CS4102 Algorithms

Fall 2019

Warm up:

Show that $P = NP$

Today's Keywords

- Reductions
- P vs NP
- NP Hard, NP Completeness
- k-Independent Set
- k-Vertex Cover
- 3SAT
- k-Clique

CLRS Readings

- Chapter 34

Homeworks

- HW9 due Thursday 12/5 at 11pm
  - Reductions, Graphs
  - Written (LaTeX)
- HW10C due Thursday 12/5 at 11pm
  - Implement a problem from HW9
  - No late submissions

Final Exam

- Monday, December 9, 7pm in Maury 209 (our section)
  - Practice exam coming soon
  - Review session likely the weekend before
  - SDAC: please schedule for some time on Monday 12/9

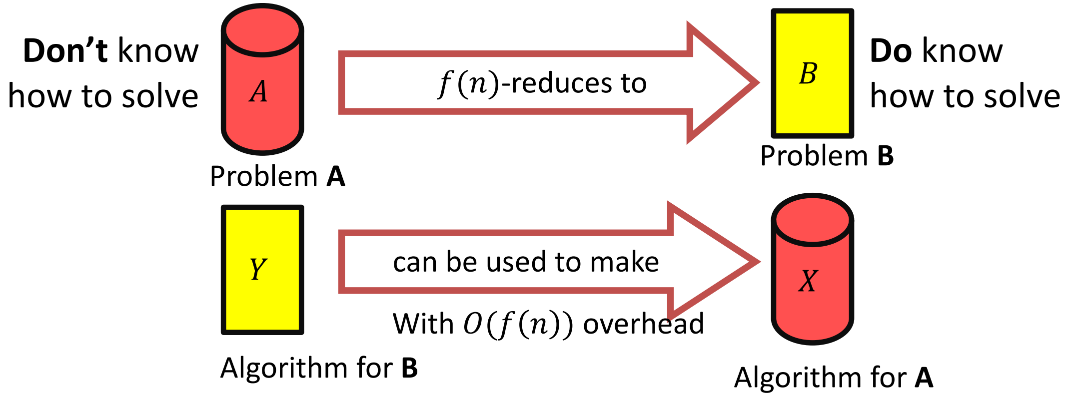
Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

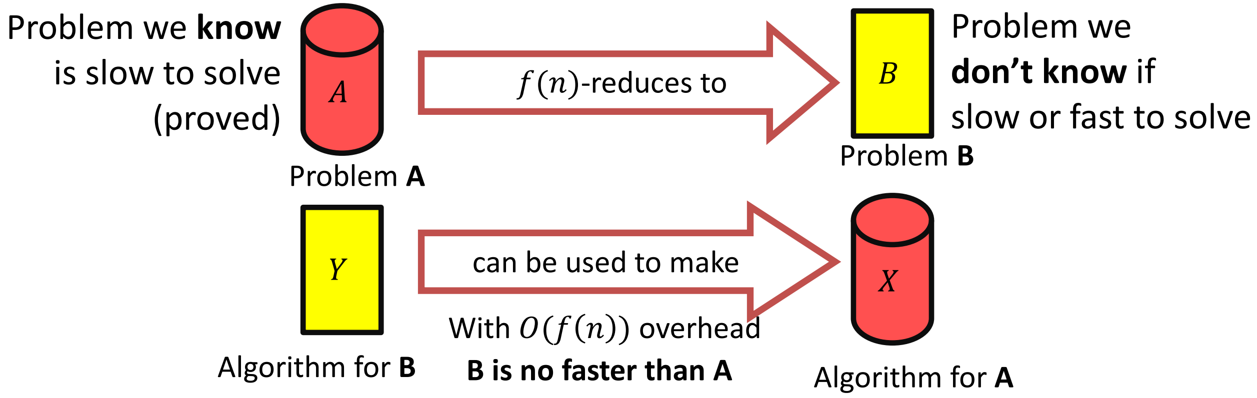
Reductions

Possible uses

- Use solver for B to solve A



- Prove lower bound for B by showing it's as hard as A



MacGyver’s Reduction

Problem we don’t know how to solve

A

Opening a door

Aim duct at door, insert keg

Solution for $\boldsymbol{A}$

Keg cannon battering ram

Problem we do know how to solve

B

Lighting a fire

How?

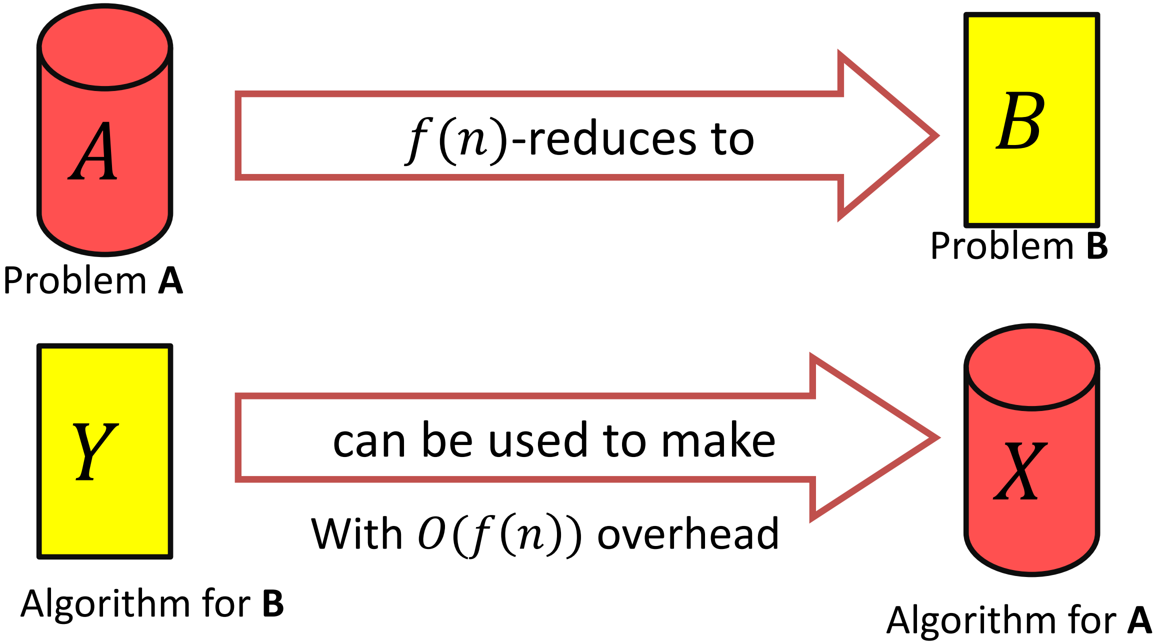
Solution for $\boldsymbol{B}$

Alcohol, wood, matches

Put fire under the Keg

Reduction

Reduction Proof Notation



A is not a harder problem than B

$$A \le B$$

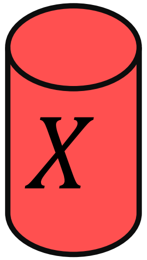
If A requires time $\Omega(f(n))$ time then B also requires $\Omega(f(n))$ time

$$A \le_{f(n)} B$$

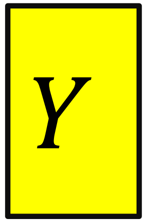
Or we could have solved A faster using B's solver!

Proof of Lower Bound by Reduction

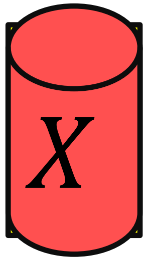
<u>To Show: Y is slow</u>



1. We know X is slow (by a proof)
(e.g., X = some way to open the door)



2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)



