CS4102 Algorithms

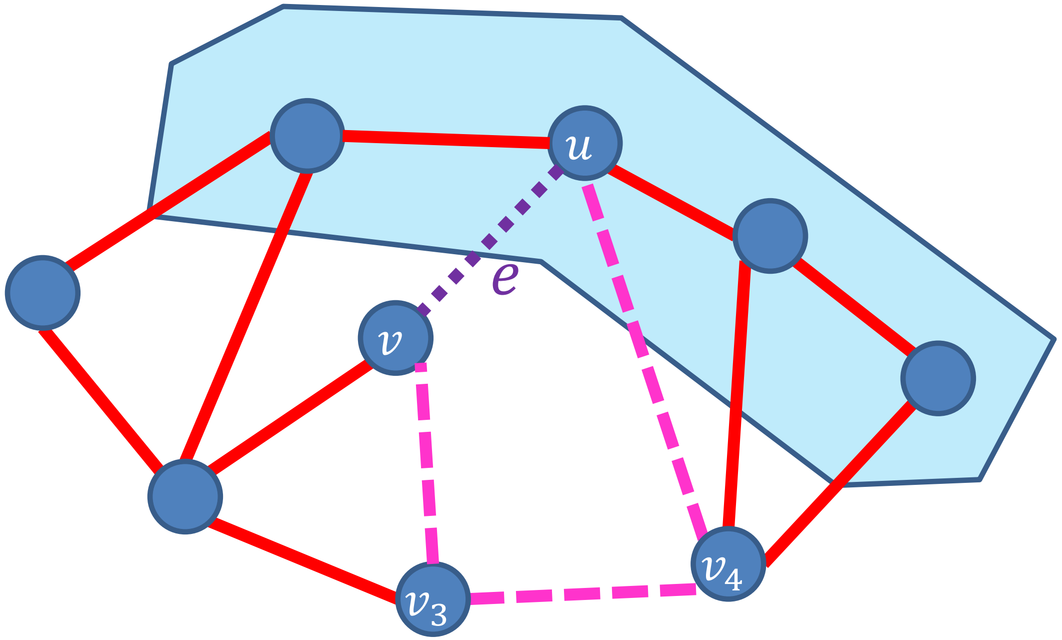
Fall 2019

Warm up:

Show that no cycle crosses a cut exactly once

no cycle crosses a cut exactly once

- Assume the cycle crosses the cut once
- Consider some edge (u, v) in the cycle which crosses the cut
- If we remove (u, v) then there is still a path from u to v which must somewhere cross the cut



Today's Keywords

- Graphs
- Minimum Spanning Tree
- Prim's Algorithm
- Shortest path
- Dijkstra's Algorithm
- Breadth-first search

CLRS Readings

- Chapter 22
- Chapter 23

Homeworks

- HW7 due Thursday, November 14 @ 11pm
  - Written (use latex)
  - Graphs!
- HW10B also due Thursday, November 14 @ 11pm
  - No late submissions allowed
- Reminder: I will not have office hours Monday
  - Tuesday 11-1 instead

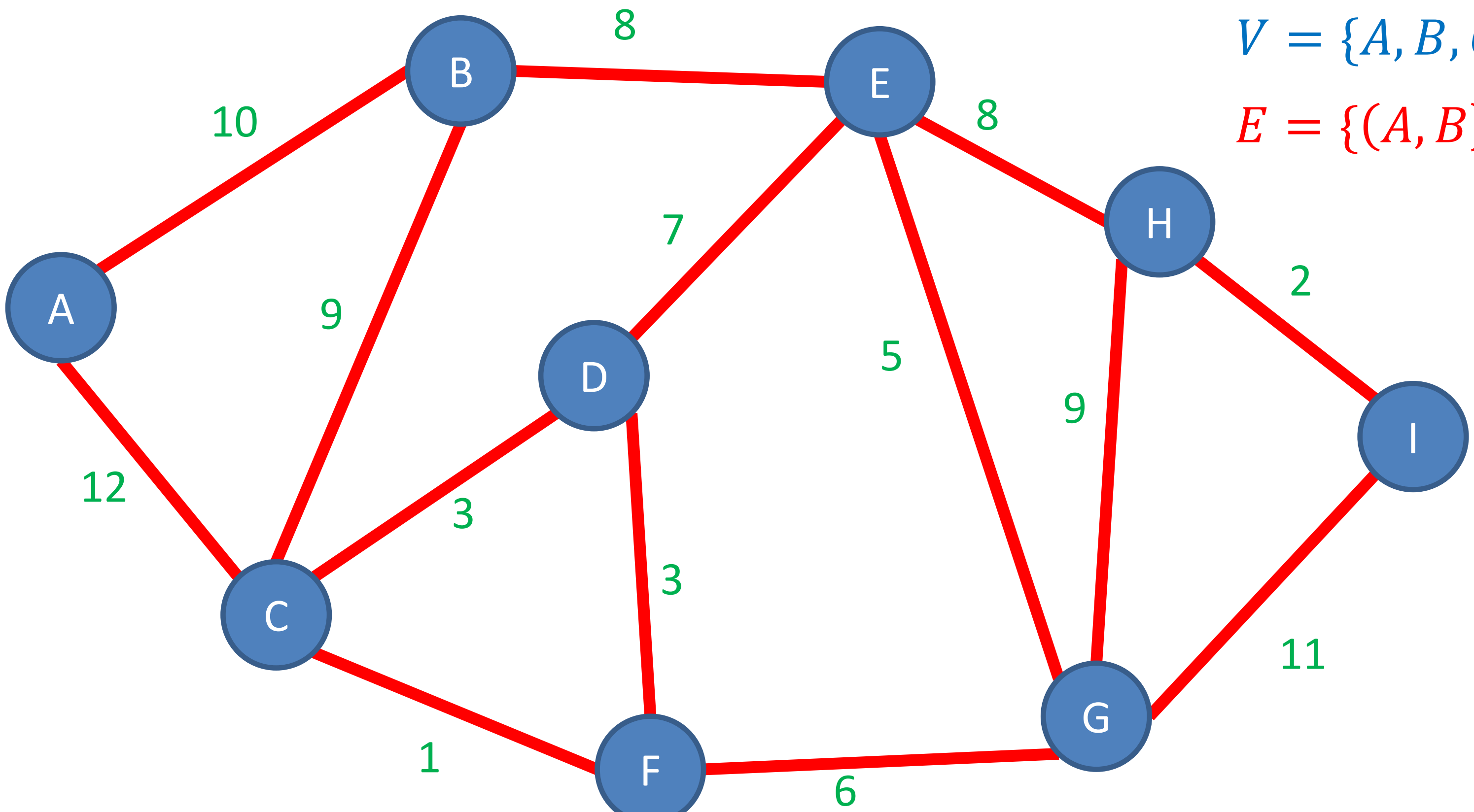
Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges

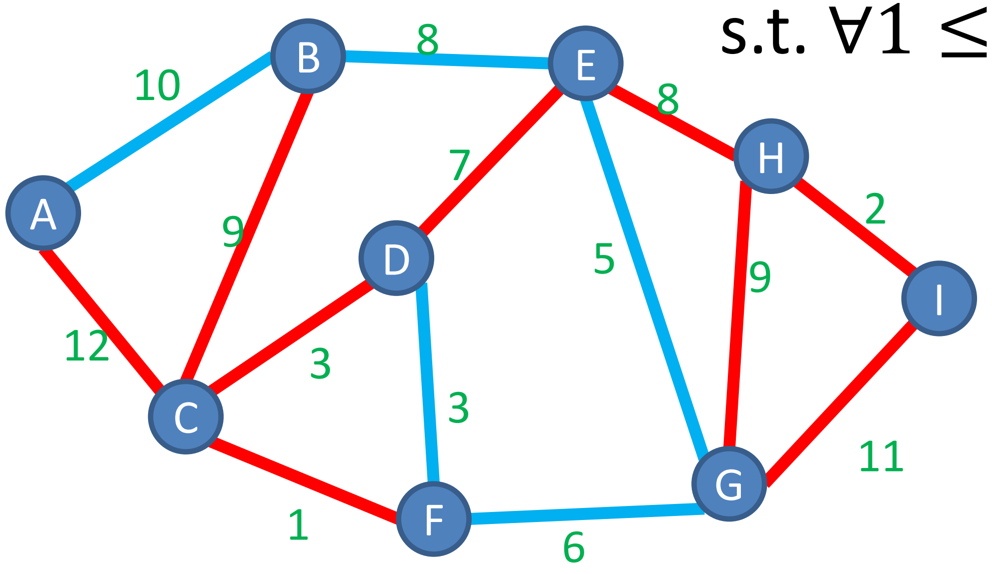
$w(e)$ = weight of edge e



$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{(A, B), (A, C), (B, C), \ldots\}$

Definition: Path



A sequence of nodes $(v_1, v_2, \ldots, v_k)$
s.t. $\forall 1 \leq i \leq k-1, (v_i, v_{i+1}) \in E$

