CS 4102: Algorithms

Lecture 4: Karatsuba, Induction, and Master Theorem

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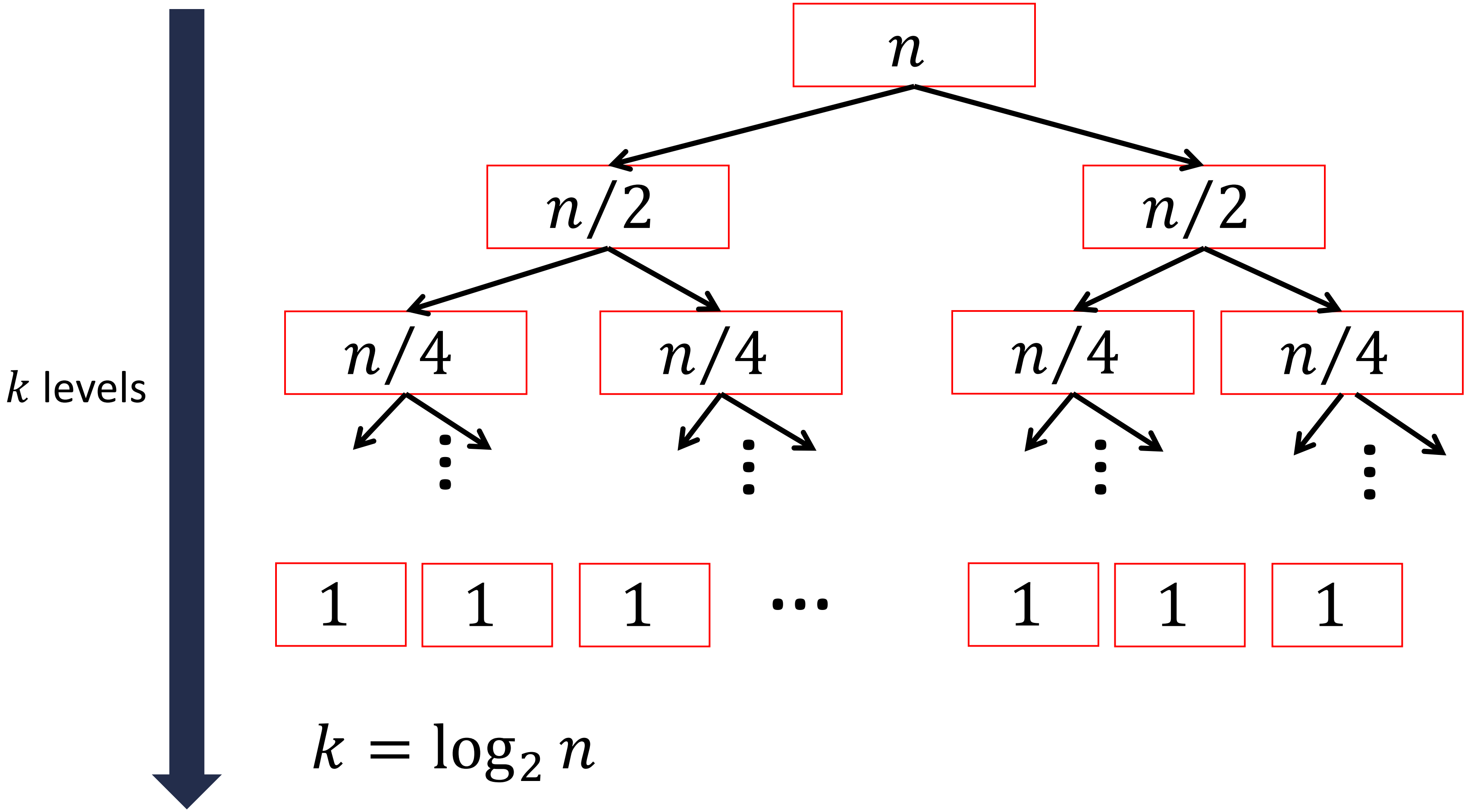
Warm Up

What is the asymptotic run time of MergeSort if its recurrence is

$$T(n) = 2T\left(\frac{n}{2}\right) + 209n$$

Tree Method

$$T(n) = 2T\left(\frac{n}{2}\right) + 209n$$



Number of subproblems	Cost of subproblem
1	$209n$
2	$209n/2$
4	$209n/4$
2^k	$209n/2^k$

Tree Method

$$T(n) = 2T(n/2) + 209n$$

What is the cost?

