Factorization Bandits for Interactive Recommendation

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In this supplementary document, we provide detailed proofs for Lemma 1 and Theorem 1 in our paper. We use the same notations as those in the paper.

Proof of Lemma 1

Consider the regularized quadratic loss specified in the paper. By taking the gradient with respect to both $\mathbf{\Theta}$ and $\mathbf{v}_a$ and applying our enhanced reward generation assumption, we have,

$$\begin{aligned}\mathbf{A}_t(vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)) = &\sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a_{t'}}, \hat{\mathring{\mathbf{V}}}_{a_{t'}})\mathbf{W}^\mathsf{T})\eta_{t'} \\ &+ \sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a_{t'}}, \hat{\mathring{\mathbf{V}}}_{a_{t'}})\mathbf{W}^\mathsf{T})\big(vec((0, \mathring{\mathbf{V}}^*_{a_{t'}} - \hat{\mathring{\mathbf{V}}}_{a_{t'}})\mathbf{W}^\mathsf{T})vec(\mathbf{\Theta}^{*\mathbf{v}})\big) \\ &- \lambda_1 vec(\mathbf{\Theta}^*)\end{aligned} \tag{1}$$

in which $\eta_{t'}$ is the Gaussian noise at trial t' in reward generation. Then we can derive the uncertainty on latent user factor estimation as follows,

$$\begin{aligned}&\|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t} \\ =&\|\sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a'_t}, \hat{\mathring{\mathbf{V}}}_{a'_t})\mathbf{W}^\mathsf{T})\big(vec((0, \mathring{\mathbf{V}}^*_{a'_t} - \hat{\mathring{\mathbf{V}}}_{a'_t})\mathbf{W}^\mathsf{T})vec(\mathbf{\Theta}^{*\mathbf{v}})\big) \\ &+ \sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a'_t}, \hat{\mathring{\mathbf{V}}}_{a'_t})\mathbf{W}^\mathsf{T})\eta_{t'} - \lambda_1 vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t^{-1}} \\ \leq& \sum_{t'=1}^{t}\|vec((\mathring{\mathbf{X}}_{a'_t}, \hat{\mathring{\mathbf{V}}}_{a_{t'}})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}}\|(\mathbf{v}^*_{a_{t'},u} - \hat{\mathbf{v}}_{a_{t'},u})\mathbf{w}_u^\mathsf{T}\|_2\|vec(\mathbf{\Theta}^{*\mathbf{v}})\|_2 \\ &+ \|\sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a_{t'}}, \hat{\mathring{\mathbf{V}}}_{a_{t'}})\mathbf{W}^\mathsf{T})\eta_{t'}\|_{\mathbf{A}_t^{-1}} + \|\lambda_1 vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t^{-1}} \\ \leq& \frac{LS}{\sqrt{\lambda_1}}\sum_{t'=1}^{t}\|\mathbf{v}^*_{a_{t'},u} - \hat{\mathbf{v}}_{a_{t'},u}\|_2 + \|\sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a_{t'}}, \hat{\mathring{\mathbf{V}}}_{a'_t})\mathbf{W}^\mathsf{T})\eta_{t'}\|_{\mathbf{A}_t^{-1}} \\ &+ \sqrt{\lambda_1}\|vec(\mathbf{\Theta}^*)\|\end{aligned} \tag{2}$$

in which the second term on the right-hand side is bounded by the property of self-normalized vector-valued martingales (Abbasi-yadkori, Pál, and Szepesvári 2011) as follows,

$$\|\sum_{t'=1}^{t} vec((\mathring{\mathbf{X}}_{a'_t}, \hat{\mathring{\mathbf{V}}}_{a'_t})\mathbf{W}^\mathsf{T})\eta_{t'}\|_{\mathbf{A}_t^{-1}} \leq \sqrt{\ln\big(\frac{det(\mathbf{A}_t)}{\delta\lambda_1}\big)} \tag{3}$$

because $\|(\mathbf{x}_a, \mathbf{v}_a)\|_2 \leq L$ and η_t only has a finite variance.