3. Show how to use Y to perform X quickly

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick

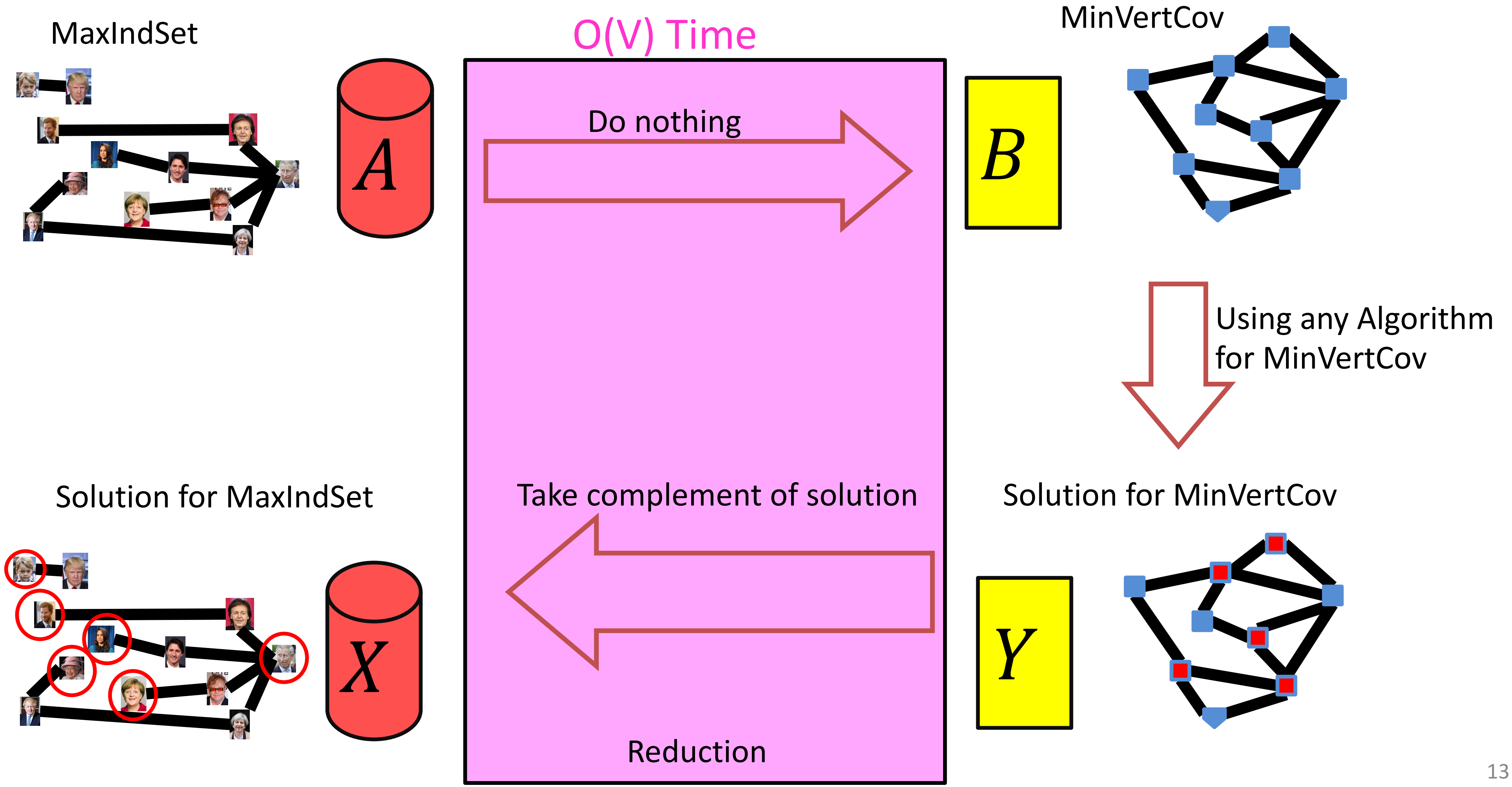
Maximum Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

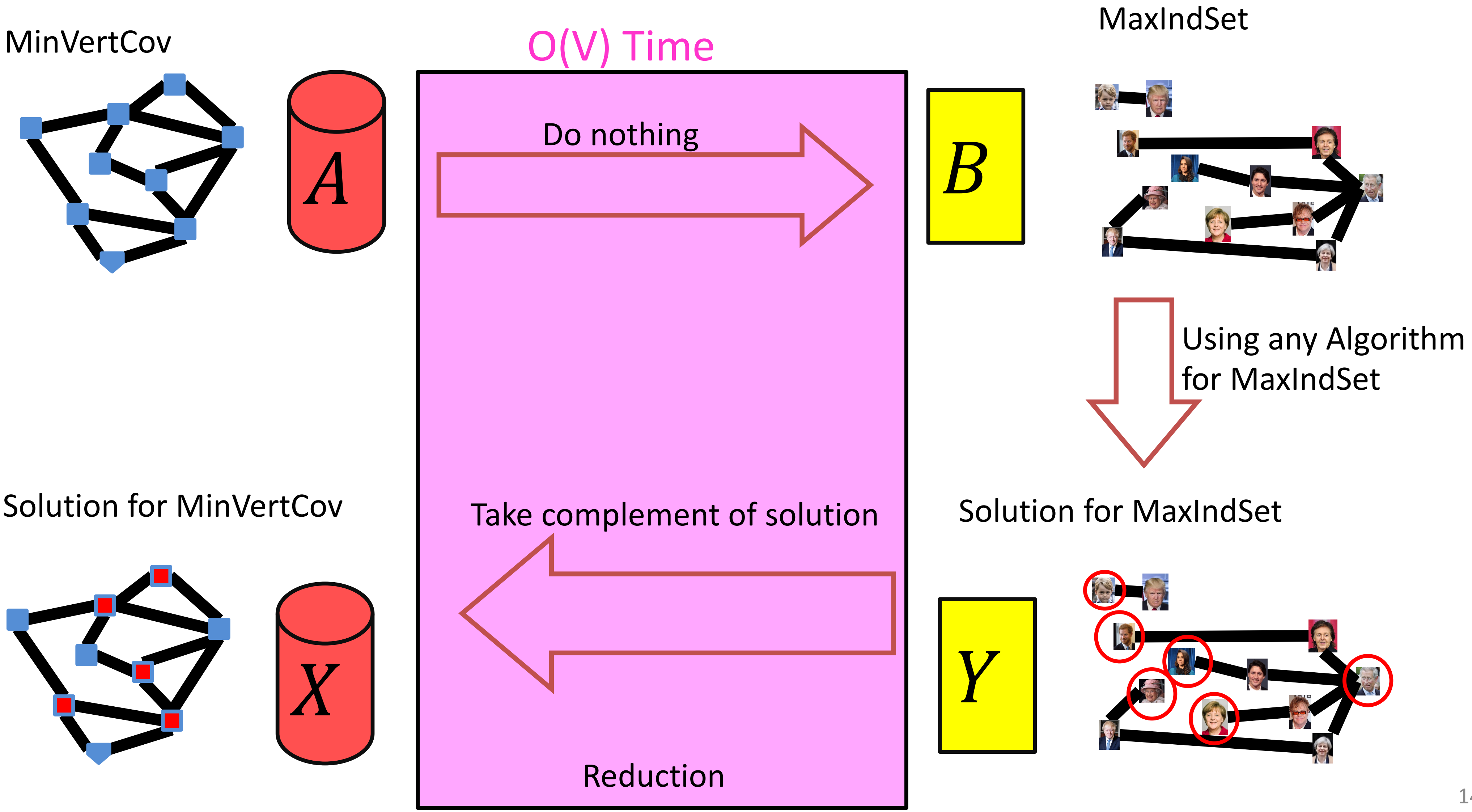
Minimum Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover: Given a graph $G = (V, E)$ find the minimum vertex cover C

MaxIndSet V-Time Reducible to MinVertCover



MinVertCover V-Time Reducible to MaxIndSet



Conclusion

- MaxIndSet and MinVertCov are either both fast, or both slow
  - Spoiler alert: We don't know which!
    - (But we think they're both slow)
  - Both problems are NP-Complete

k Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- k Independent Set Problem: Given a graph $G = (V, E)$ and a number k, **determine whether there is an independent set $\boldsymbol{S}$ of size $\boldsymbol{k}$**

k Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- k Vertex Cover: Given a graph $G = (V, E)$ and a number k, **determine whether there is a vertex cover $\boldsymbol{C}$ of size $\boldsymbol{k}$**

Problem Types

- Decision Problems:
  - Is there a solution?
    - Output is True/False
  - Is there a vertex cover of size k?