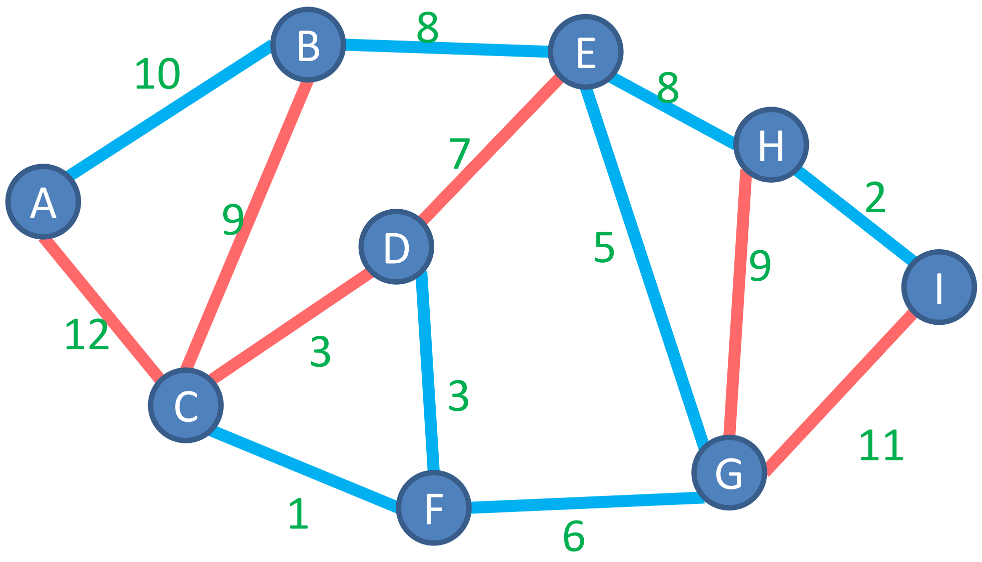
Simple Path:
A path in which each node appears at most once

Cycle:
A path of > 2 nodes in which $v_1 = v_k$

Definition: Minimum Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

$$Cost(T) = \sum_{e \in E_T} w(e)$$

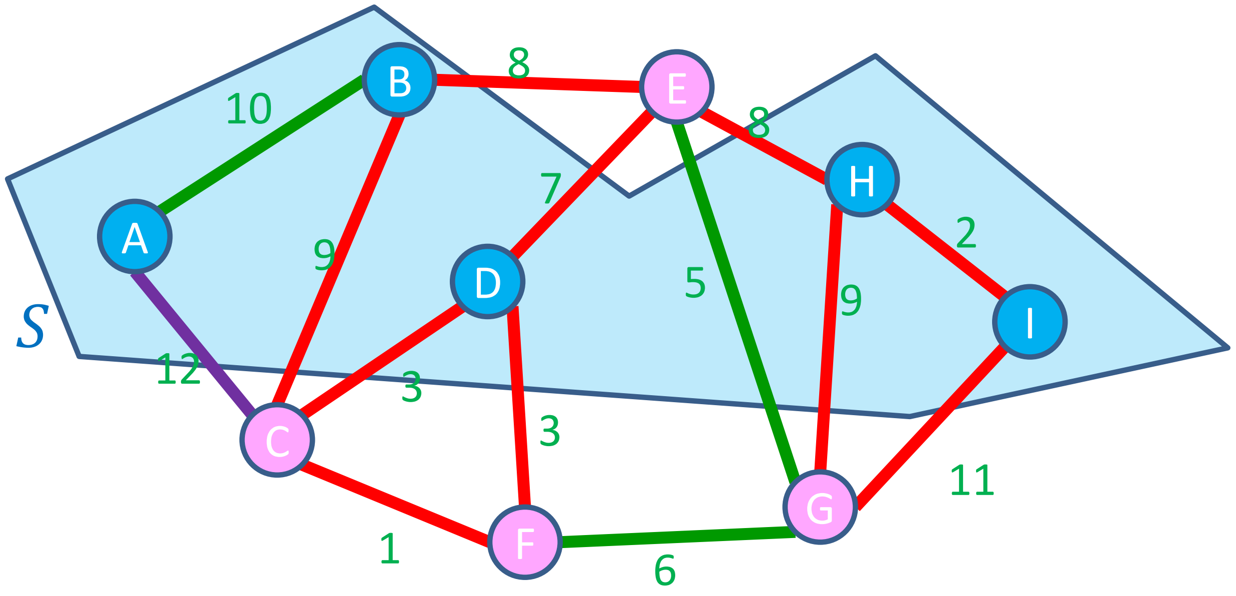


How many edges does T have?

$V - 1$

Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$

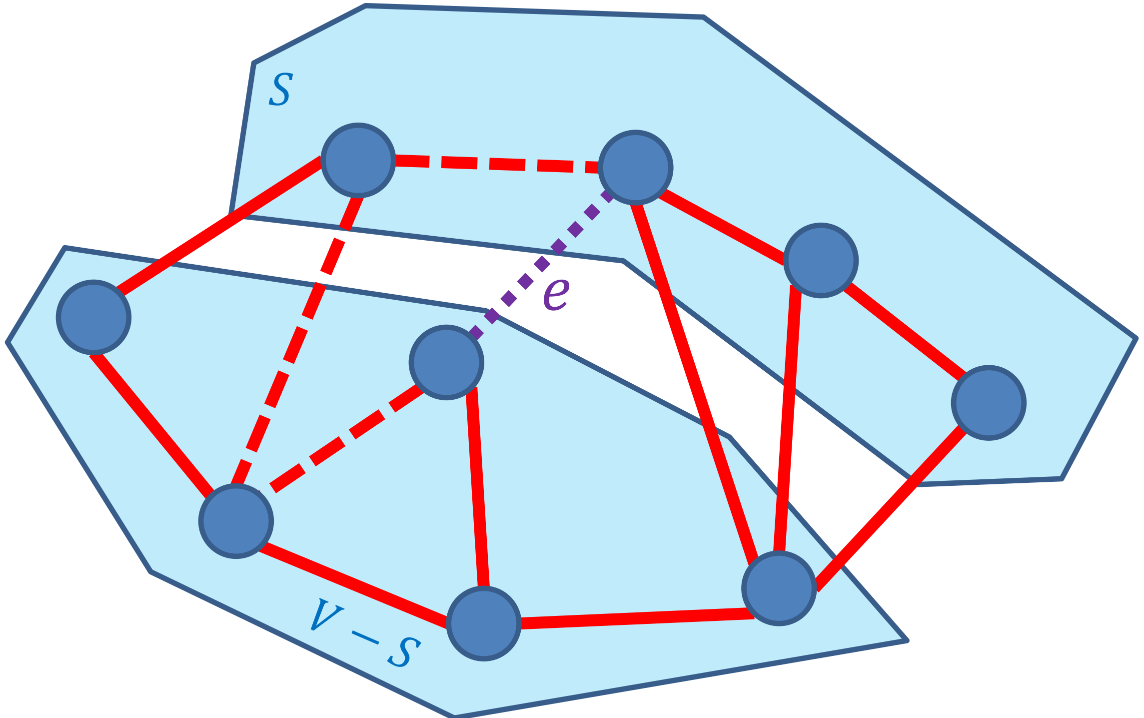


Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

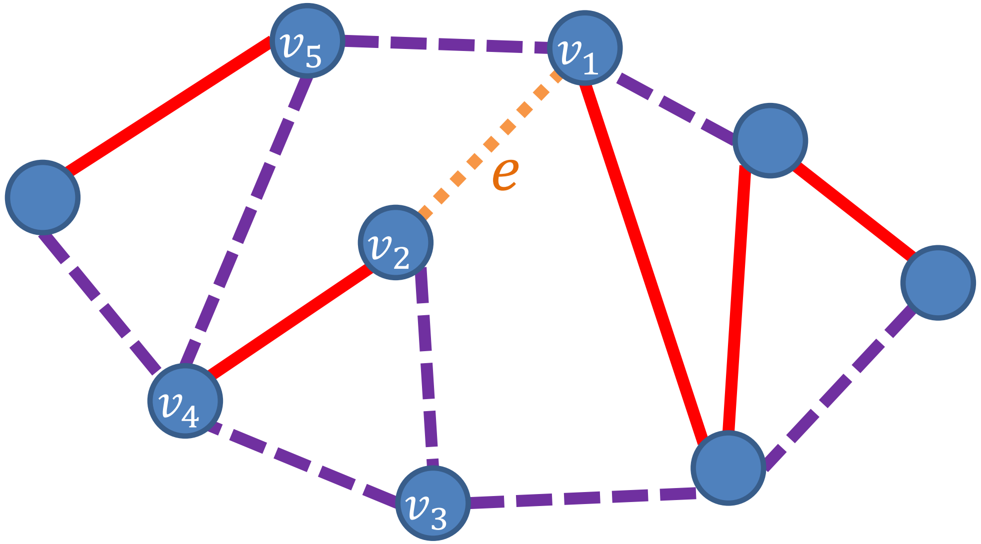
Cut Property

Consider any cut $(S, V - S)$ in a graph $G = (V, E)$, the minimum weight edge crossing that cut is in *some* MST of G



Warm up 2gether: Cycle Theorem

Consider any cycle in a graph $G = (V, E)$, the maximum weight edge on that cycle is *not* in *some* MST of G



What is our strategy?

