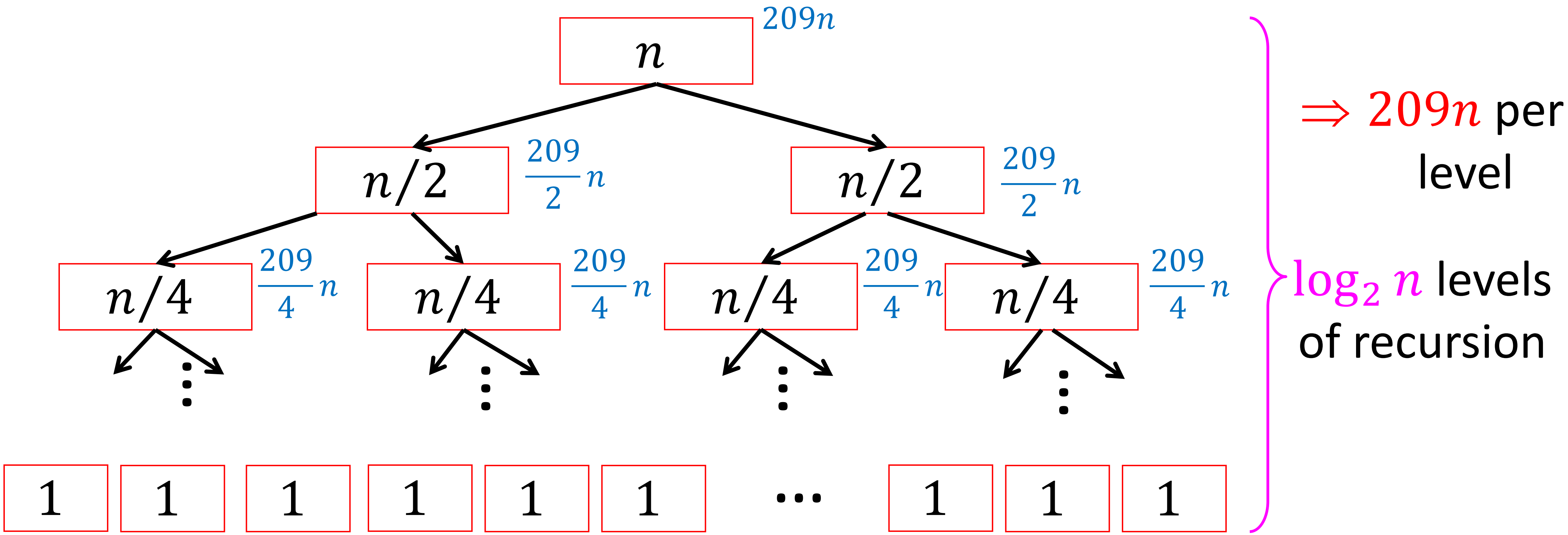
Warm Up

What is the asymptotic run time of MergeSort if its recurrence is

$$T(n) = 2T\left(\frac{n}{2}\right) + 209n$$

Tree method

$$T(n) = 2T(\frac{n}{2}) + 209n$$



$$T(n) = 209n \sum_{i=1}^{\log n} 1 = 209n \log_2 n$$

Tree Method

$$T(n) = 2T(n/2) + 209n$$

Number of subproblems	Cost of subproblem
1	$209n$
2	$209n/2$
4	$209n/4$
2^k	$209n/2^k$

What is the cost?

Cost at level i: $2^i \cdot \dfrac{209n}{2^i} = 209n$

Total cost: $T(n) = \displaystyle\sum_{i=0}^{\log_2 n} 209n$

$$= 209n \sum_{i=0}^{\log_2 n} 1 \quad = n \log_2 n$$

$$= \Theta(n \log n)$$

Today’s Keywords

- Karatsuba (finishing up)
- Guess and Check Method
- Induction
- Master Theorem

CLRS Readings

- Chapter 4

Homeworks

- Hw1 due Thursday, September 12 at 11pm
  - Start early!
  - Written (use Latex!) – Submit BOTH pdf and zip!
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer