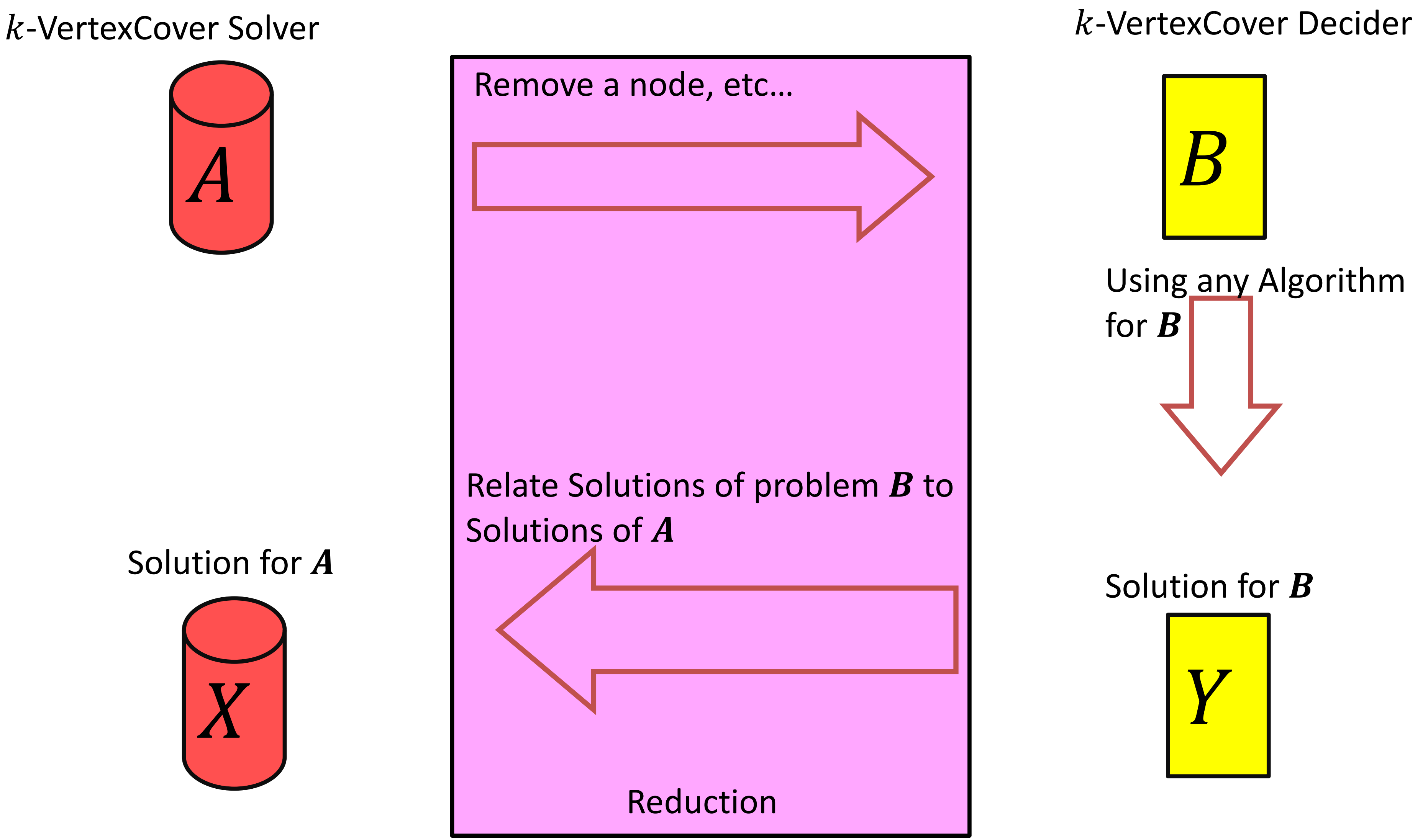
If we can solve this

- Search Problems:
  - Find a solution
    - Output is complex
  - Give a vertex cover of size k

Then we can solve this

- Verification Problems:
  - Given a potential solution, is it valid?
    - Output is True/False
  - Is **this** a vertex cover of size k**?**

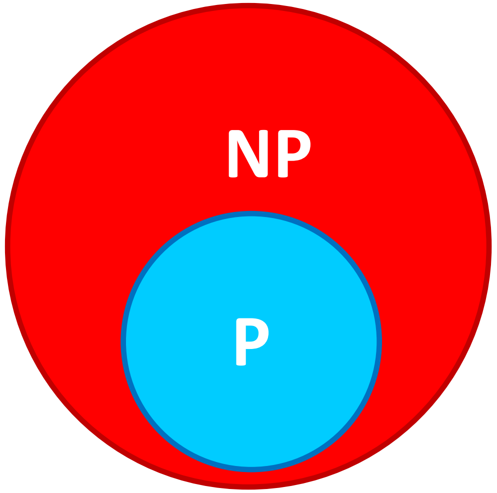
Reduction



k-VertexCover Solver
A
Remove a node, etc…
k-VertexCover Decider
B
Using any Algorithm for $\boldsymbol{B}$
Relate Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$
Solution for $\boldsymbol{A}$
X
Solution for $\boldsymbol{B}$
Y
Reduction

P vs NP

- P
  - Deterministic Polynomial Time
  - Problems solvable in polynomial time
    - $O(n^p)$ for some number p
- NP
  - Non-Deterministic Polynomial Time
  - Problems verifiable in polynomial time
    - $O(n^p)$ for some number p
- Open Problem: Does P=NP?
  - Certainly $P \subseteq NP$



k-Independent Set is NP

- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

  How can we verify it?

  1. Check that it's of size k $O(V)$
  2. Check that it's an independent set $O(V^2)$

k-Vertex Cover is NP

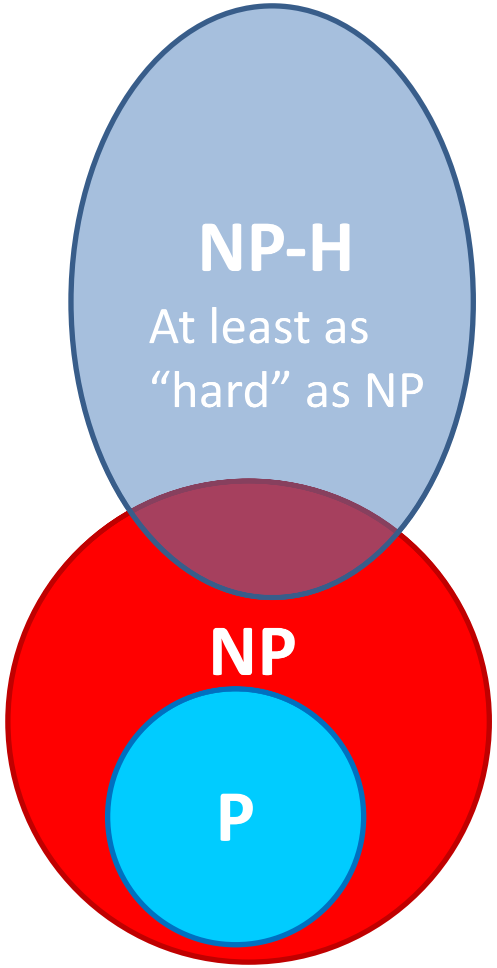
- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

    How can we verify it?

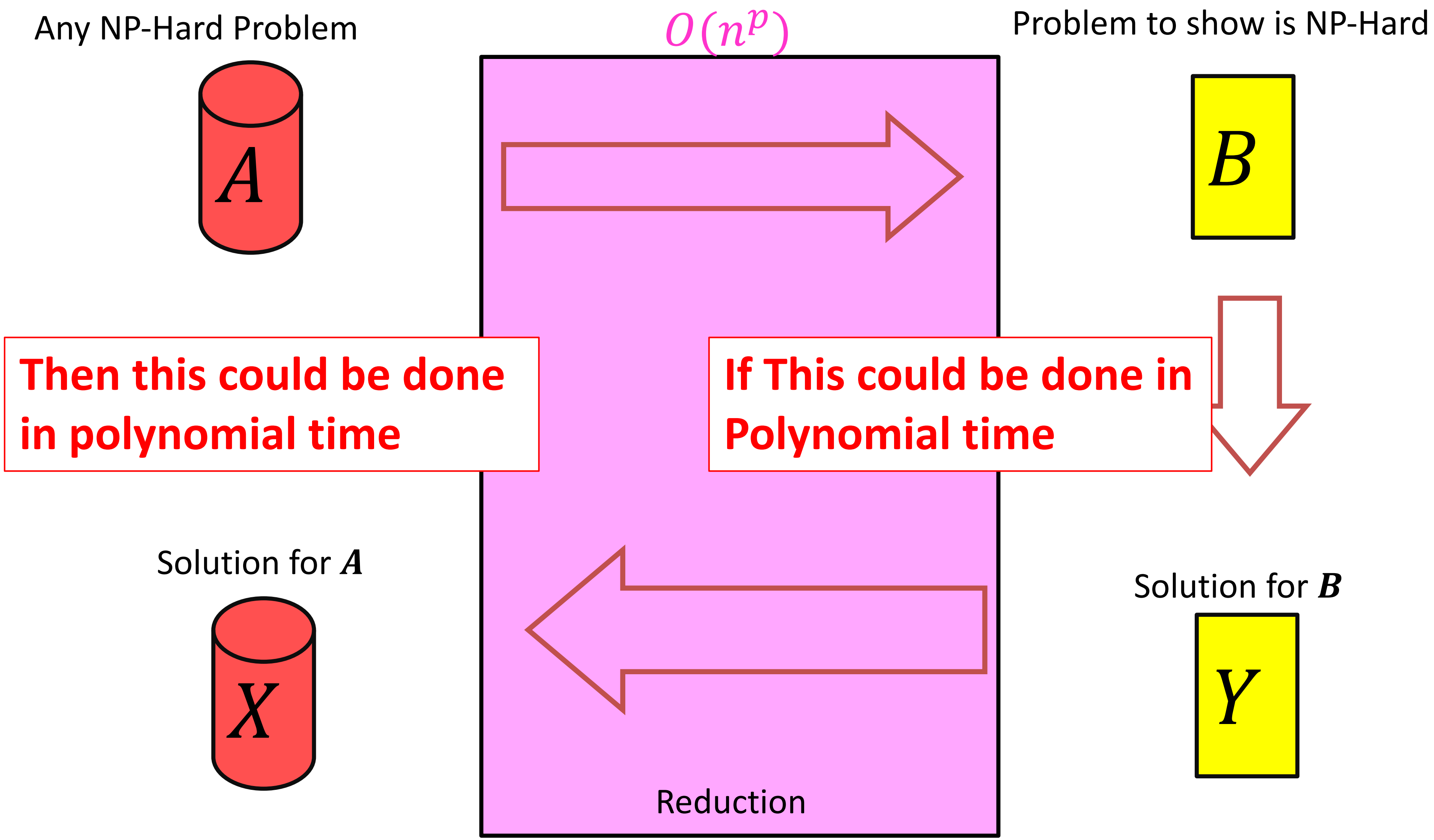
    1. Check that it's of size k $O(V)$
    2. Check that it's a Vertex Cover $O(E)$