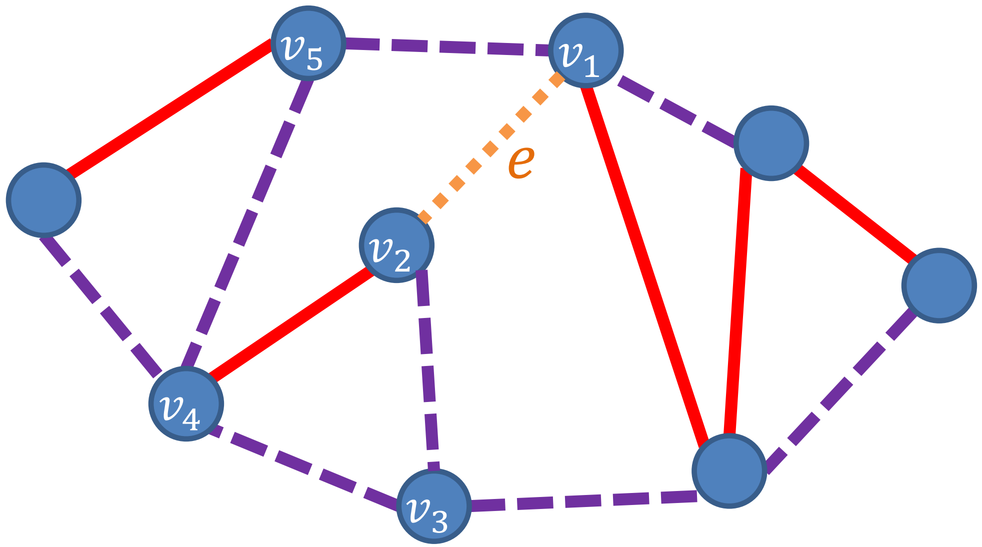
Assume we have a MST Already:

2 cases:

1. *Tree **does not** have max weight edge*
2. *Tree **has** max weight edge*

Cycle Theorem: Case 1

Consider any cycle c $= (v_1, v_2, \dots v_k, v_1)$ in a graph $G = (V, E)$, the maximum weight edge e on that cycle is *not* in *some* MST of G



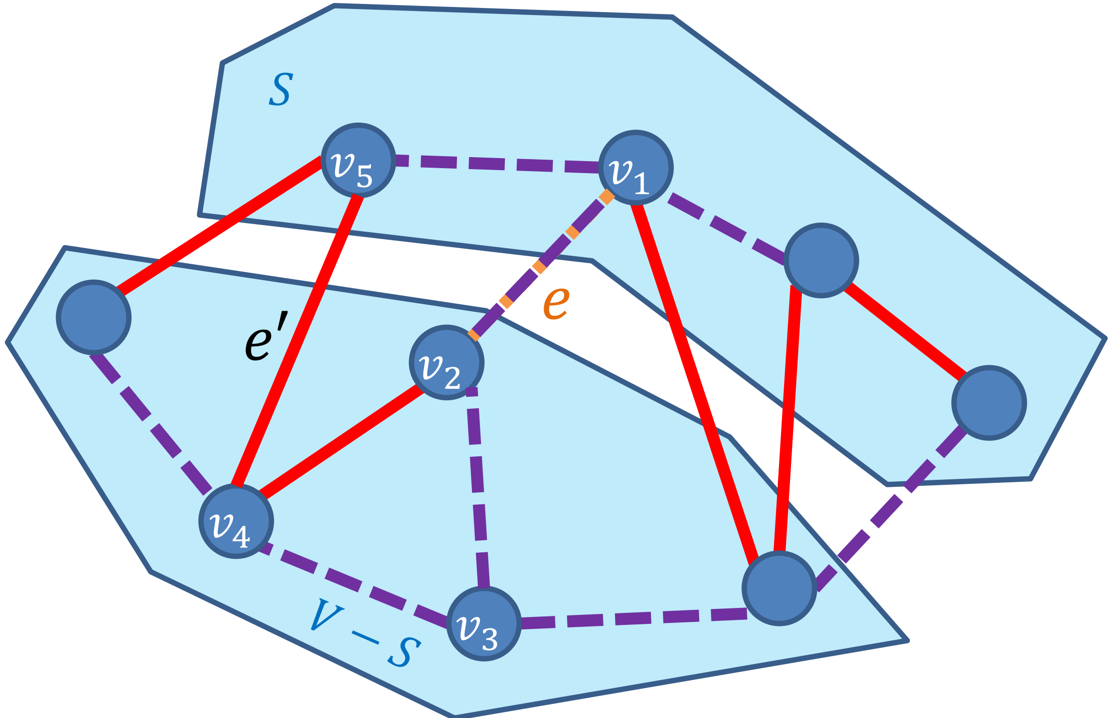
Consider some MST T,

Case 1: (the easy case)

If $e \notin T$ Then claim holds

Cycle Theorem: Case 2

Consider any cycle $c = (v_1, v_2, \ldots v_k, v_1)$ in a graph $G = (V, E)$, the maximum weight edge e on that cycle is *not* in *some* MST of G



Consider some MST T,

Case 2:

Consider if $e = (v_1, v_2) \in T$

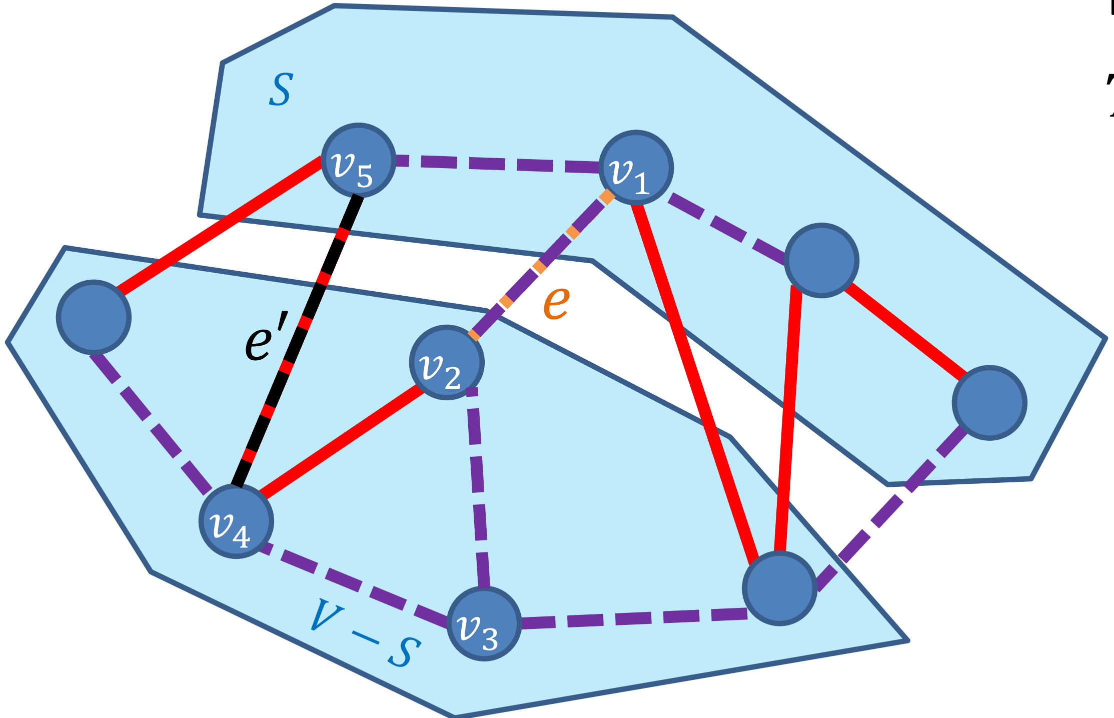
Let $(S, V - S)$ be a cut which e crosses

There is some other edge e' not in T which crosses $(S, V - S)$

Build tree T' by exchanging e' for e

Cycle Theorem: Case 2

Consider any cycle $c = (v_1, v_2, \ldots v_k, v_1)$ in a graph $G = (V, E)$, the maximum weight edge e on that cycle is *not* in *some* MST of G



Consider some MST T,

Case 2:

if $e = (v_1, v_2) \in T$

$T' = T$ with edge e' instead of e

We assumed $w(e) \geq w(e')$

$w(T') = w(T) - w(e) + w(e')$

$w(T') \leq w(T)$

So T' is also a MST!