Cost at level i: $2^i \cdot \frac{209n}{2^i} = 209n$

Total cost: $T(n) = \sum_{i=0}^{\log_2 n} 209n$

$$= 209n \sum_{i=0}^{\log_2 n} 1 = 209n \log_2 n$$

$$= \Theta(n \log n)$$

Number of subproblems	Cost of subproblem
1	$209n$
2	$209n/2$
4	$209n/4$
2^k	$209n/2^k$

Today's Keywords

Karatsuba's Algorithm

Guess and Check Method

Induction

Master Theorem

CLRS Readings: Chapter 4

Homeworks

HW1 due **Thursday, September 12, 11pm**

- Start early!
- Written (use Latex!) – Submit both **zip** and **pdf**!
- Asymptotic notation
- Recurrences
- Divide and Conquer

Karatsuba Multiplication

1. Break into smaller subproblems

$$
\begin{array}{r}
\boxed{a}\,\boxed{b} = 10^{\frac{n}{2}}\boxed{a} + \boxed{b} \\
\times\ \boxed{c}\,\boxed{d} = 10^{\frac{n}{2}}\boxed{c} + \boxed{d}
\end{array}
$$

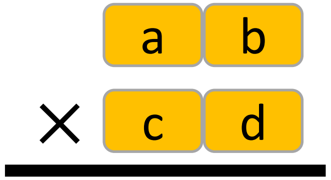
$$= \quad 10^{n}(\boxed{a} \times \boxed{c}) + $$

$$10^{\frac{n}{2}}(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}) + $$

$$(\boxed{b} \times \boxed{d})$$

Recall: previous divide-and-conquer recursively computed ac, ad, bc, bd

Karatsuba Multiplication



$$10^{n}(\boxed{ac}) + 10^{\frac{n}{2}}(\boxed{ad + bc}) + \boxed{bd}$$

Can’t avoid these

This can be simplified!

$$(a + b)(c + d) =$$

$$\boxed{ac} + \boxed{ad + bc} + \boxed{bd}$$

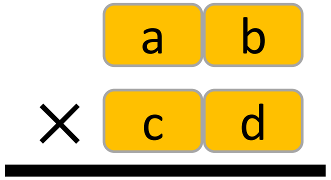
$$\boxed{ad + bc} = \boxed{(a + b)(c + d) - \boxed{ac} - \boxed{bd}}$$

Two multiplications

One multiplication

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time



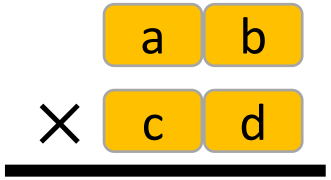
$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

Recursively solve

$$T(n) =$$

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time



$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

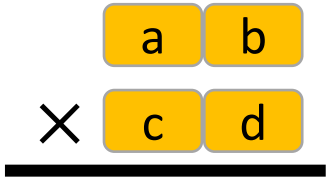
Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right)$$

Need to compute **3** multiplications, each of size $n/2$: ac, bd, $(a+b)(b+c)$

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time



$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

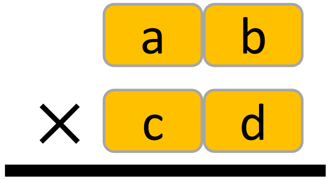
Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Need to compute 3 multiplications, each of size $n/2$: ac, bd, $(a+b)(b+c)$

2 shifts and 6 additions on n-bit values

Karatsuba Multiplication



Divide:

- Break n-digit numbers into four numbers of $^{n}/_{2}$ digits each (call them a, b, c, d)

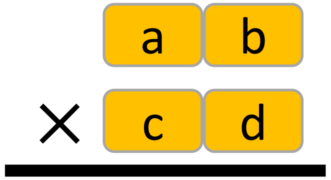
Conquer:

- If $n > 1$:
  - Recursively compute $ac, bd, (a+b)(c+d)$
- If $n = 1$:
  - Compute $ac, bd, (a+b)(c+d)$ directly (base case)

Combine:

- $10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$

Karatsuba Multiplication



1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

<u>Pseudocode:</u>

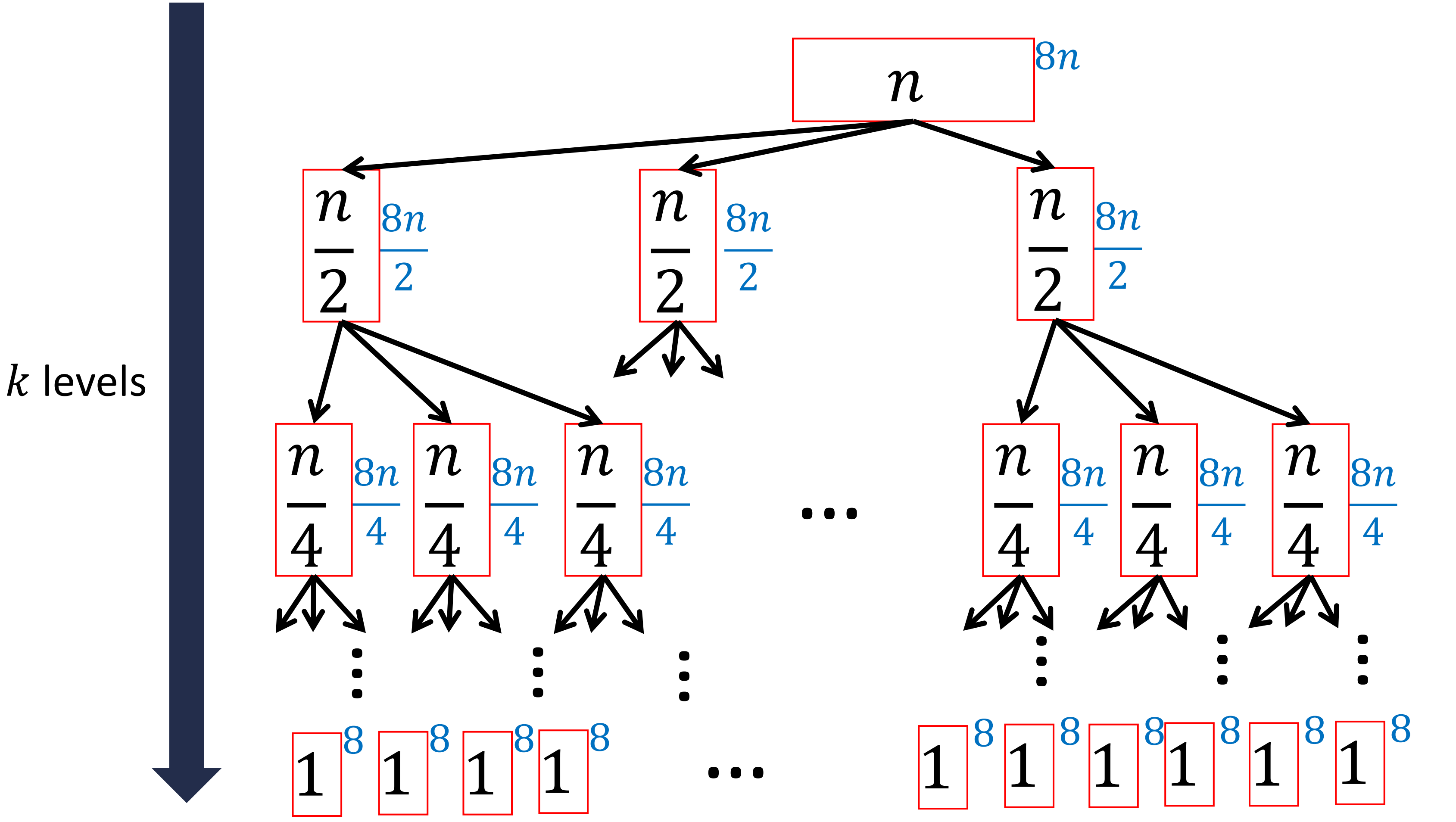
1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(a, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^n x + 10^{n/2} z + y$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 3T(n/2) + 8n$$



Number of subproblems	Cost of subproblem
1	$8n$
3	$\frac{8n}{2}$
9	$\frac{8n}{4}$
3^k	$\frac{8n}{2^k} = 8$

Karatsuba Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 3T(n/2) + 8n$$

How many levels?

Problem size at k^{th} level: $\frac{n}{2^k}$

Base case: $n = 1$

At level k, it should be the case that $\frac{n}{2^k} = 1$

$$n = 2^k \Rightarrow k = \log_2 n$$

Number of subproblems	Cost of subproblem
1	$8n$
3	$\frac{8n}{2}$
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3^k	$\frac{8n}{2^k} = 8$

Karatsuba Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 3T(n/2) + 8n$$

$$k = \log_2 n$$

What is the cost?