For the first term on the right-hand side in Eq (2), if the regularization parameter λ_1 is sufficiently large, the Hessian matrix of the loss function specified in the paper is positive definite at the optimizer based on the property of alternating least square (Uschmajew 2012). The estimation of $\mathbf{\Theta}$ and $\mathbf{v}_a$ is thus locally q-linearly convergent to the optimizer. This indicates that for every $\epsilon_1 > 0$, we have,

$$\|\hat{\mathbf{v}}_{a,t+1} - \mathbf{v}^*_a\|_2 \leq (q_1 + \epsilon_1)\|\hat{\mathbf{v}}_{a,t} - \mathbf{v}^*_a\|_2 \tag{4}$$

where $0 < q_1 < 1$. As a conclusion, we have for any $\delta > 0$, with probability at least $1 - \delta$,

$$\begin{aligned}\|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t} \leq& \frac{2}{\sqrt{\lambda_1}}\frac{(q_1 + \epsilon_1)(1 - (q_1 + \epsilon_1)^t)}{1 - (q_1 + \epsilon_1)} \\ &+ \sqrt{\ln\big(\frac{det(\mathbf{A}_t)}{\delta\lambda_1}\big)} + \sqrt{\lambda_1}S\end{aligned} \tag{5}$$

In addition, since $\|(\mathbf{x}_a, \mathbf{v}_a)\|_2 \leq L$, we have

$$Trace(\mathbf{A}_t) \leq \lambda(d + l)N + \sum_{t'=1}^{t} w^2_{u_{t'},j}L^2 \tag{6}$$

so that

$$det(\mathbf{A}_t) \leq \Big(\frac{Trace(\mathbf{A}_t)}{(d + l)N}\Big)^{(d+l)N} \leq \big(\lambda_1 + \frac{L^2\sum_{t'=1}^{t} w^2_{u_{t'},j}}{(d + l)N}\big)^{(d+l)N} \tag{7}$$

Putting all these together, Eq (2) can be further bounded by,

$$\begin{aligned}\|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t} \leq& \frac{2}{\sqrt{\lambda_1}}\frac{(q_1 + \epsilon_1)(1 - (q_1 + \epsilon_1)^t)}{1 - (q_1 + \epsilon_1)} \\ &+ \sqrt{(d + l)N\ln\big(1 + \frac{L^2\sum_{t'=1}^{t} w^2_{u_{t'},j}}{\delta\lambda_1(d + l)N}\big)} + \sqrt{\lambda_1}S\end{aligned} \tag{8}$$

The same proof techniques apply to the proof of $\|\hat{\mathbf{v}}_{a,t} - \mathbf{v}^*_a\|_{\mathbf{C}_{a,t}}$.

Proof of Theorem 1

According to the definition of cumulated regret in our paper, the regret at time t can be written as,

$$
\begin{aligned}
&R_t = r_{a_t^*,u} - r_{a_t,u} \\
=&vec((\mathring{\mathbf{X}}_{a_t^*}, \mathring{\mathbf{V}}_{a_t^*}^*)\mathbf{W}^\mathsf{T})^\mathsf{T} vec(\mathbf{\Theta}^*) - vec((\mathring{\mathbf{X}}_{a_t}, \mathring{\mathbf{V}}_{a_t}^*)\mathbf{W}^\mathsf{T})^\mathsf{T} vec(\mathbf{\Theta}^*) \\
\leq&vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})^\mathsf{T} vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} + \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&- vec((\mathring{\mathbf{X}}_{a_t}, \mathring{\mathbf{V}}_{a_t}^*)\mathbf{W}^\mathsf{T})^\mathsf{T} vec(\mathbf{\Theta}^*) \\
=&vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})^\mathsf{T} (vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)) \\
&+ \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} \\
&+ \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_u\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&+ vec(0, (\hat{\mathring{\mathbf{V}}}_{a_t,t} - \mathring{\mathbf{V}}_{a_t}^*)\mathbf{w}^\mathsf{T})^\mathsf{T} vec(\mathbf{\Theta}^{*\mathbf{v}}) \\
\leq&\|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t} \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&+ \alpha_t^a \|\mathbf{\Theta}^{*\mathbf{v}} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} + \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} \\
\leq&2\alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + 2\alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} \\
&+ \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} + \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}}
\end{aligned}
\tag{9}
$$

in which the first inequality is based on the following two inequalities.