Thus the claim holds

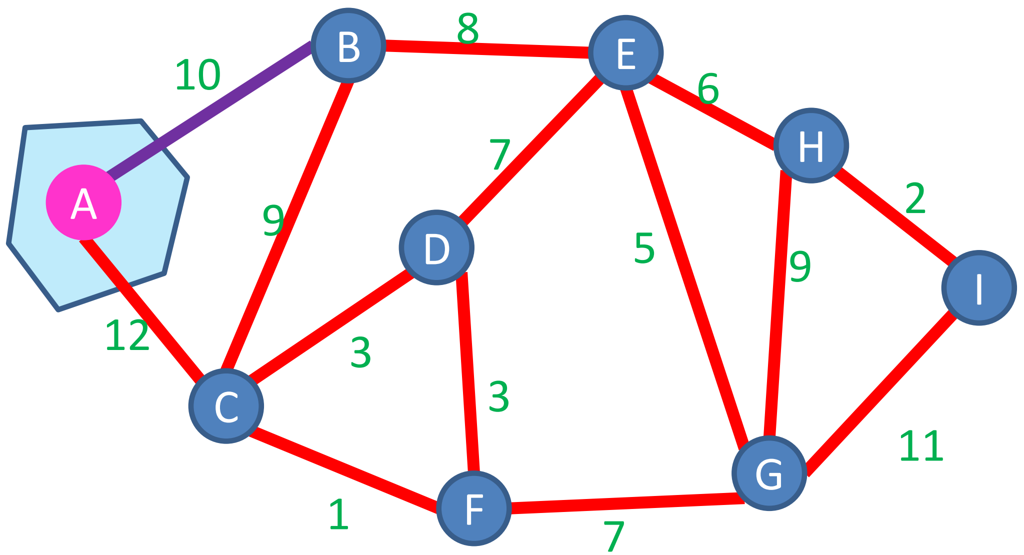
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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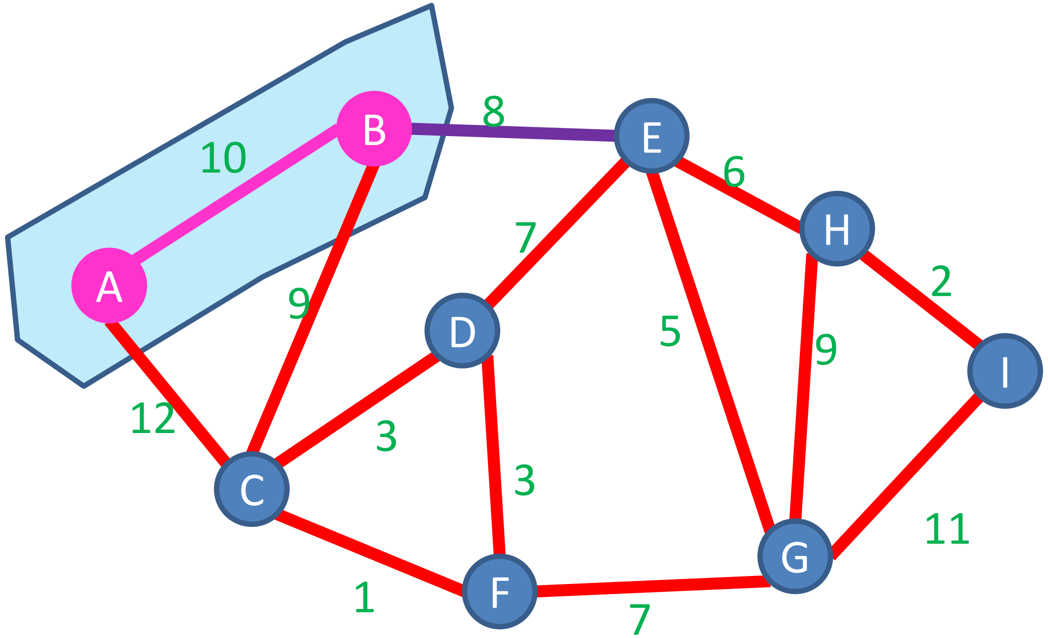

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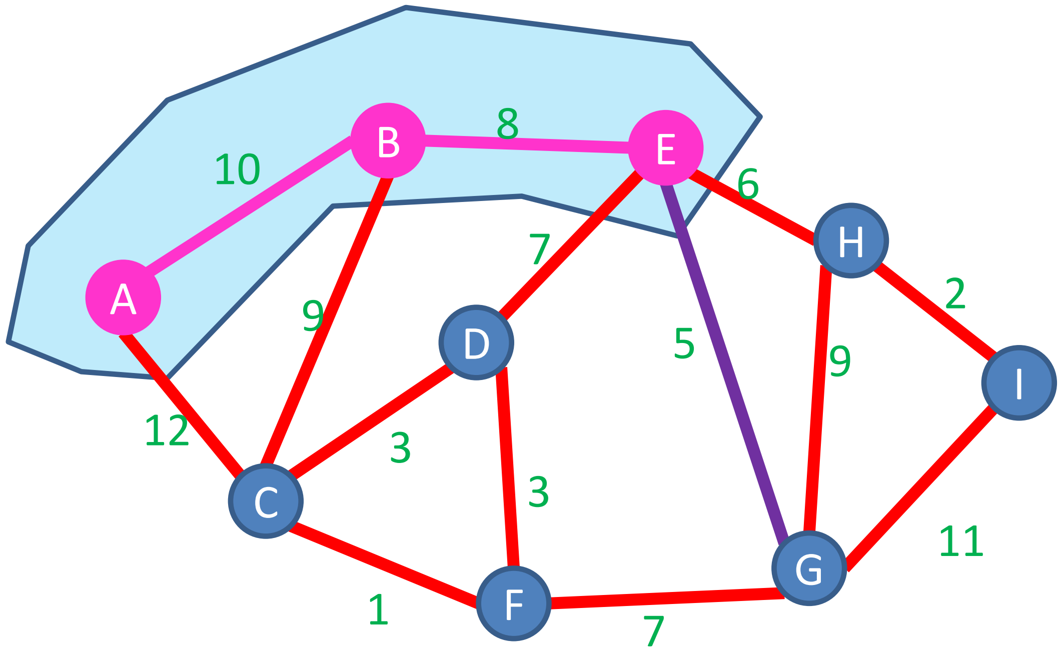

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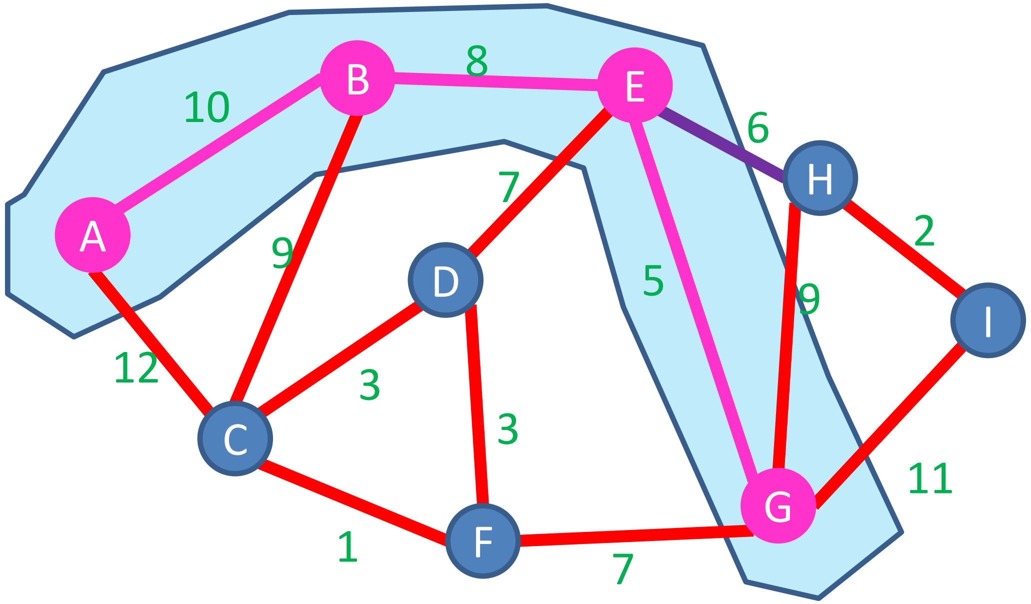

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Pick a start node

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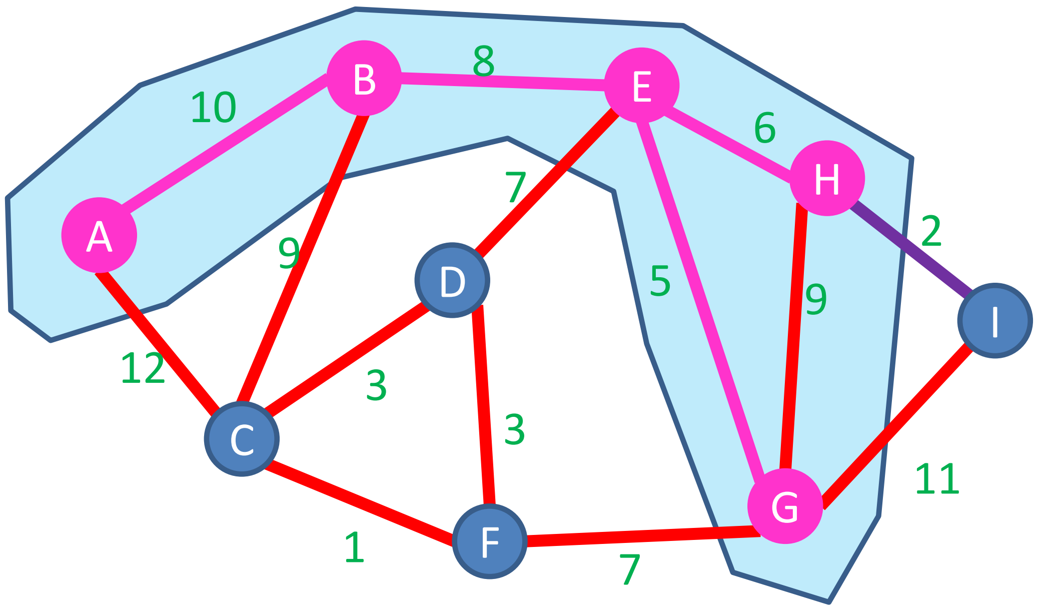
Keep edges in a Heap