Karatsuba

1. Break into smaller subproblems

$$
\begin{array}{r}
\boxed{a}\boxed{b} = 10^{\frac{n}{2}}\boxed{a} + \boxed{b} \\
\times\ \boxed{c}\boxed{d} = 10^{\frac{n}{2}}\boxed{c} + \boxed{d} \\
\hline
\end{array}
$$

$$
10^{n}(\boxed{a} \times \boxed{c}) + \\
10^{\frac{n}{2}}(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}) + \\
(\boxed{b} \times \boxed{d})
$$

Recall: previous divide-and-conquer recursively computed ac, ad, bc, bd

Karatsuba lets us reuse ac, bd to compute $(ad + bc)$ in one multiply

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time

	a	b
$\times$	c	d

$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right)$$

Need to compute **3** multiplications, each of size $n/2$: ac, bd, $(a+b)(c+d)$

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time

$$\begin{array}{r} a \; b \\ \times \; c \; d \\ \hline \end{array}$$

$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Need to compute **3** multiplications, each of size $n/2$: ac, bd, $(a+b)(c+d)$

2 shifts and 6 additions on n-bit values

Karatsuba Algorithm

$$\begin{array}{r} \boxed{a}\,\boxed{b} \\ \times\ \boxed{c}\,\boxed{d} \\ \hline \end{array}$$

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

$$\boxed{T(n) = 3T\left(\frac{n}{2}\right) + 8n}$$

<u>Pseudo-code</u>

1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(b, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^n x + 10^{n/2} z + y$

Karatsuba Example

$$n = 4$$

$$\begin{array}{r} 4\,1\,0\,2 \\ \times 1\,8\,1\,9 \\ \hline \end{array}$$

$$\left.\begin{array}{r} a = 41 \\ b = 02 \\ c = 18 \\ d = 19 \end{array}\right\} \text{Constant time divide} \qquad \Theta(1)$$

$$\left.\begin{array}{r} a + b = 43 \\ \mathrm{c} + d = 37 \end{array}\right\} \text{2 preliminary additions} \qquad \Theta(2n)$$

$$\left.\begin{array}{r} ac = 41 \times 18 = 738 \\ bd = 02 \times 19 = 38 \\ (a+b)(c+d) = 43 \times 37 = 1591 \end{array}\right\} \text{3 recursive Karatsuba calls}$$

each size ${}^{n}/_{2} = 2$ $\qquad 3\mathrm{T}(n/2)$

Karatsuba Example

$n = 4$

$$\left.\begin{aligned} ac &= 41 \times 18 = 738 \\ bd &= 02 \times 19 = 38 \\ (a+b)(c+d) &= 43 \times 37 = 1591 \end{aligned}\right\}$$

3 recursive Karatsuba calls

each size $n/_2 = 2$

$3\mathrm{T}(n/2)$

$$\left.\begin{aligned} &10^n(ac) + 10^{\frac{n}{2}}\big((a+b)(c+d) - ac - bd\big) + bd \\ &10^4(ac) + 10^{\frac{4}{2}}\big((a+b)(c+d) - ac - bd\big) + bd \\ &10000(ac) + 100\big((a+b)(c+d) - ac - bd\big) + bd \\ &10000(738) + 100(1591 - 738 - 38) + 38 \\ &10000(738) + 100(815) + 38 \end{aligned}\right\}$$

Combine step

Karatsuba Example

$$10000(738) + 100(815) + 38$$

$$7380000 + 81500 + 38$$

$$7461538$$

Combine step $\Theta(6n)$

$n = 4$

$$\begin{array}{r} 4\,1\,0\,2 \\ \times\,1\,8\,1\,9 \\ \hline 7\,4\,6\,1\,5\,3\,8 \end{array}$$