NP-Hard

- How can we try to figure out if P=NP?
- Identify problems at least as "hard" as NP
  - If any of these "hard" problems can be solved in polynomial time, then all NP problems can be solved in polynomial time.
- Definition: NP-Hard:
  - B is NP-Hard if $\forall A \in NP, A \leq_p B$
  - $A \leq_p B$ means A reduces to B in polynomial time

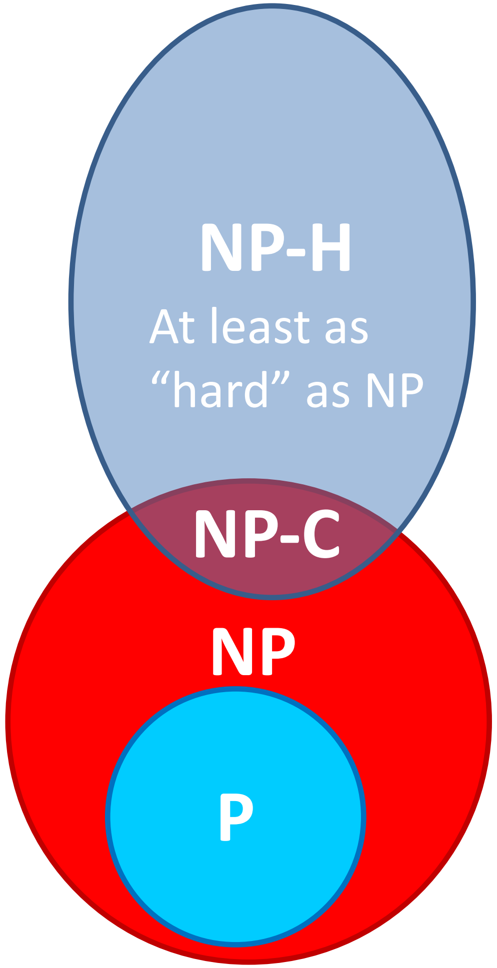


NP-Hardness Reduction



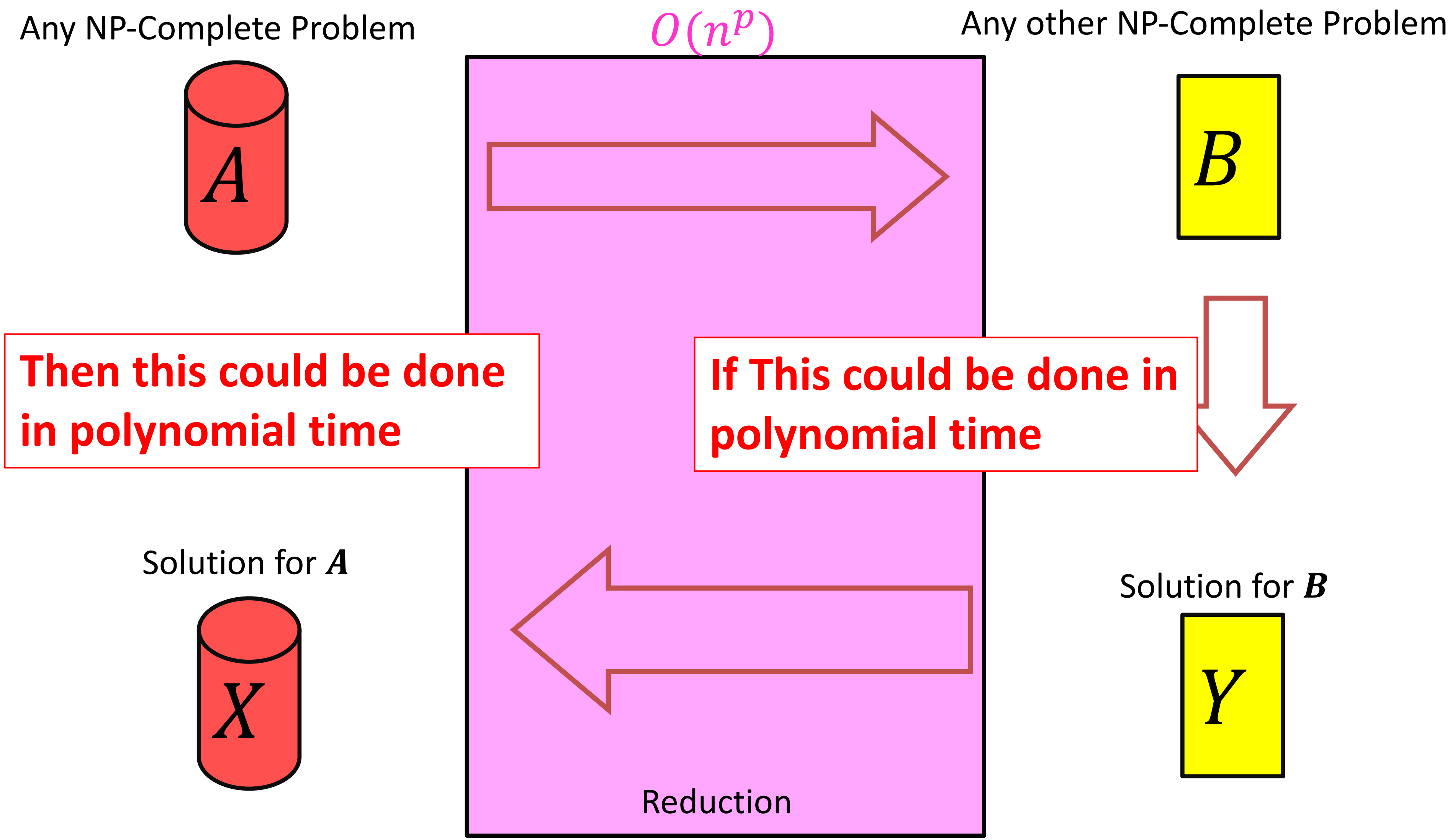
NP-Complete

- "Together they stand, together they fall"
- Problems solvable in polynomial time iff ALL NP problems are
- NP-Complete = NP ∩ NP-Hard
- **How to show a problem is NP-Complete?**
  - Show it belongs to NP
    - Give a polynomial time verifier
  - Show it is NP-Hard
    - Give a reduction from another NP-H problem