$O(E \log V)$



Prim's Algorithm

Initialize $d_v = \infty$ for each node v

Keep a priority queue PQ of nodes, using d_v as key

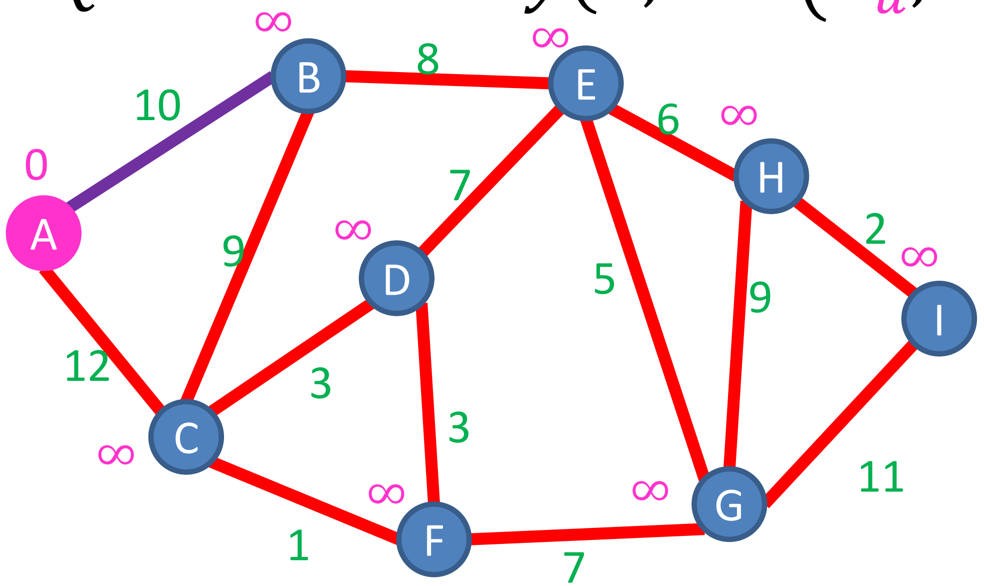
Pick a start node s, set $d_s = 0$

While PQ is not empty:

$\quad v = PQ.extractmin()$

$\quad$ for each $u \in V$ s.t. $(v, u) \in E$:

$\quad\quad PQ.decreaseKey(u, \min(d_u, w(v, u)))$



d_u is the cost to add node u to the tree with only one edge

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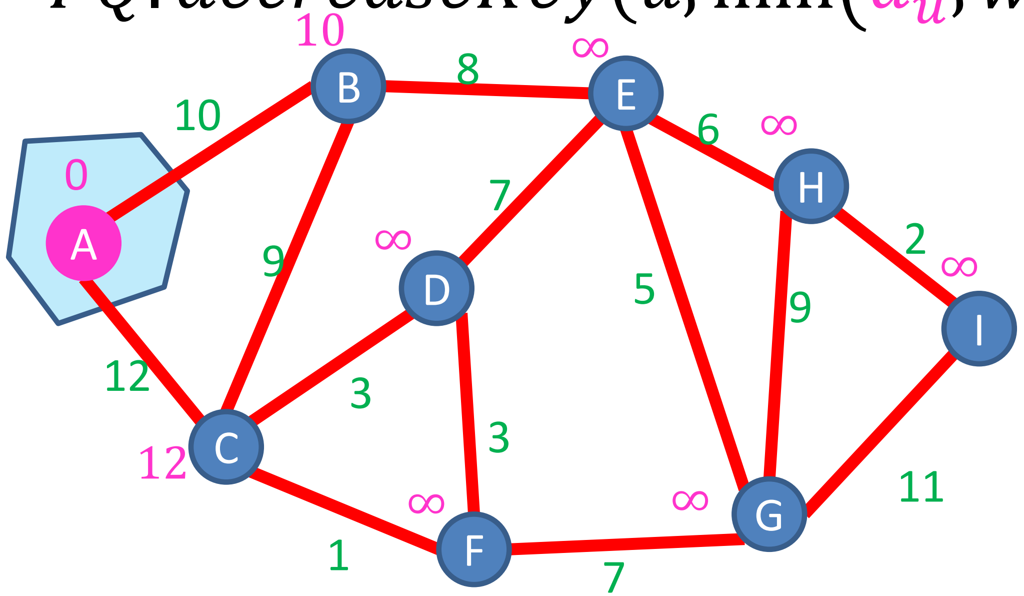

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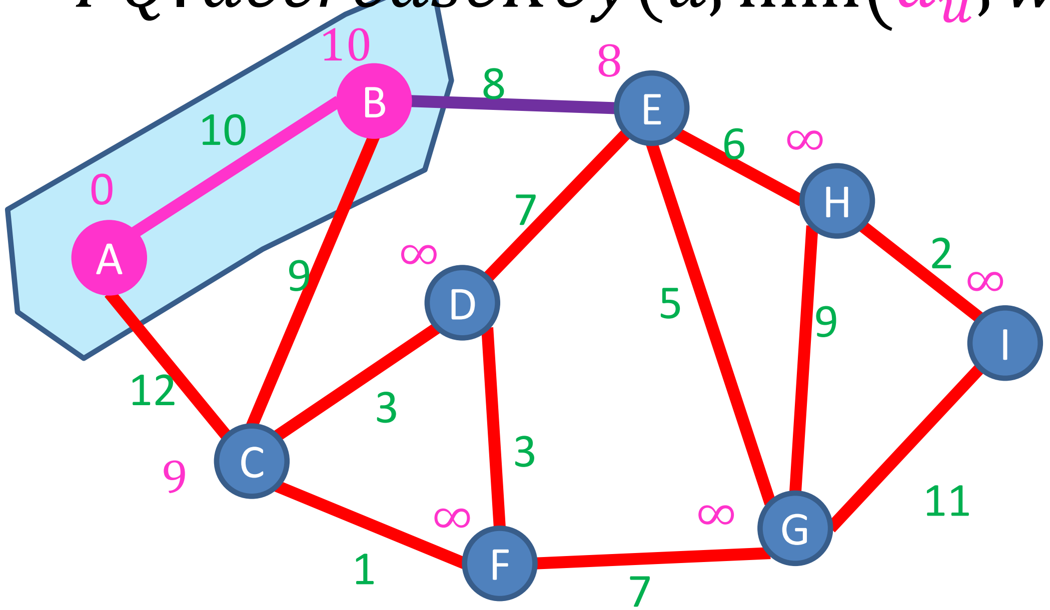


