <= maxweight \
          and sum (s) > sum (best)
        best = s
 return best
```

Just like brute force knapsack:
Running time $\in \Theta(n2^n)$

Greedy Subset Sum?

- Pick largest item that fits
  - Bad: $I = \{ 4, 5, 7 \}$ $W = 9$
- Pick smallest item
  - Bad: $I = \{ 4, 5, 7 \}$ $W = 7$
- Doesn't prove there is no myopic criteria that works

Note: Subset Sum is known to be NP-Complete, so finding one would prove P = NP

Charge

- More greedy algorithm examples in Section this week
- PS3: greedy phylogeny algorithm
  - Not optimal (prove in Question 8)
  - Usually reasonably good (similar to algorithms used in practice)