Karatsuba Algorithm

$$
\begin{array}{r}
a \quad b \\
\times \quad c \quad d \\
\hline
\end{array}
$$

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

<u>Pseudo-code</u>

$$\boxed{T(n) = 3T\left(\frac{n}{2}\right) + 8n}$$

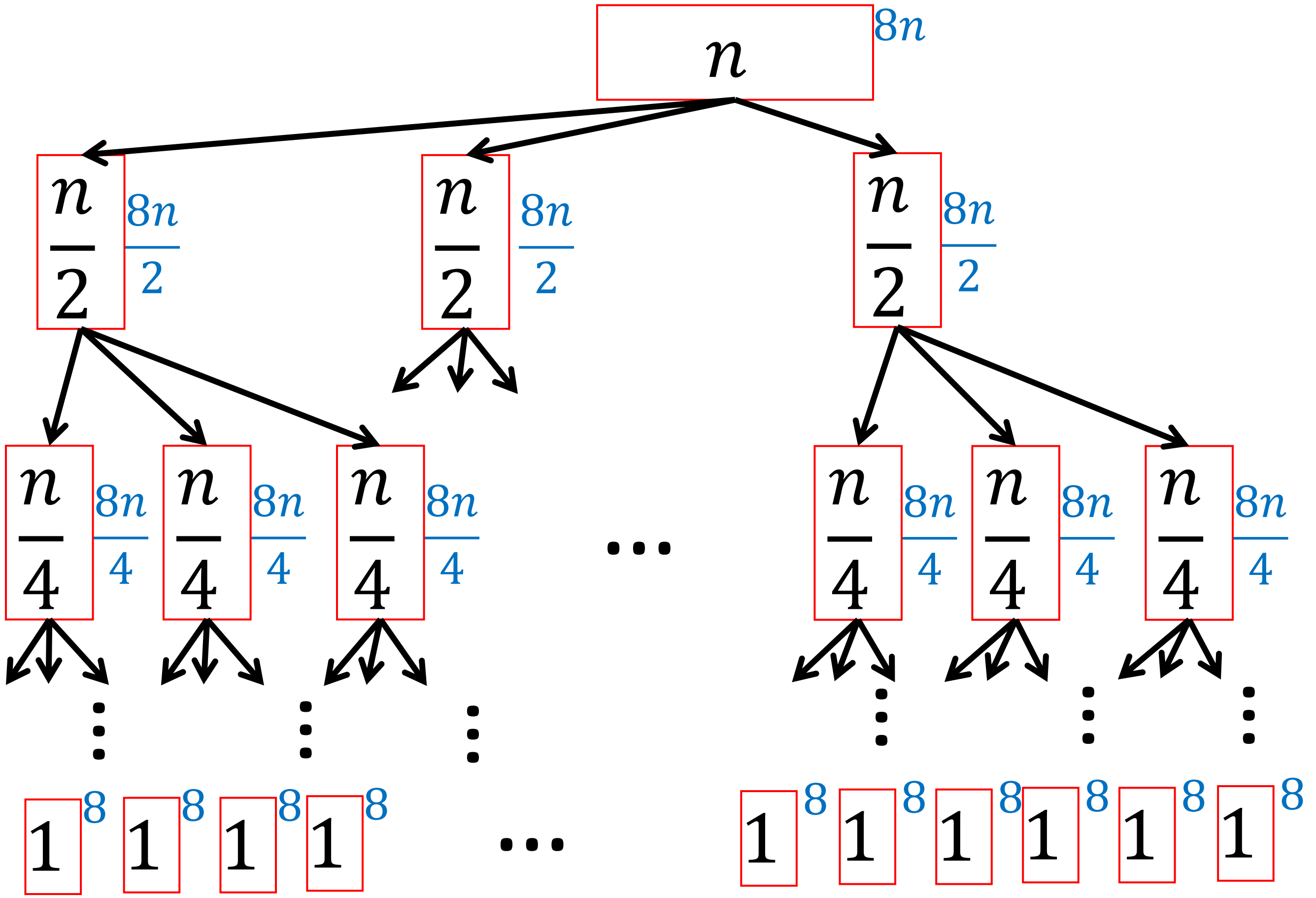
1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(b, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^n x + 10^{n/2} z + y$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$\boxed{T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}\!/_{2}\right)^{i}}$$



$$8n \cdot 1$$

$$8n \cdot \frac{3}{2}$$

$$8n \cdot \frac{9}{4}$$

$$\vdots$$

$$8n \cdot \frac{3^{\log_2 n}}{2^{\log_2 n}}$$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