Prim's Algorithm

Initialize $d_v = \infty$ for each node v

Keep a priority queue PQ of nodes, using d_v as key

Pick a start node s, set $d_s = 0$

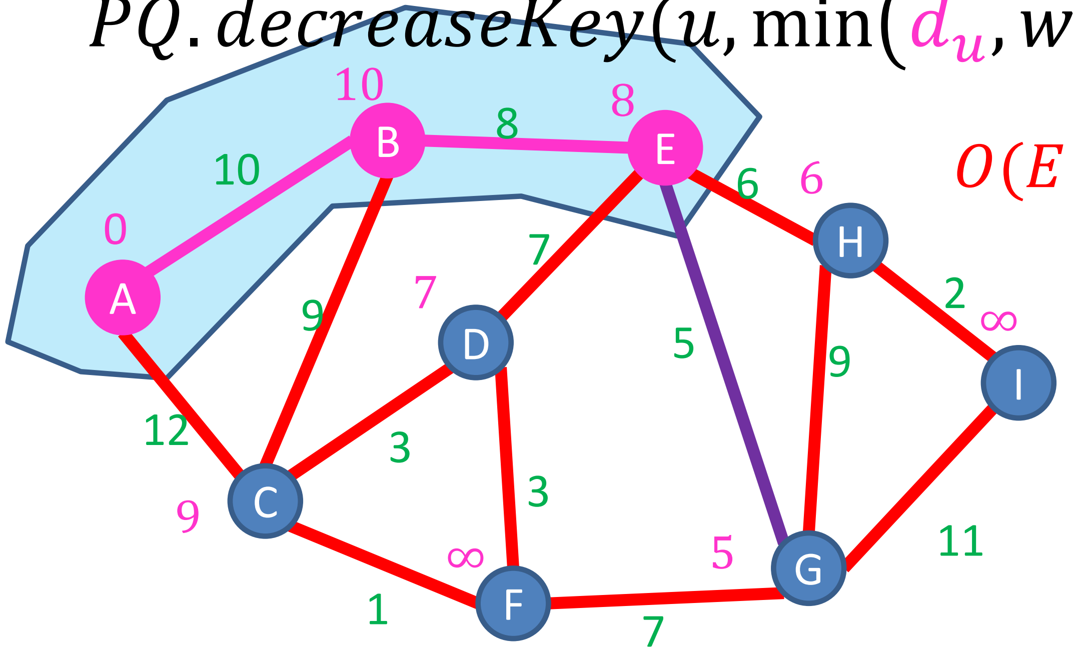
While PQ is not empty: V loops

$v = PQ.extractmin()$ $O(\log V)$

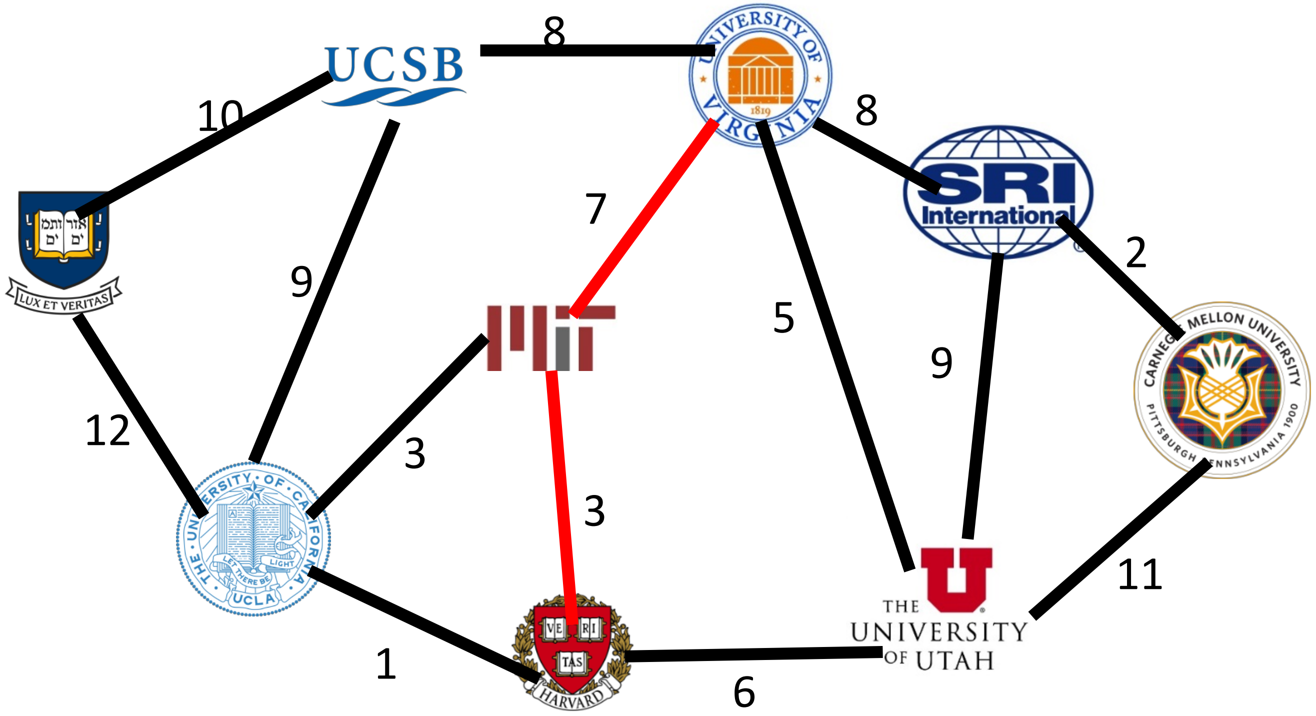
for each $u \in V$ s.t. $(v, u) \in E$: E times total

$PQ.decreaseKey(u, \min(d_u, w(v, u)))$ $O(\log V)$

$O(E \log V + V \log V)$



Single-Source Shortest Path



Find the quickest way to get from UVA to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \to v$ (call this weight $\delta(s, v)$)

(assumption: all edge weights are positive)

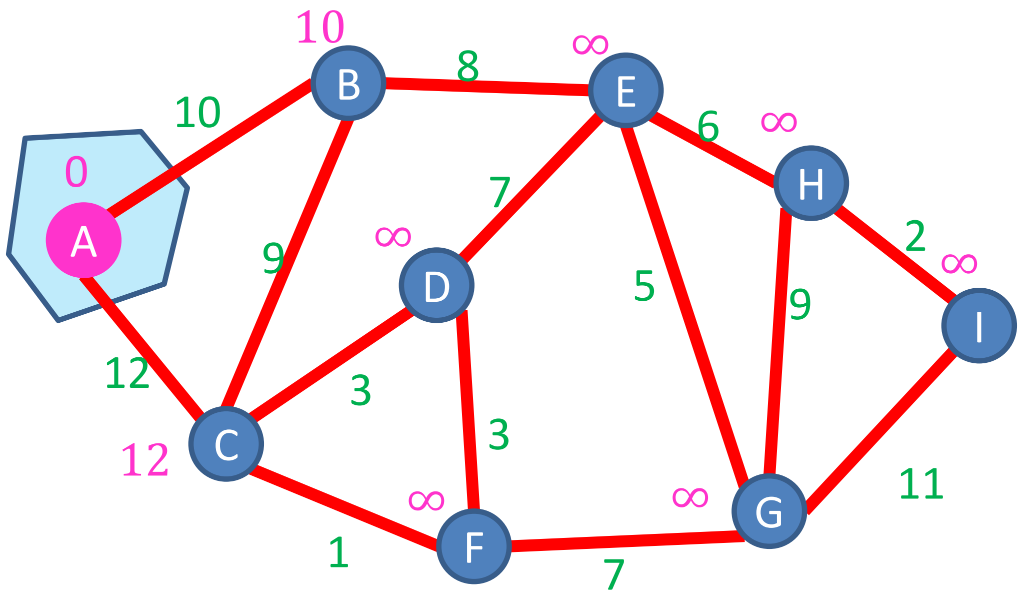
Dijkstra's Algorithm

Given some start node s

Start with an empty tree A

Repeat $V - 1$ times:

    Add the "nearest" node not yet in A



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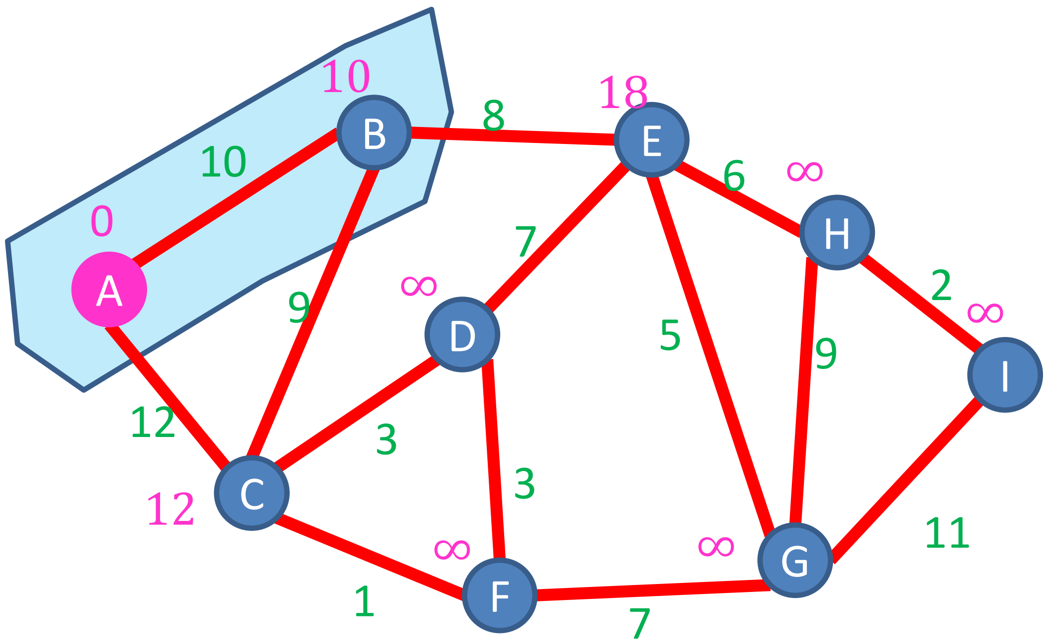

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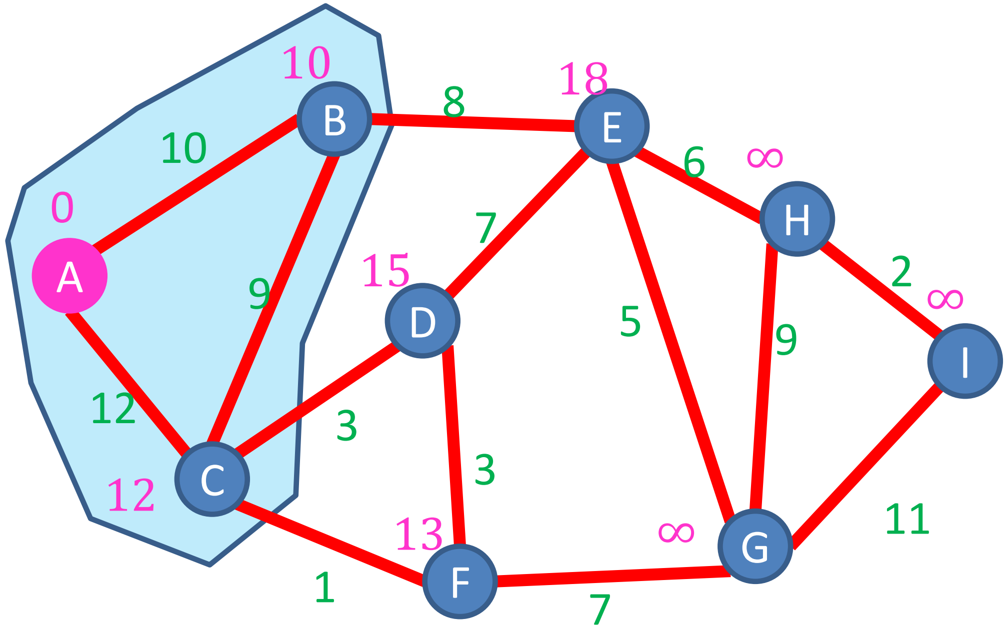

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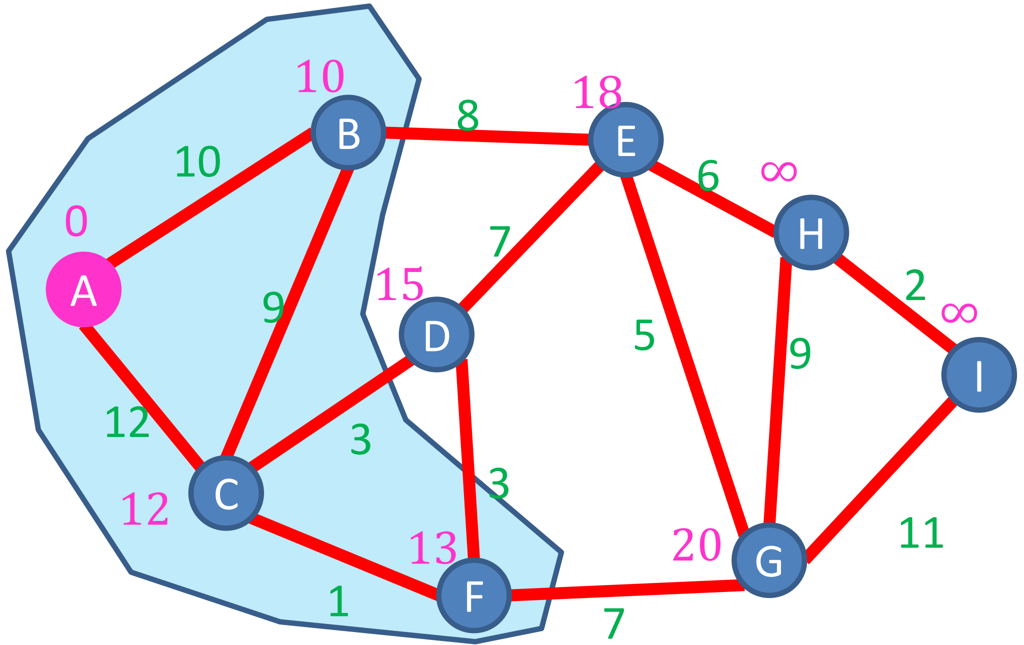
VERY similar to Prim's!