Cost at level i: $3^i \cdot \frac{8n}{2^i} = \left(\frac{3}{2}\right)^i \cdot 8n$

Total cost: $T(n) = \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i \cdot 8n = 8n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i$

Number of subproblems	Cost of subproblem
1	$8n$
3	$\frac{8n}{2}$
9	$\frac{8n}{4}$
3^k	$\frac{8n}{2^k} = 8$

Karatsuba Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 3T(n/2) + 8n$$

$$= 8n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i$$

$$= 8n \frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1}$$

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$$

Karatsuba Multiplication

$$T(n) = 8n \frac{\left(3/_2\right)^{\log_2 n+1} - 1}{3/_2 - 1}$$

How to simplify this
(using asymptotic notation)?

Drop constant multiples

Karatsuba Multiplication

$$T(n) = 8n \frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1}$$

How to simplify this (using asymptotic notation)?

$$= \Theta\left(n\left((3/2)^{\log_2 n+1} - 1\right)\right)$$

Drop constant multiples

$$= \Theta\left(\frac{3}{2}n \cdot (3/2)^{\log_2 n} - n\right)$$

Distribute terms

Karatsuba Multiplication

$$T(n) = 8n \frac{(3/2)^{\log_2 n + 1} - 1}{3/2 - 1}$$

How to simplify this (using asymptotic notation)?

$$= \Theta\left(n\left((3/2)^{\log_2 n+1} - 1\right)\right)$$

Drop constant multiples

$$= \Theta\left(\frac{3}{2} n \cdot (3/2)^{\log_2 n} - n\right)$$

Distribute terms

$$= \Theta\left(n \cdot (3/2)^{\log_2 n}\right)$$

Drop constants and low-order terms

Karatsuba Multiplication

$$T(n) = \Theta\left(n \cdot \left({}^{3}/_{2}\right)^{\log_2 n}\right)$$

How to simplify this (using asymptotic notation)?

Properties of logarithms:

$$2^{\log_2 n} = n$$

$$3^{\log_2 n} = 2^{\log_2\left(3^{\log_2 n}\right)} = 2^{(\log_2 n)(\log_2 3)} = \left(2^{\log_2 n}\right)^{\log_2 3} = n^{\log_2 3}$$