Karatsuba

$$T(n) = 8n\left(\frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1}\right) = 16n\left(\left(\frac{3}{2}\right)^{\log_2 n+1} - 1\right)$$

$$= 16n\left(\left(\frac{2^{\log_2 3}}{2}\right)^{\log_2 n+1} - 1\right)$$

$$= 16n\left(\left(2^{\log_2 3 - 1}\right)^{\log_2 n+1} - 1\right)$$

$$= 16n\left(2^{\log_2 3 \log_2 n + \log_2 3 - \log_2 n - 1} - 1\right)$$

$$= 16n\left(2^{\log_2 3 \log_2 n} \cdot 2^{\log_2 3} \cdot \frac{1}{2^{\log_2 n}} \cdot \frac{1}{2} - 1\right)$$

$$= 16n\left(\left(2^{\log_2 n}\right)^{\log_2 3} \cdot 3 \cdot \frac{1}{n} \cdot \frac{1}{2} - 1\right)$$

$$= 16n\left(n^{\log_2 3} \cdot 3 \cdot \frac{1}{n} \cdot \frac{1}{2} - 1\right) = 16\cancel{n} \cdot n^{\log_2 3} \cdot \frac{1}{\cancel{n}} \cdot \frac{3}{2} - 16n = 24n^{\log_2 3} - 16n$$

Side Notes

$$3 = 2^{\log_2 3}$$

$$n = 2^{\log_2 n}$$

Karatsuba

3. Use asymptotic notation to simplify

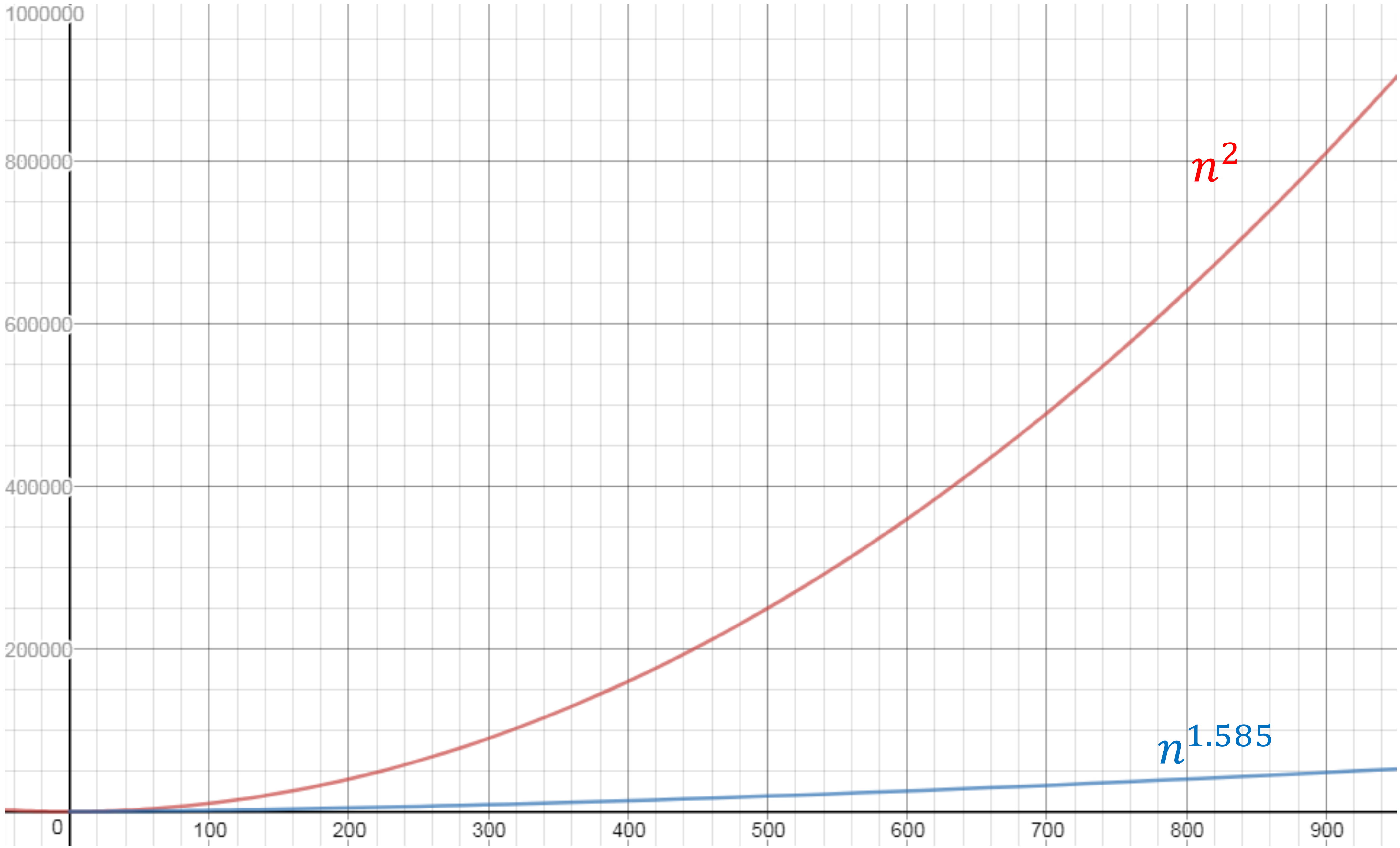
$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}\!/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}\!/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}\!/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