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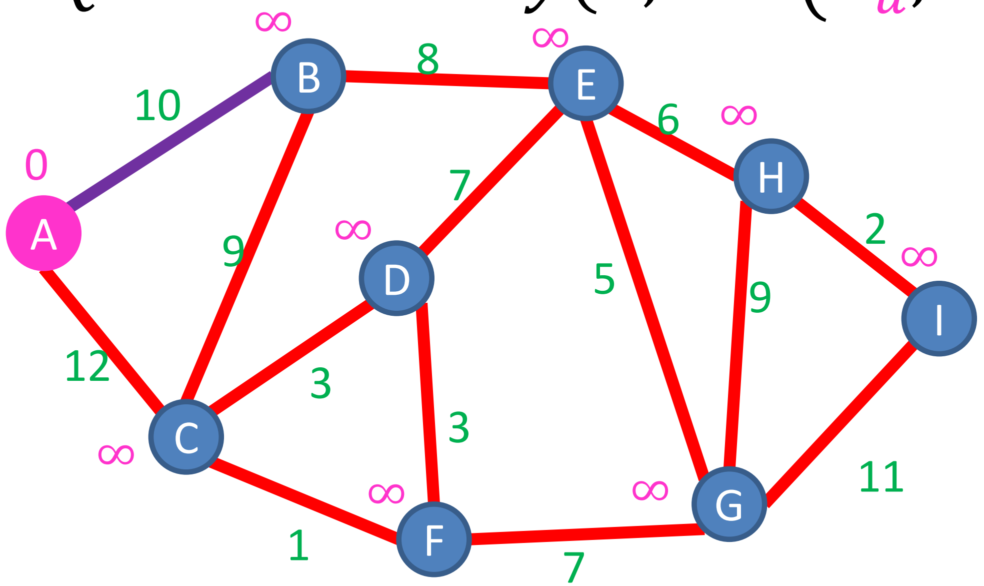

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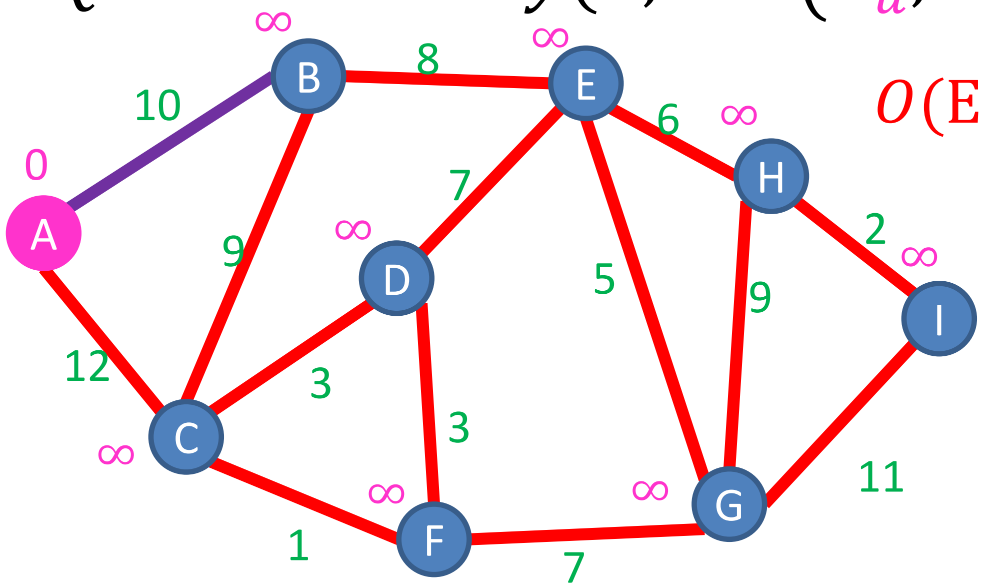
$\quad v = PQ.extractmin()$ $O(\log V)$

$\quad$ for each $u \in V$ s.t. $(v, u) \in E$: E times total

$\quad\quad PQ.decreaseKey(u, \min(d_u, d_v + w(v, u)))$ $O(\log V)$

$O(\mathrm{E} \log V + V \log V)$



Dijkstra's Algorithm Proof Strategy

- Proof by induction
- Idea: show that when node u is removed from the priority queue, $d_u = \delta(s, u)$
  - Claim 1: when u is removed from the queue, $d_u \geq \delta(s, u)$
    - i.e. d_u is at least the length of the shortest path
  - Claim 2: if we consider any path $(s, \ldots, u)$, $w(s, \ldots, u) \geq d_u$
    - i.e. d_u is no longer than any other path from s to u, including the shortest one

Proof of Dijkstra's

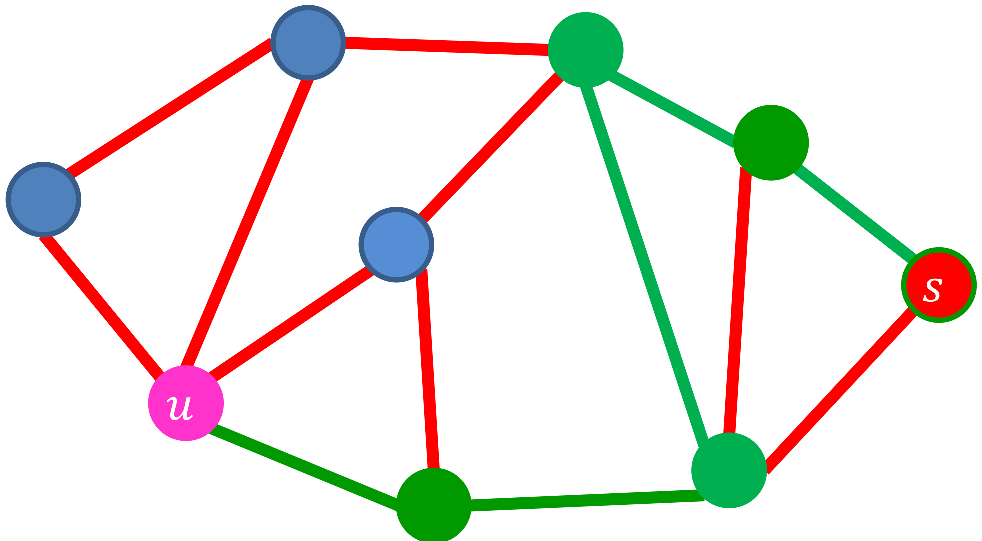
- Base case:
  - $i = 0, u = v_1 = s, \delta(s, v_1) = 0$
- Assume that nodes $v_1 = s, \dots, v_i$ have been removed from PQ already, and for each of them $d_{v_i} = \delta(s, v_i)$
- Let node u be the $(i + 1)^{th}$ node extracted

Proof of Dijkstra's: Claim 1

- Let node u be the $(i+1)^{th}$ node extracted
- Claim 1: $d_u \geq \delta(s,u)$
  - Proof: node u has a path of weight d_u from s
    - Discovering a path was how we updated the key!
  - Since d_u is the weight of SOME path, its weight is at least that of the SHORTEST path

Proof of Dijkstra's: Claim 2

- Let node u be the $(i+1)^{th}$ node extracted
- for any path $(s, \ldots, u)$, $w(s, \ldots, u) \geq d_u$
- Extracted nodes define a cut of the graph
- Let edge (x, y) be the last edge in this path which crosses the cut