1) According to our arm selection criterion described in Algorithm 1 of our paper, if arm a is chosen, it satisfies:

$$
\begin{aligned}
&vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} \\
\geq&vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}}
\end{aligned}
\tag{10}
$$

2) The difference between righthand side of Eq (10) and true reward can be bounded by,

$$
\begin{aligned}
&vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} - vec((\mathring{\mathbf{X}}_{a_t^*}, \mathring{\mathbf{V}}_{a_t^*}^*)\mathbf{W}^\mathsf{T}) vec(\boldsymbol{\theta}^*) \\
=&vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})(vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)) \\
&+ vec((\mathbf{0}, (\hat{\mathring{\mathbf{V}}}_{a_t^*} - \mathring{\mathbf{V}}_{a_t^*}^*))\mathbf{W}^\mathsf{T})^\mathsf{T} vec(\mathbf{\Theta}^*) \\
&+ \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + \alpha_t^a \|\hat{\mathbf{\Theta}}_t \mathbf{W}_u^\mathbf{v}\|_{\mathbf{C}_{a_t^*,t}^{-1}}
\end{aligned}
$$

, and using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} - vec((\mathring{\mathbf{X}}_{a_t^*}, \mathring{\mathbf{V}}_{a_t^*}^*)\mathbf{W}^\mathsf{T}) vec(\boldsymbol{\theta}^*) \\
\geq&- \|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t} \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&- \|(\hat{\mathbf{v}}_{a_t^*,t} - \mathbf{v}_{a_t^*}^*)\|_{\mathbf{C}_{a_t^*,t}} \|\mathbf{\Theta}^{*\mathbf{v}} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&+ \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
\geq&- \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} - \alpha_t^a \|\mathbf{\Theta}^{*\mathbf{v}} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&+ \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} + \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
&- \alpha^{a_t} \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}}
\end{aligned}
\tag{11}
$$

Combining Eq (10) and Eq (11), we have the following inequality,

$$
\begin{aligned}
&vec((\mathring{\mathbf{X}}_{a_t^*}, \hat{\mathring{\mathbf{V}}}_{a_t^*})\mathbf{W}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} \\
\leq&vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{w}^\mathsf{T}) vec(\hat{\mathbf{\Theta}}_t) + \alpha_t^u \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}} \\
&+ \alpha_t^a \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} + \alpha^{a_t} \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_u\|_{\mathbf{C}_{a_t^*,t}^{-1}}
\end{aligned}
\tag{12}
$$

which corresponds to the first inequality in Eq (9). The second inequality is also based on Cauchy-Schwarz inequality. The third inequality in Eq (9) holds because α_t^u and α_t^a are defined as the upper bound of $\|vec(\hat{\mathbf{\Theta}}_t) - vec(\mathbf{\Theta}^*)\|_{\mathbf{A}_t}$ and $\|\hat{\mathbf{v}}_{a,t} - \mathbf{v}_a^*\|_{\mathbf{C}_{a,t}}$ respectively.

Then according to the the the definition of accumulated regret, the accumulated regret of Factor-UCB over T trials can be derived as,