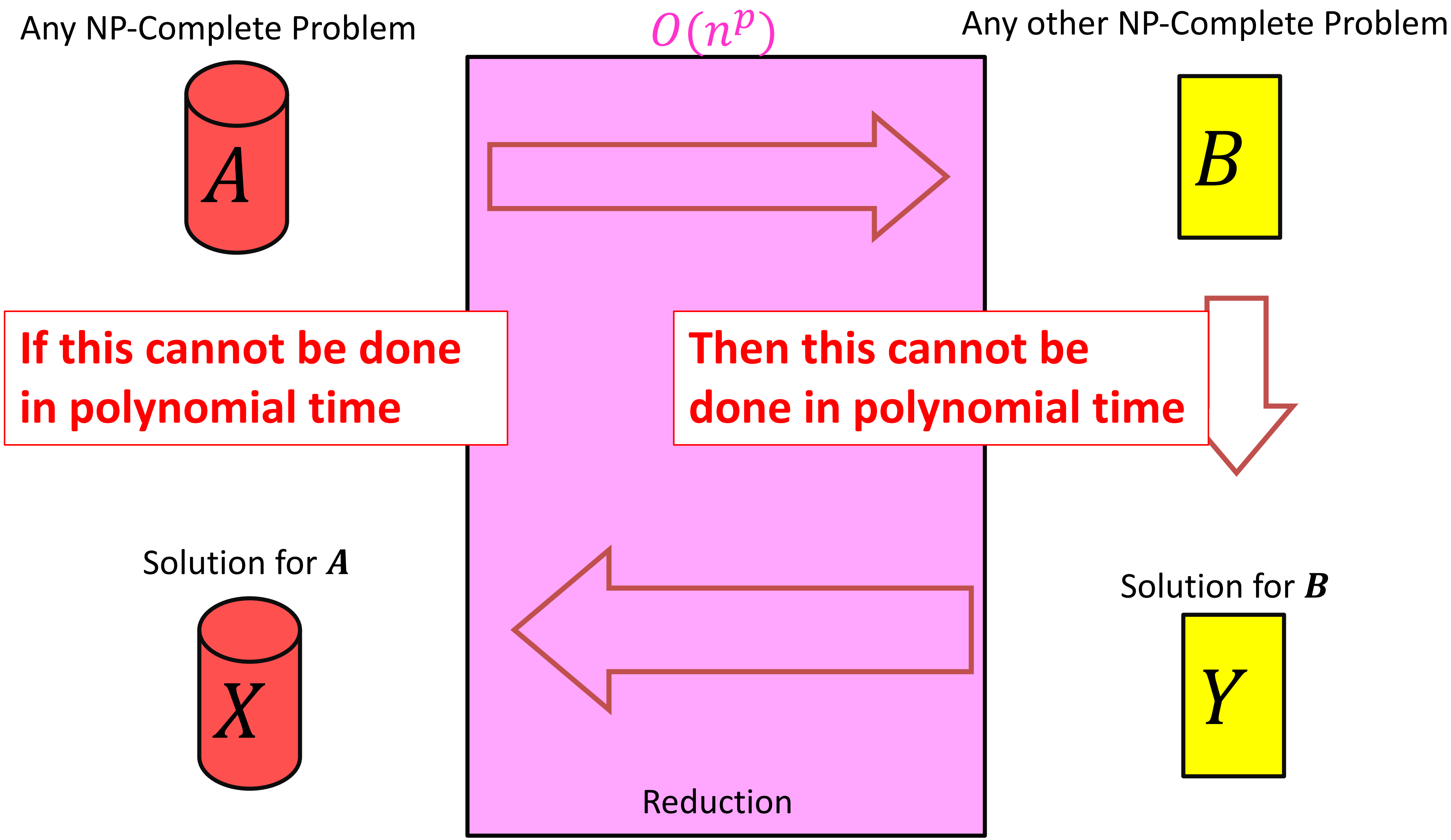


We now just need a FIRST NP-Hard problem

NP-Completeness



NP-Completeness



3-SAT

- Shown to be NP-Hard by Cook and Levin (independently)
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), Is there an assignment of true/false to each variable to make the formula true?

$$\underbrace{(x \vee y \vee z)}_{\text{Clause}} \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

Variables

$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

k-Independent Set is NP-Complete

1. Show that it belongs to NP
   – Give a polynomial time verifier
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – Show $3SAT \leq_p kIndSet$

Remember: k-Independent Set is NP

- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's an independent set $O(V^2)$

k-Independent Set is NP-Complete

1. Show that it belongs to NP
   – Give a polynomial time verifier
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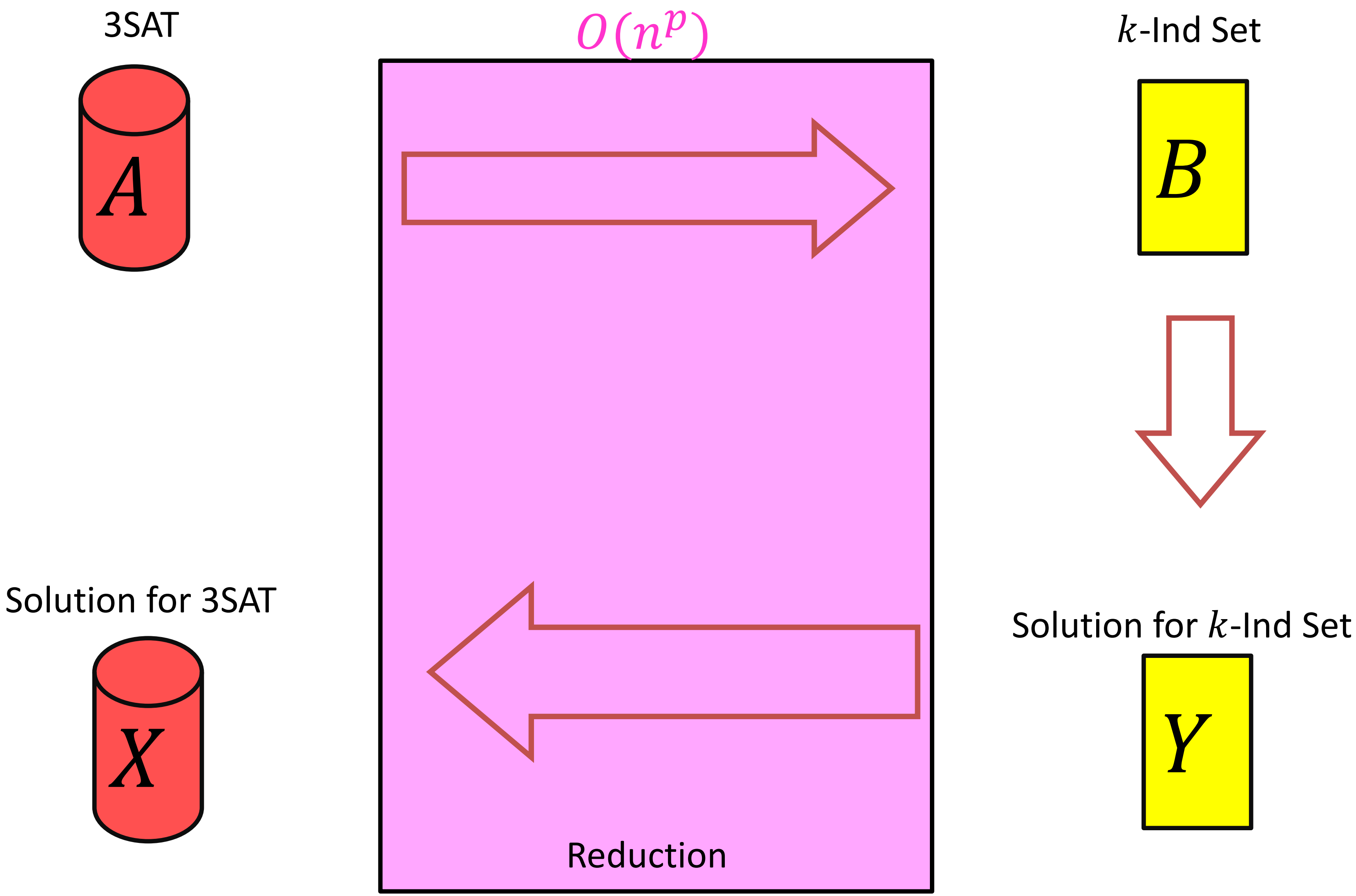
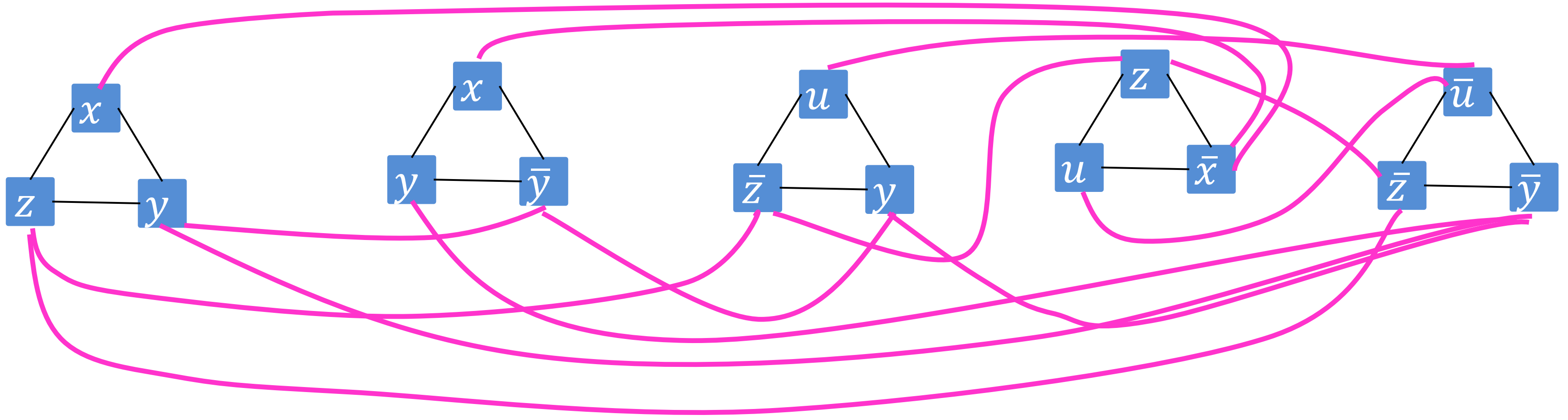
$3SAT \leq_p kIndSet$