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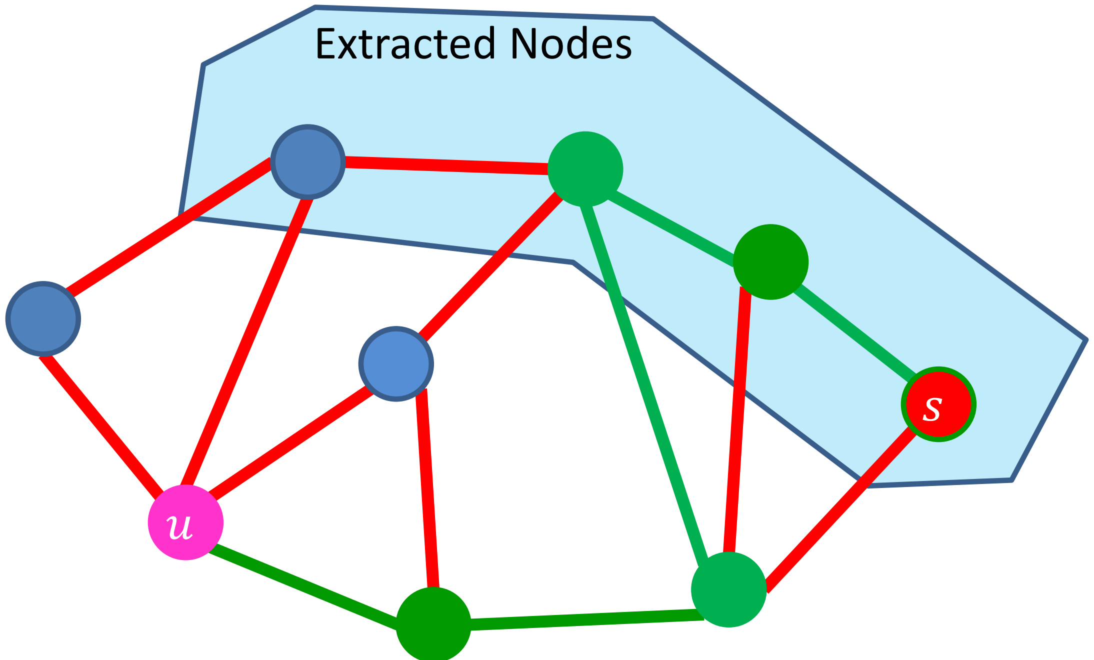

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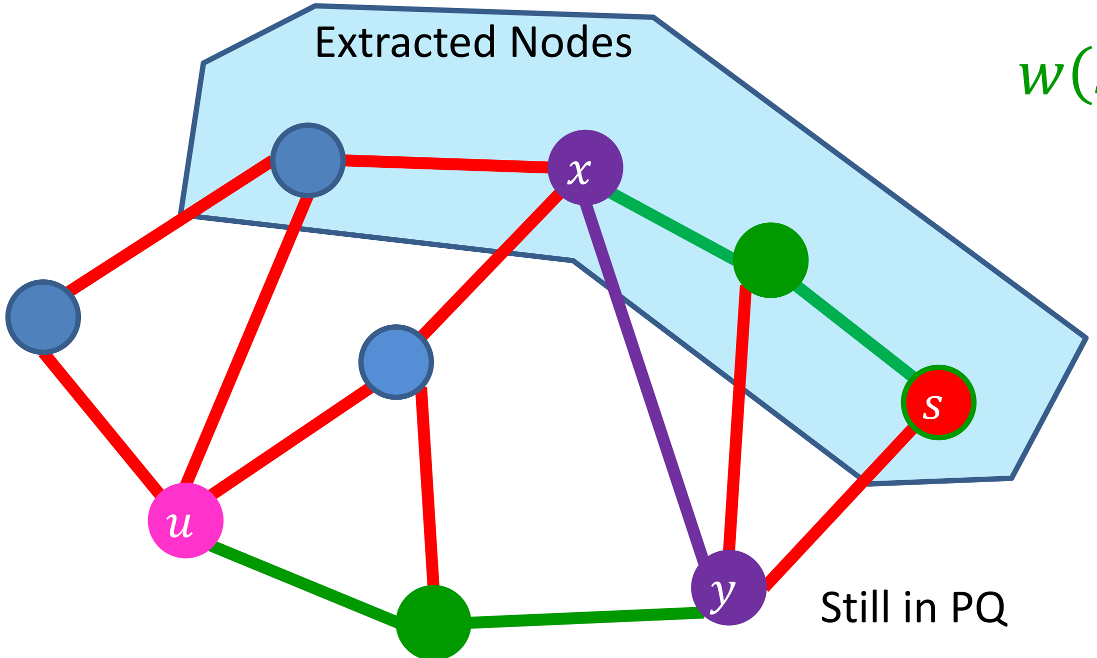


We updated y's key d_y when we extracted x **if** $d_x + w(x, y) < d_y$

$d_x = \delta(s, x)$

$$w(s, \dots, u) \geq \delta(s, x) + w(x, y) + w(y, \dots, u)$$
$$\geq d_y + w(y, \dots, u)$$ By definition
$$\geq d_u + w(y, \dots, u)$$ Because otherwise, u would not be next extracted
$$\geq d_u$$ No negative edge weights

Proof of Dijkstra's: Finale

- Claim 1: $d_u \geq \delta(s, u)$
- Claim 2: $d_u \leq w(s, \ldots, u)$ for any path from s to u (including the shortest one)
- 1&2 Together: $w(s, \ldots, u) \geq d_u \geq \delta(s, u)$
  - therefore $\delta(s, u) \geq d_u \geq \delta(s, u)$
  - $d_u = \delta(s, u)$

Breadth-First Search

- Input: a node s
- Behavior: Start with node s, visit all neighbors of s, then all neighbors of neighbors of s, …
- Output: lots of choices!
  - Is the graph connected?
  - Is there a path from s to u?
  - $\boxed{\text{Shortest number of “hops” from } s \text{ to } u}$

Sounds like Dijkstra’s!

Dijkstra's Algorithm

Initialize $d_v = \infty$ for each node v

Keep a priority queue PQ of nodes, using d_v as key

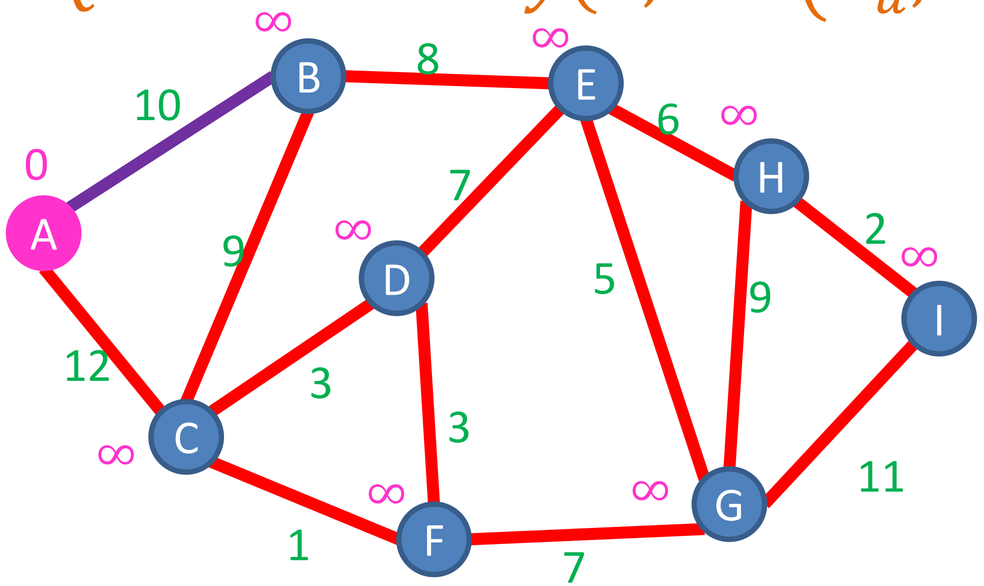
Pick a start node s, set $d_s = 0$

While PQ is not empty: Replace with a (plain-old) Queue

$$v = PQ.extractmin()$$

for each $u \in V$ s.t. $(v, u) \in E$:

$$PQ.decreaseKey(u, \min(d_u, d_v + w(v, u)))$$



BFS

Keep a queue Q of nodes

Pick a start node s

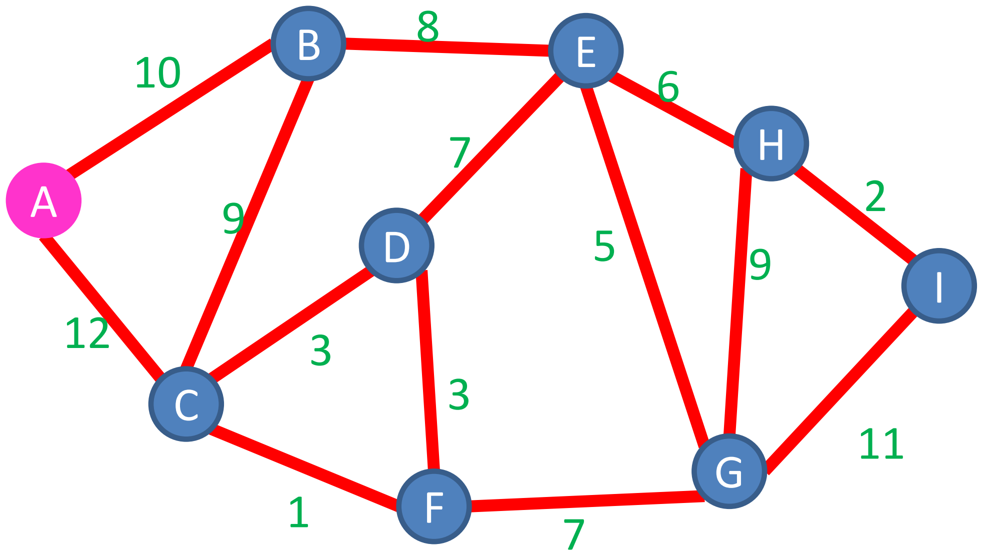
$Q.enqueue(s)$

While Q is not empty:

  $v = Q.dequeue()$

  for each "unvisited" $u \in V$ s.t. $(v, u) \in E$:

    $Q.enqueue(u)$



BFS: Shortest "Hops" Path

Keep a queue Q of nodes

Pick a start node s

$Q.enqueue(s)$

$hops = 0$

While Q is not empty:

    $v = Q.dequeue()$

    $hops \mathrel{+}= 1$

    for each "unvisited" $u \in V$ s.t. $(v, u) \in E$:

        $u.hops = hops$

        $Q.enqueue(u)$