$$
\begin{aligned}
&\mathbf{R}(T) = \sum_{t=1}^T R_t \\
\leq&\sqrt{T \sum_{t=1}^T 4(\alpha_t^u)^2 \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}}} \\
&+ \sqrt{T \sum_{t=1}^T 4(\alpha_t^u)^2 \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}}} \\
&+ \sum_{t=1}^T \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t^*,t}^{-1}} + \sum_{t=1}^T \alpha_t^a \|(\mathbf{\Theta}^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}} \\
\leq&2\alpha_T^u \sqrt{T \sum_{t=1}^T \|vec((\mathring{\mathbf{X}}_{a_t}, \hat{\mathring{\mathbf{V}}}_{a_t})\mathbf{W}^\mathsf{T})\|_{\mathbf{A}_t^{-1}}} \\
&+ 2\alpha_T^a \sqrt{T \sum_{t=1}^T \|\hat{\mathbf{\Theta}}_t^\mathbf{v} \mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}}} \\
&+ 2\alpha_T^a \frac{1}{\sqrt{\lambda_2}} \sum_{t=1}^T \|(\mathbf{\Theta}_t^{*\mathbf{v}} - \hat{\mathbf{\Theta}}_t^\mathbf{v})\mathbf{w}_{u_t}\|_2
\end{aligned}
\tag{13}
$$

For the first and second term in the right hand side of Eq (13), using conclusions of Lemma 11 in (Abbasi-yadkori, Pál, and Szepesvári 2011) and our previous proof, we have,

$$
\begin{aligned}
&2\alpha_T^u\sqrt{T\sum_{t=1}^{T}\|vec((\mathring{\hat{\mathbf{X}}}_{a_t},\mathring{\hat{\mathbf{V}}}_{a_t})\mathbf{W}^{\mathsf{T}})\|_{\mathbf{A}_t^{-1}}}\\
&+2\alpha_T^a\sqrt{T\sum_{t=1}^{T}\|\hat{\mathbf{\Theta}}_t^{\mathbf{v}}\mathbf{w}_{u_t}\|_{\mathbf{C}_{a_t,t}^{-1}}}\\
\le&2\alpha_T^u\sqrt{2(d+l)NT\ln(\frac{det(\mathbf{A}_t)}{\delta det(\lambda_1\mathbf{I})})}\\
&+2\alpha_T^a\sqrt{2lT\ln(\frac{det(\mathbf{C}_{a,t})}{\delta det(\lambda_2\mathbf{I})})}\\
\le&2\alpha_T^u\sqrt{2(d+l)NT\ln(1+\frac{L^2\sum_{t=1}^{T}\sum_{j}^{N}w_{u_t,j}^2}{\delta\lambda_1(d+l)N})}\\
&+2\alpha_T^a\sqrt{2lT\ln(1+\frac{S^2\sum_{t=1}^{T}\sum_{j}^{N}w_{u_t,j}^2}{\delta\lambda_2 l})}
\end{aligned}
\tag{14}
$$

For the third term, again according to the q-linear convergence property, we have,

$$
\begin{aligned}
&2\alpha_T^a\frac{1}{\sqrt{\lambda_2}}\sum_{t=1}^{T}\|(\mathbf{\Theta}_t^{*\mathbf{v}}-\hat{\mathbf{\Theta}}_t^{\mathbf{v}})\mathbf{w}_{u_t}\|_2\\
\le&\ 2\alpha_T^a\frac{1}{\sqrt{\lambda_2}}\sum_{t=1}^{T}\|vec(\mathbf{\Theta}_t^{*\mathbf{v}})-vec(\hat{\mathbf{\Theta}}_t^{\mathbf{v}}))\|_2\|\mathbf{w}_{u_t}\|_2\\
\le&\ 2\alpha_T^a\frac{S}{\sqrt{\lambda_2}}\sum_{t=1}^{T}(q_2+\epsilon_2)^t\\
\le&\ 2\alpha_T^a\frac{1}{\sqrt{\lambda_2}}\frac{(q_2+\epsilon_2)(1-(q_2+\epsilon_2)^T)}{1-(q_2+\epsilon_2)}
\end{aligned}
\tag{15}
$$

Submitting the two inequalities in Eq (14) and Eq (15) to Eq (13) finishes the proof of Theorem 1.

References

Abbasi-yadkori, Y.; Pál, D.; and Szepesvári, C. 2011. Improved algorithms for linear stochastic bandits. In *NIPS*. 2312–2320.

Uschmajew, A. 2012. Local convergence of the alternating least squares algorithm for canonical tensor approximation. *SIAM Journal on Matrix Analysis and Applications* 33(2):639–652.