$a^{bc} = \left(a^b\right)^c$

$2^{\log_2 n} = n$

$\log a^b = b \log a$

$2^{\log_2 n} = n$

Karatsuba Multiplication

$$T(n) = \Theta\left(n \cdot \left({}^{3}/_{2}\right)^{\log_2 n}\right)$$

$$= \Theta\left(n \cdot \left(\frac{3^{\log_2 n}}{2^{\log_2 n}}\right)\right)$$

$$= \Theta\left(n \cdot \left(\frac{n^{\log_2 3}}{n}\right)\right)$$

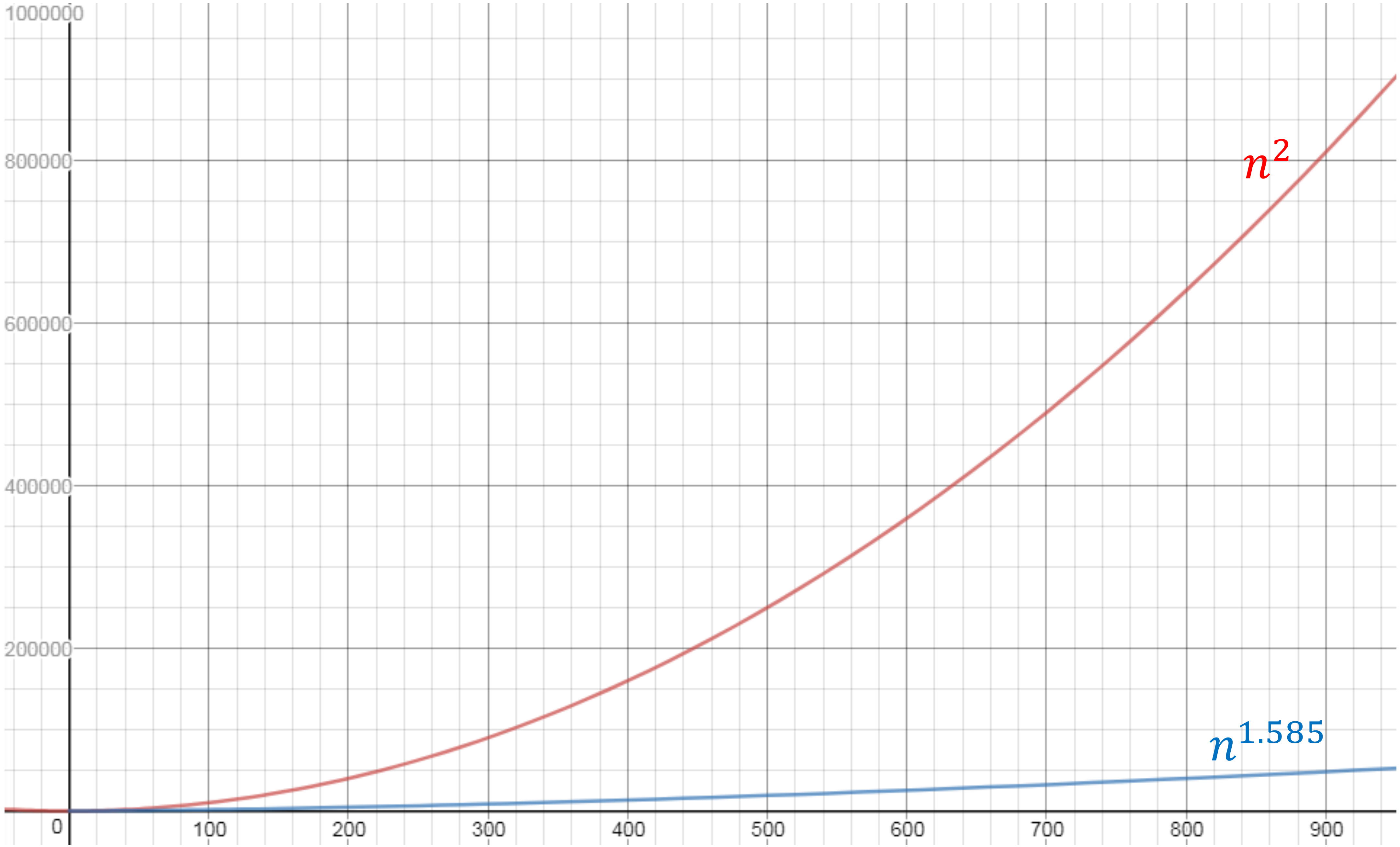
$$= \Theta\left(n^{\log_2 3}\right) \approx \Theta(n^{1.585})$$

How to simplify this (using asymptotic notation)?

$$2^{\log_2 n} = n$$

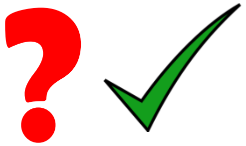
$$3^{\log_2 n} = n^{\log_2 3}$$

Strictly better than schoolbook method!


1000000
800000
600000
400000
200000
0
100
200
300
400
500
600
700
800
900
n^2
$n^{1.585}$

Recurrence Solving Techniques

Tree



Guess/Check (induction)

“Cookbook”

Substitution

Induction (Review)

Goal: $\forall k \in \mathbb{N}, P(k)$ holds

Base case(s): $P(1)$ holds

Technically, called *strong induction*

Hypothesis: $\forall x \leq x_0, P(x)$ holds

Inductive step: $P(1), \ldots, P(x_0) \Rightarrow P(x_0 + 1)$

Guess and Check Blueprint

Show: $T(n) = O(g(n))$

Consider: $g_*(n) = c \cdot g(n)$ for some constant c

Goal: show $\exists n_0$ such that $\forall n > n_0, T(n) \leq g_*(n)$

- (definition of big-O)

Technique: Induction

- Base cases:
  - Show $T(1) \leq g_*(1)$ (sometimes, may need to consider additional base cases)
- Hypothesis:
  - $\forall n \leq x_0, T(n) \leq g_*(n)$
- Inductive step:
  - Show that $T(x_0 + 1) \leq g_*(x_0 + 1)$

Need to ensure that in inductive step, can either appeal to a base case or to the inductive hypothesis