$$T(n) = 24\left(n^{\log_2 3}\right) - 16n \; = \Theta\left(n^{\log_2 3}\right)$$
$$\approx \Theta\left(n^{1.585}\right)$$


1000000
800000
600000
400000
200000
0
100
200
300
400
500
600
700
800
900
n^2
$n^{1.585}$

Recurrence Solving Techniques

Tree

?✓ Guess/Check (induction)

“Cookbook”

Substitution

Induction (review)

Goal: $\forall k \in \mathbb{N}, P(k)$ holds

Base case(s): $P(1)$ holds

Technically, called *strong induction*

Hypothesis: $\forall x \leq x_0, P(x)$ holds

Inductive step: show $P(1), \dots, P(x_0) \Rightarrow P(x_0 + 1)$

Guess and Check Intuition

- **Show:** $T(n) \in O(g(n))$
- **Consider:** $g_*(n) = c \cdot g(n)$ for some constant c, i.e. pick $g_*(n) \in O(g(n))$
- **Goal:** show $\exists n_0$ such that $\forall n > n_0, T(n) \leq g_*(n)$
  - (definition of big-O)
- **Technique:** Induction
  - Base cases:
    - show $T(1) \leq g_*(1), T(2) \leq g_*(2), \dots$ for a small number of cases (may need additional base cases)
  - Hypothesis:
    - $\forall n \leq x_0, T(n) \leq g_*(n)$
  - Inductive step:
    - Show $T(x_0 + 1) \leq g_*(x_0 + 1)$

Need to ensure that in inductive step, can either appeal to a <u>base case</u> or to the <u>inductive hypothesis</u>

Karatsuba Guess and Check (Loose)

$T(n) \in O(n^{1.6})$

$g_*(n) = 3000\, n^{1.6}$

$$T(n) = 3\,T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \le 3000\, n^{1.6} = O(n^{1.6})$

Base cases: $T(1) = 8 \le 3000$

$T(2) = 3(8) + 16 = 40 \le 3000 \cdot 2^{1.6}$

… up to some small k

Hypothesis: $\forall n \le x_0, T(n) \le 3000n^{1.6}$

Inductive step: Show that $T(x_0 + 1) \le 3000(x_0 + 1)^{1.6}$

Karatsuba Guess and Check (Loose)

Hyp: $T(n) \le 3000\, n^{1.6}$ $\quad \forall n \le x_0$ $\qquad T(n) = 3T\left(\frac{n}{2}\right) + 8n$