Instance of 3SAT to Instance of kIndSet

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



For each clause, produce a triangle graph with its three variables as nodes

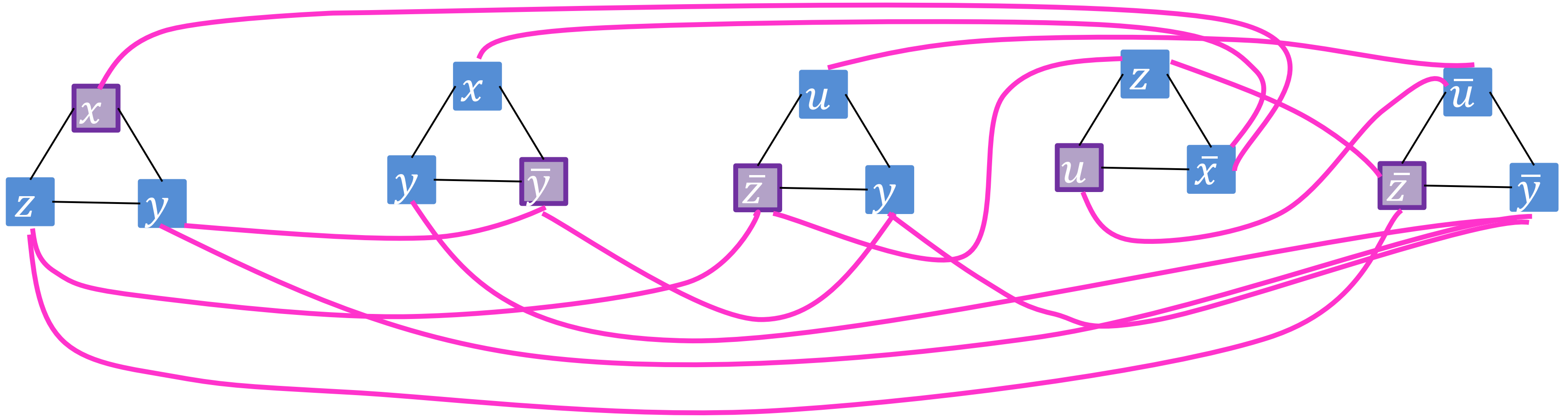
Connect each node to all of its opposites

Let k = number of clauses

There is a k-IndSet in this graph **iff** there is a satisfying assignment

kIndSet ⇒ Satisfying Assignment

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



$x = true$
$y = false$
$z = false$
$u = true$

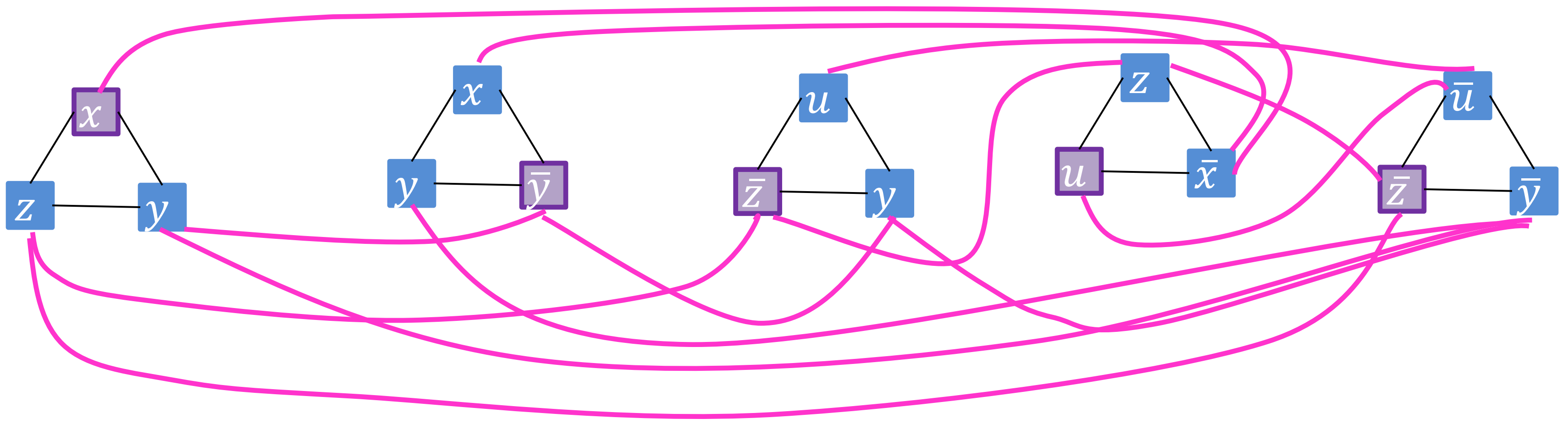
One node per triangle is in the Independent set:
because we can have exactly k total in the set, and 2 in a triangle would be adjacent

If x is selected in some triangle, $\bar{x}$ is not selected in any triangle:
Because every x is adjacent to every $\bar{x}$

Set the variable which each included node represents to "true"

Satisfying Assignment ⇒ kIndSet

$$(x \lor y \lor z) \land (x \lor \bar{y} \lor y) \land (u \lor y \lor \bar{z}) \land (z \lor \bar{x} \lor u) \land (\bar{x} \lor \bar{y} \lor \bar{z})$$



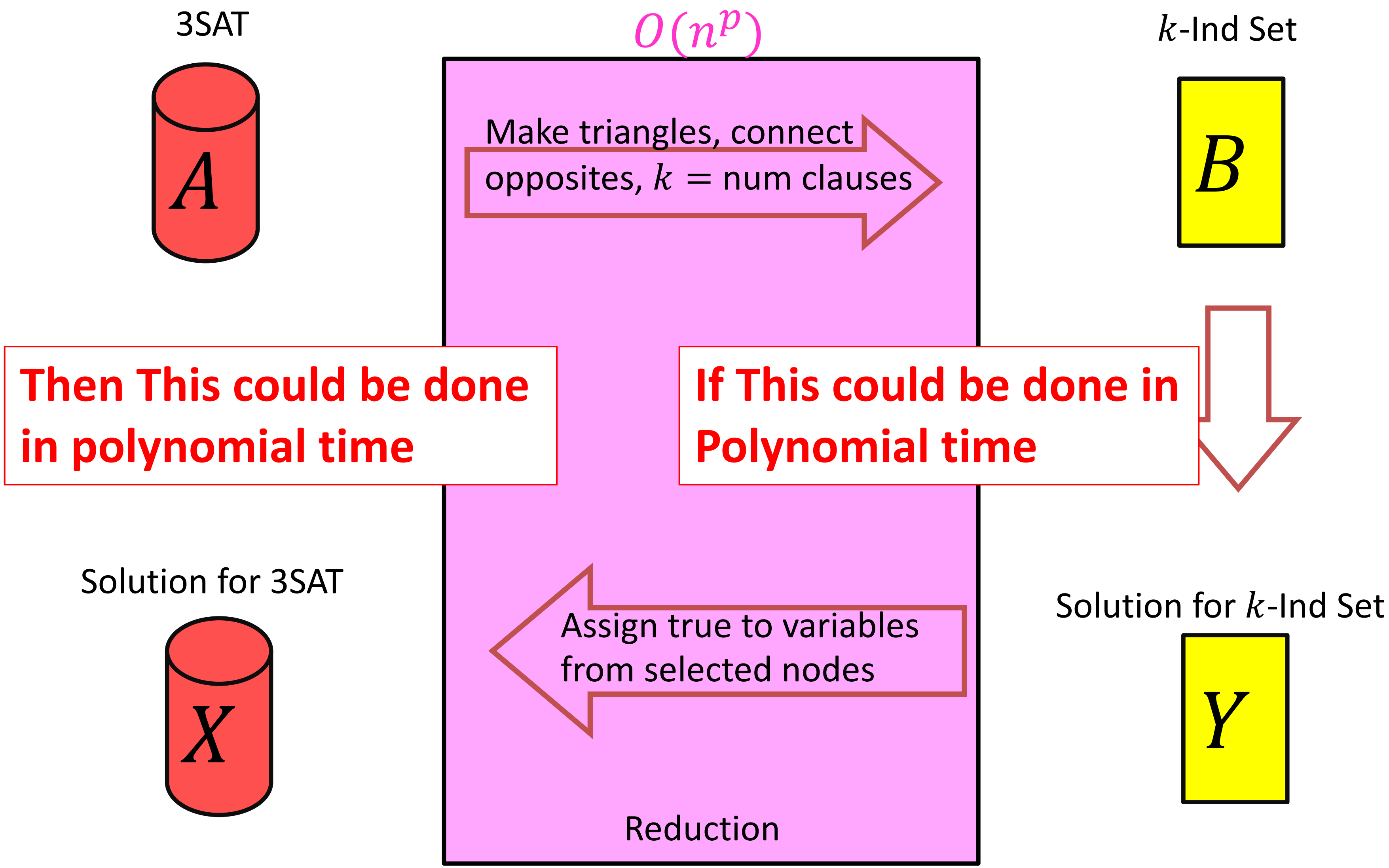
$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

