CS4102 Proofs Day 5 - Fall 2019

Master Theorem Case 1

Recurrence:

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Assumption:

$$f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

To Show:

$$T(n) = O(n^{\log_b a})$$

Proof: (let $L = \log_b n$, i.e. the height of the recurrence tree)

$$\begin{aligned}
T(n) &= f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \ldots + a^L f(\frac{n}{b^L}) \\
&\leq c((n)^{\log_b a - \varepsilon} + a(\frac{n}{b})^{\log_b a - \varepsilon} + a^2(\frac{n}{b^2})^{\log_b a - \varepsilon} + \ldots + a^{L-1}(\frac{n}{b^{L-1}})^{\log_b a - \varepsilon}) + a^L f(1) \\
&= cn^{\log_b a - \varepsilon}(1 + \frac{a}{b^{\log_b a - \varepsilon}} + \frac{a^2}{b^{2(\log_b a - \varepsilon)}} + \ldots + \frac{a^{L-1}}{b^{(L-1)(\log_b a - \varepsilon)}}) + a^L f(1) \\
&= cn^{\log_b a - \varepsilon}(1 + b^{\varepsilon} + b^{2\varepsilon} + \ldots + b^{(L-1)\varepsilon}) + a^L f(1) \\
&= cn^{\log_b a - \varepsilon}(\frac{b^{\varepsilon L} - 1}{b^{\varepsilon} - 1}) + a^L f(1) \\
&= cn^{\log_b a - \varepsilon}(\frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}) + a^L f(1) \\
&= cn^{\log_b a - \varepsilon}((n^{\varepsilon} - 1) \cdot \frac{1}{b^{\varepsilon} - 1}) + a^{\log_b n} f(1) \\
&= cn^{\log_b a - \varepsilon}((n^{\varepsilon} - 1) \cdot c_2) + n^{\log_b a} \cdot c_3 \\
&= c_4 n^{\log_b a} - c_4 n^{\log_b a - \varepsilon} + n^{\log_b a} \cdot c_3 \\
&= O(n^{\log_b a})
\end{aligned}$$