Show: $T(x_0+1) \le 3000\,(x_0+1)^{1.6}$

$\frac{x_0+1}{2} < x_0$

$$T(x_0+1) = 3T\left(\boxed{\frac{x_0+1}{2}}\right) + 8(x_0+1)$$

$$\le 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1) \quad \leftarrow \text{use inductive hypothesis}$$

$$= \frac{3}{2^{1.6}}\left(3000\,(x_0+1)^{1.6}\right) + 8(x_0+1)$$

$$\le 0.997 \cdot 3000\,(x_0+1)^{1.6} + 8(x_0+1)$$

$$= (1 - .003)\left(3000\,(x_0+1)^{1.6}\right) + 8(x_0+1)$$

$$= 3000(x_0+1)^{1.6} - 9(x_0+1)^{1.6} + 8(x_0+1)$$

$9 > 8$

$$\le 3000\,(x_0+1)^{1.6}$$

$T(n) \in O(n^{1.6})$

Mergesort Guess and Check

$$T(n) = 2\,T\left(\frac{n}{2}\right) + n$$

Goal: $T(n) \leq n \log_2 n \;= O(n \log_2 n)$

Base cases: $T(1) = 0$

$T(2) = 2 \leq 2 \log_2 2$

… up to some small k

Hypothesis: $\forall n \leq x_0 \; T(n) \leq n \log_2 n$

Inductive step: $T(x_0 + 1) \leq (x_0 + 1) \log_2 (x_0 + 1)$

Math, math, and more math…(on board, see lecture supplemental)

Mergesort Guess and Check

Goal: $T(n) \le n \log n$ $\quad T(1), T(2) \ldots$ $\quad T(n) = 2T(\frac{n}{2}) + n$

Assume hypothesis: $\forall n < x_0 \quad T(n) \le n \log n$

show $T(x_0+1) \le (x_0+1) \log_2 (x_0+1)$

$\frac{x_0+1}{2} < x_0$

$$T(x_0+1) = 2T\left(\boxed{\frac{x_0+1}{2}}\right) + (x_0+1)$$

$$\le \cancel{2}\left(\frac{x_0+1}{\cancel{2}}\right) \log_2\left(\frac{x_0+1}{2}\right) + (x_0+1)$$

$$= (x_0+1)\left(\log_2(x_0+1) - \overset{1}{\cancel{\log_2(2)}}\right) + (x_0+1)$$

$$= (x_0+1)\log_2(x_0+1) \cancel{- (x_0+1)} \cancel{+ x_0+1}$$

Therefore $T(n) \le n \log n \rightarrow T(n) \in O(n \log n)$

Karatsuba Guess and Check

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

Math, math, and more math…(on board, see lecture supplemental)

Karatsuba Guess and Check

Hyp: $\forall n \le x_0 \quad T(n) \le 24 n^{\log_2 3} - 16n$ $\qquad T(n) = 3T\left(\frac{n}{2}\right) + 8n$

Show $T(x_0+1) \le 24(x_0+1)^{\log_2 3} - 16(x_0+1)$

$$T(x_0+1) = 3T\left(\boxed{\frac{x_0+1}{2}}\right) + 8(x_0+1)$$

$$\le 3\left(24\left(\frac{x_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{x_0+1}{2}\right)\right) + 8(x_0+1)$$

$$= 3 \cdot 24\left(\frac{x_0+1}{2}\right)^{\log_2 3} - 24(x_0+1) + 8(x_0+1)$$

$$= \cancel{3} \cdot 24 \frac{(x_0+1)^{\log_2 3}}{\cancel{2^{\log_2 3}}\ \cancel{3}} - 16(x_0+1)$$

$$= 24(x_0+1)^{\log_2 3} - 16(x_0+1).$$

What if we leave out the $-16n$?

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

What we wanted: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3}$ Induction failed!

What we got: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} + 8(x_0 + 1)$

"Bad Mergesort" Guess and Check

$$T(n) = 2\,T\left(\frac{n}{2}\right) + 209n$$

Goal: $T(n) \le 209n \log_2 n \;= O(n \log_2 n)$

Base cases: $T(1) = 0$

$T(2) = 518 \le 209 \cdot 2 \log_2 2$

… up to some small k

Hypothesis: $\forall n \le x_0, T(n) \le 209n \log_2 n$

Inductive step: $T(x_0 + 1) \le 209(x_0 + 1) \log_2(x_0 + 1)$