Use one true variable from the assignment for each triangle

The independent set has k nodes, because there are k clauses

If any variable x is true then $\bar{x}$ cannot be true

$3SAT \leq_p kIndSet$



k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   – Give a polynomial time verifier
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – We showed $kIndSet \leq_p kVertCov$

Remember: k-Vertex Cover is NP

- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

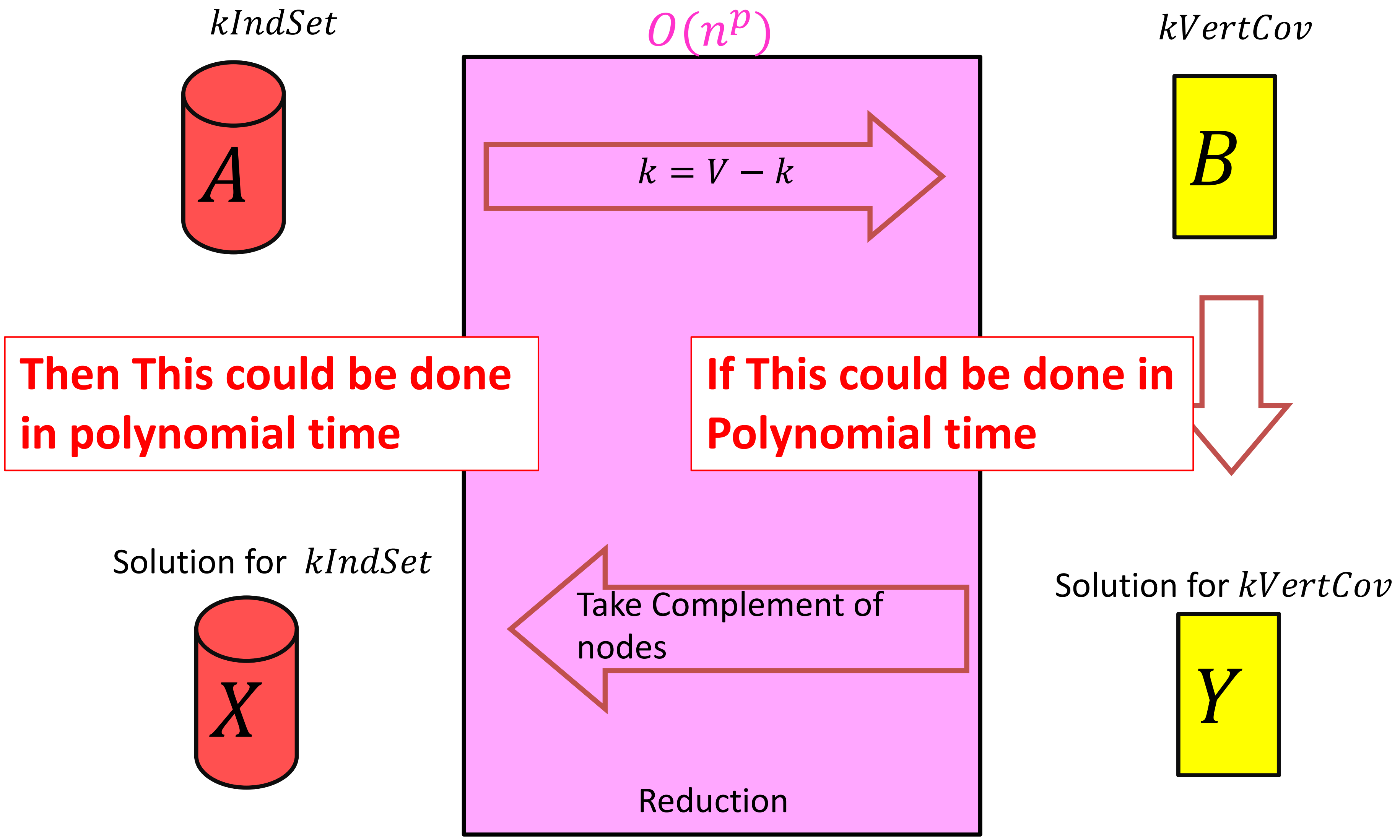
How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's a Vertex Cover $O(E)$

k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We showed $kIndSet \leq_p kVertCov$

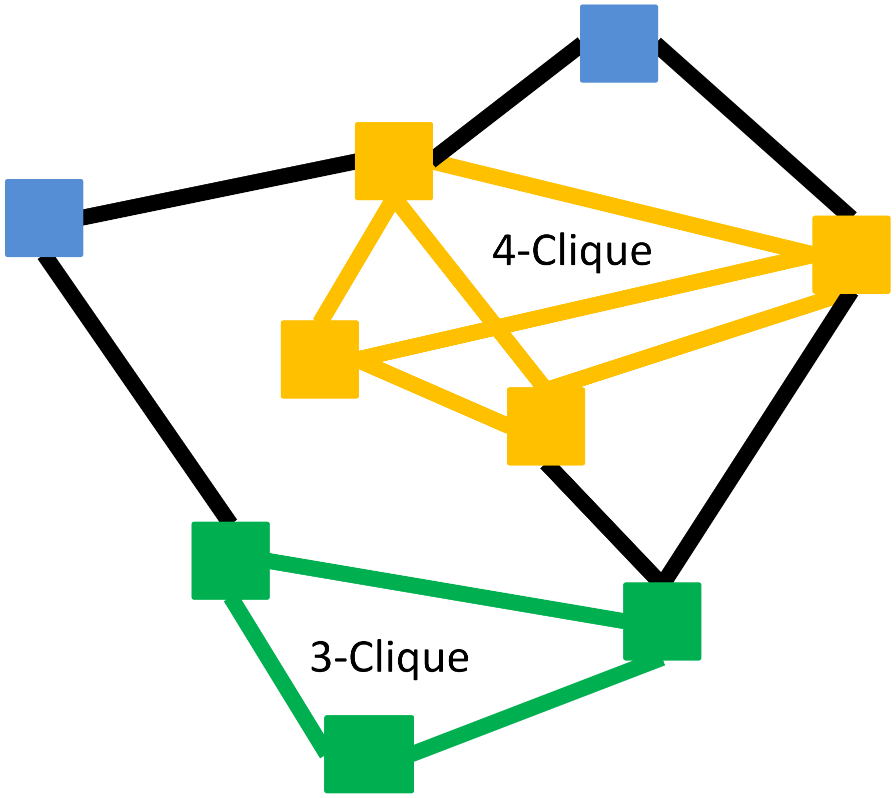
Remember: kIndSet $\leq_p$ kVertCov



k-Clique Problem

Given a graph G and a number k, is there a *clique* of size k?

- Clique: A complete subgraph



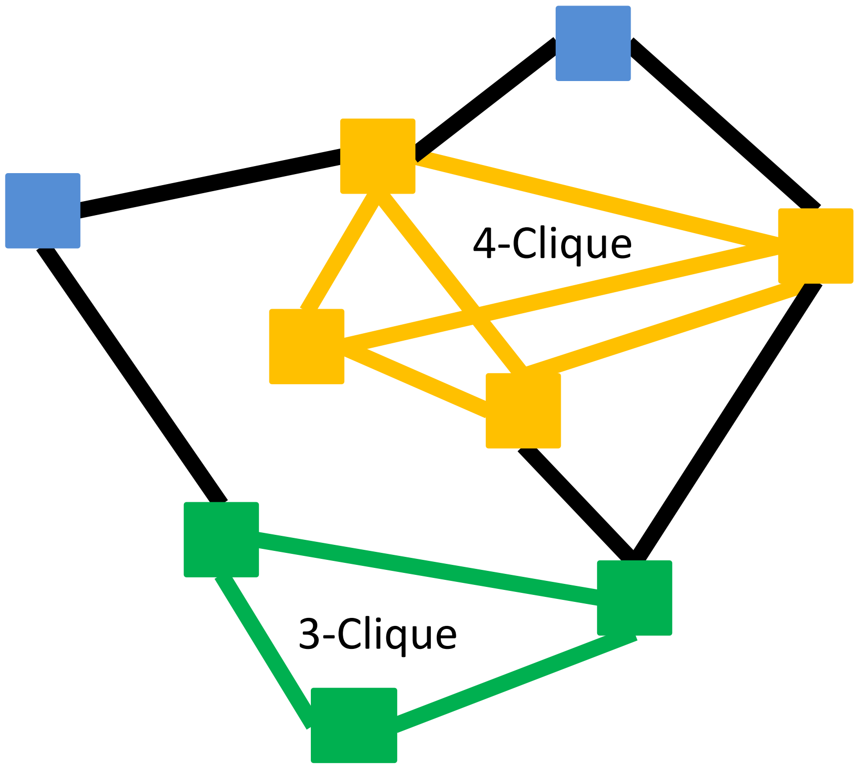
k-Clique is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We will show $3SAT \leq_p kClique$

