

CS4102 Day 4 Recurrence Proofs - Fall 2019

Karatsuba Recurrence: Tree Method

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Using the tree method for solving the recurrence, we obtained the sum:

$$\begin{aligned} T(n) &= 8n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i \\ &= 8n \frac{\left(\frac{3}{2}\right)^{\log_2 n+1} - 1}{\frac{3}{2} - 1} \\ &= 8n \frac{\left(\frac{3}{2}\right)^{\log_2 n+1} - 1}{\frac{1}{2}} \\ &= 16n \left(\left(\frac{3}{2}\right)^{\log_2 n+1} - 1 \right) \\ &= 16n(2^{\log_2 3-1})^{\log_2 n+1} - 16n \\ &= 16n(2^{\log_2 3 \cdot \log_2 n - \log_2 n + \log_2 3 - 1}) - 16n \\ &= 16n((2^{\log_2 n})^{\log_2 3} \cdot 2^{-\log_2 n} \cdot 2^{\log_2 3} \cdot 2^{-1}) - 16n \\ &= 16n \left(n^{\log_2 3} \cdot \frac{1}{n} \cdot 3 \cdot \frac{1}{2} \right) - 16n \\ &= 24n^{\log_2 3} - 16n \\ &= \Theta(n^{\log_2 3}) \\ &\approx \Theta(n^{1.585}) \end{aligned}$$

Karatsuba, Guess and Check, Loose Bound

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal:

$$T(n) \leq 3000n^{1.6}$$

Base Case:

$$T(1) = 8 \leq 3000$$

Hypothesis:

$$\forall n < x_0, T(n) \leq 3000n^{1.6}$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &\leq 3 \left(3000 \left(\frac{x_0 + 1}{2}\right)^{1.6} \right) + 8(x_0 + 1) \\ &= \frac{3}{2^{1.6}} \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &\leq 0.997 \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &= (1 - 0.003) \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 0.003 \cdot 3000(x_0 + 1)^{1.6} \\ &= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 9(x_0 + 1)^{1.6} \\ &\leq 3000(x_0 + 1)^{1.6} \end{aligned}$$

MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Goal:

$$T(n) \leq n \log_2 n$$

Base Case: by inspection

Hypothesis:

$$\forall n < x_0, T(n) \leq n \log_2 n$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 2T\left(\frac{x_0 + 1}{2}\right) + (x_0 + 1) \\ &\leq 2\left(\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}\right) + x_0 + 1 \\ &= (x_0 + 1) \log_2 \frac{x_0 + 1}{2} + x_0 + 1 \\ &= (x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + x_0 + 1 \\ &= (x_0 + 1)(\log_2(x_0 + 1) - 1) + x_0 + 1 \\ &= (x_0 + 1) \log_2(x_0 + 1) - (x_0 + 1) + x_0 + 1 \\ &= (x_0 + 1) \log_2(x_0 + 1) \end{aligned}$$

Karatsuba, Guess and Check, Tight Bound

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal:

$$T(n) \leq 24n^{\log_2 3} - 16n$$

Base Case: by inspection

Hypothesis:

$$\forall n < x_0, T(n) \leq 24n^{\log_2 3} - 16n$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &\leq 3\left(24\left(\frac{x_0 + 1}{2}\right)^{\log_2 3} - 16\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &= 3\left(\frac{24}{3}(x_0 + 1)^{\log_2 3} - 8(x_0 + 1)\right) + 8(x_0 + 1) \\ &= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1) \\ &= 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1) \end{aligned}$$