Karatsuba Analysis using Guess and Check

$$T(n) = 3T(n/2) + 8n$$

Goal: $T(n) \leq 3000\, n^{1.6} = O(n^{1.6})$

Base case: $T(1) = 8 \leq 3000$

Hypothesis: $\forall n \leq x_0,\ \ T(n) \leq 3000n^{1.6}$

Inductive step: Show $T(x_0 + 1) \leq 3000(x_0 + 1)^{1.6}$

Karatsuba Guess and Check (Loose)

$$T(n) = 3T(n/2) + 8n$$

Hypothesis: $\forall n \leq x_0$: $T(n) \leq 3000n^{1.6}$

Show: $T(x_0 + 1) \leq 3000(x_0 + 1)^{1.6}$

$$T(x_0 + 1) = 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1)$$ Recurrence definition

$$\leq 3\left(3000\left(\frac{x_0 + 1}{2}\right)^{1.6}\right) + 8(x_0 + 1)$$ Inductive hypothesis

Karatsuba Guess and Check (Loose)

$$T(x_0+1) = 3T\left(\frac{x_0+1}{2}\right) + 8(x_0+1)$$ Recurrence definition

$$\leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1)$$ Inductive hypothesis

$$\leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1)^{1.6}$$ $\forall x \geq 0: x^{1.6} \geq x$

$$= \left(\frac{9000}{2^{1.6}} + 8\right)(x_0+1)^{1.6}$$ Distributive property

$$\leq 3000(x_0+1)^{1.6}$$ $\frac{9000}{2^{1.6}} + 8 \leq 3000$

Show: $T(x_0+1) \leq 3000(x_0+1)^{1.6}$

Mergesort Guess and Check

$$T(n) = 2T(n/2) + n$$

Goal: $T(n) \le n \log_2 n \; = O(n \log_2 n)$

Base case: $T(1) = 0 = 1 \log_2 1 = 0$

Hypothesis: $\forall n \le x_0 \colon T(n) \le n \log_2 n$

Inductive step: $T(x_0 + 1) \le (x_0 + 1) \log_2 (x_0 + 1)$

Mergesort Guess and Check

$$T(n) = 2T(n/2) + n$$

Hypothesis: $\forall n \leq x_0 \colon T(n) \leq n \log_2 n$

Show: $T(x_0 + 1) \leq (x_0 + 1) \log_2 (x_0 + 1)$

$$T(x_0 + 1) = 2T\left(\frac{x_0 + 1}{2}\right) + (x_0 + 1)$$

$$\leq 2\left(\frac{x_0 + 1}{2}\right) \log_2 \left(\frac{x_0 + 1}{2}\right) + (x_0 + 1)$$

$$= (x_0 + 1)(\log_2(x_0 + 1) - 1) + (x_0 + 1)$$

$$= (x_0 + 1) \log_2 (x_0 + 1)$$

Karatsuba Guess and Check

$$T(n) = 3T(n/2) + 8n$$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base case: $T(1) = 8 = 24(1)^{\log_2 3} - 16(1) = 24 - 16 = 8$

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

Karatsuba Guess and Check

$$T(n) = 3T(n/2) + 8n$$

Hypothesis: $\forall n \leq x_0 : T(n) \leq 24n^{\log_2 3} - 16n$

Show: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

$$T(x_0 + 1) = 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1)$$

$$\leq 3\left[24\left(\frac{x_0 + 1}{2}\right)^{\log_2 3} - 16\left(\frac{x_0 + 1}{2}\right)\right] + 8(x_0 + 1)$$

$$= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1)$$

$$2^{\log_2 3} = 3$$

$$= 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$$

Karatsuba Guess and Check

$$T(n) = 3T(n/2) + 8n$$

What if we leave this out?

Hypothesis: $\forall n \leq x_0 : T(n) \leq 24n^{\log_2 3} - 16n$

Show: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

$$T(x_0 + 1) = 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1)$$

$$\leq 3\left[24\left(\frac{x_0 + 1}{2}\right)^{\log_2 3} - 16\left(\frac{x_0 + 1}{2}\right)\right] + 8(x_0 + 1)$$

$$= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1)$$

$$= 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$$

Karatsuba Guess and Check

$$T(n) = 3T(n/2) + 8n$$

What if we leave this out?

Hypothesis: $\forall n \le x_0: T(n) \le 24n^{\log_2 3} - 16n$

Show: $T(x_0 + 1) \le 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

$$T(x_0 + 1) = 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1)$$

$$\le 3\left[24\left(\frac{x_0 + 1}{2}\right)^{\log_2 3} - 16\left(\frac{x_0 + 1}{2}\right)\right] + 8(x_0 + 1)$$

$$= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1)$$

$$= 24(x_0 + 1)^{\log_2 3} + 8(x_0 + 1)$$

Extra term does not vanish! Induction fails.