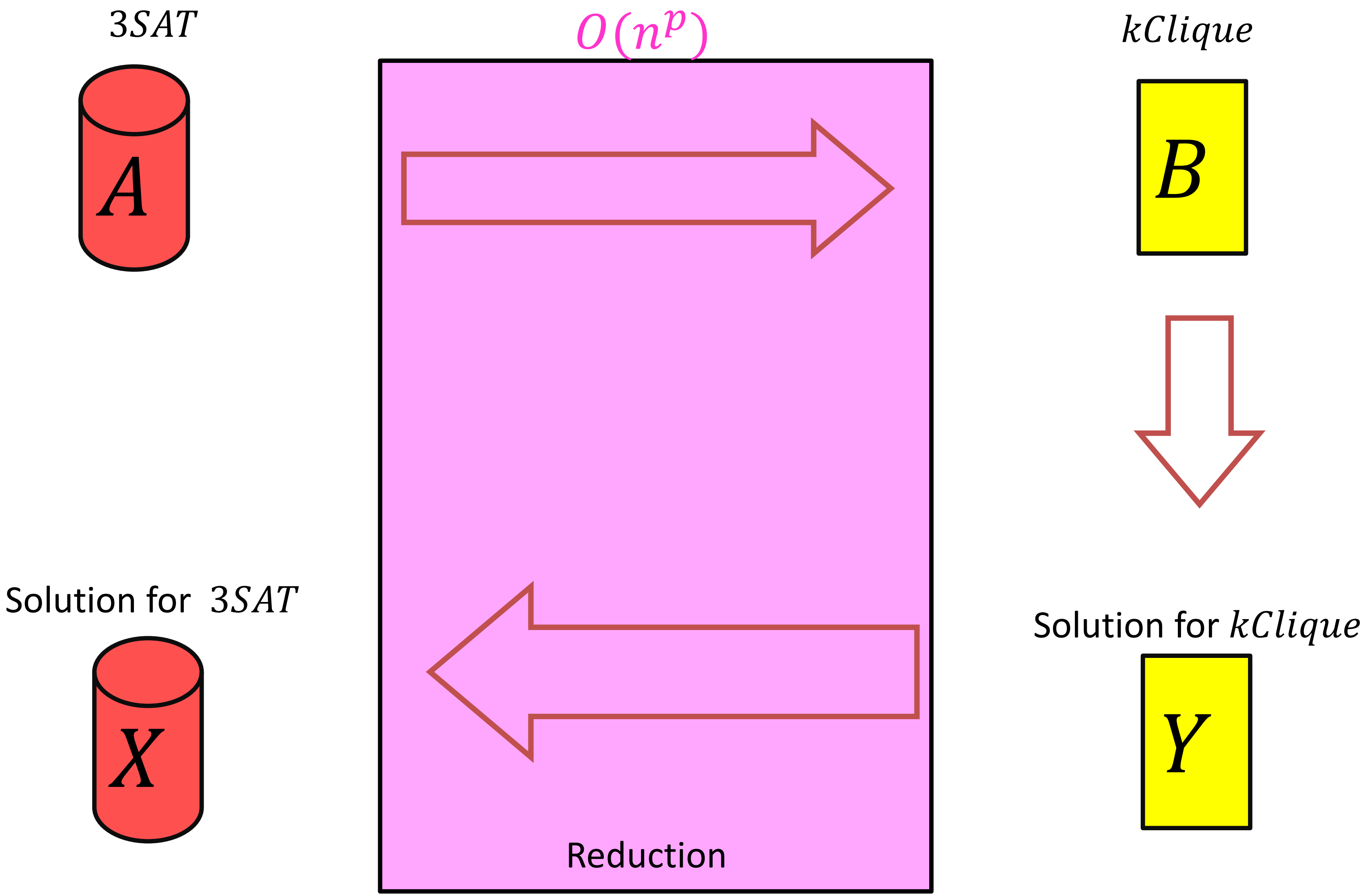
k-Clique is NP

Given a Graph, k, and a potential solution

1. Check that the solution has k nodes
2. Check that every pair of nodes share an edge

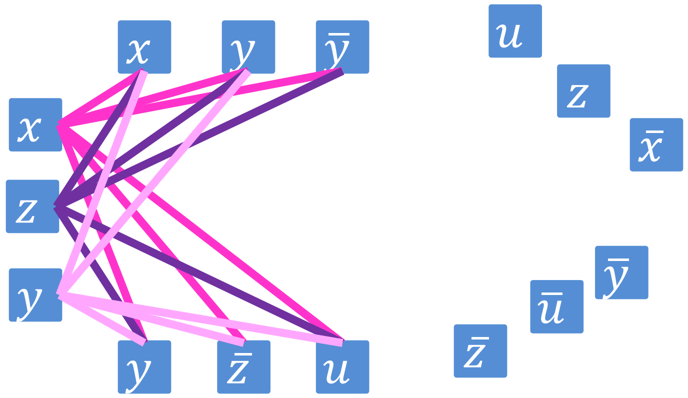


$3SAT \leq_p kClique$



Instance of 3SAT to Instance of kClique

$$(x \lor y \lor z) \land (x \lor \bar{y} \lor y) \land (u \lor y \lor \bar{z}) \land (z \lor \bar{x} \lor u) \land (\bar{x} \lor \bar{y} \lor \bar{z})$$



(also do this for the other clauses, omitted due to clutter)

For each clause, produce a node for each of its three variables

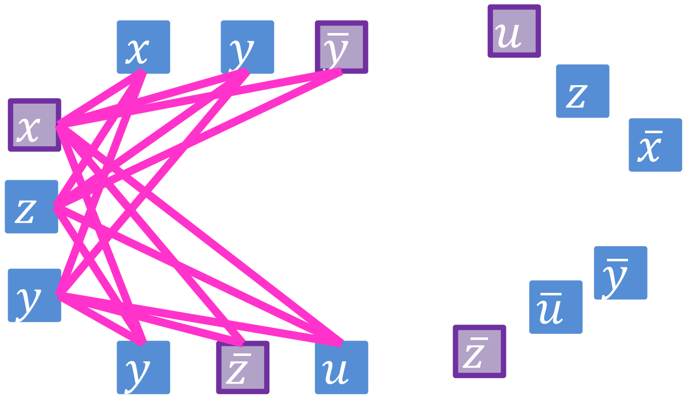
Connect each node to all non-contradictory nodes in the other clauses (i.e., anything that's not its negation)

Let k = number of clauses

There is a k-Clique in this graph **iff** there is a satisfying assignment

kClique ⇒ Satisfying Assignment

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

There are k triplets in the graph, and no two nodes in the same triplet are adjacent

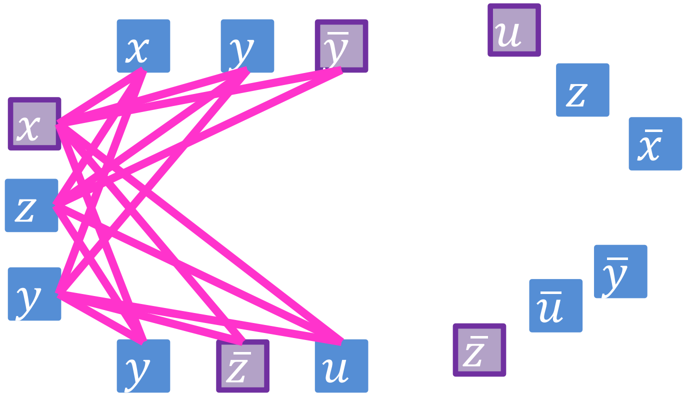
To have a k-Clique, must have one node from each triplet

Cannot select a node for both a variable and its negation

Therefore selection of nodes is a satisfying assignment

Satisfying Assignment ⇒ kClique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

Select one node for a true variable from each clause

There will be k nodes selected

We can't select both a node and its negation

All nodes will be non-contradictory, so they will be pairwise adjacent

$3SAT \leq_p kClique$