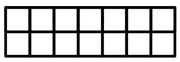
CS4102 Algorithms
Spring 2019

From Last Time

How many ways are there to tile a $2\times n$ board with dominoes?

How many ways to tile this:



With these?

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Today's Keywords

- Dynamic Programming
- Log Cutting

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CLRS Readings

- Chapter 15

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Homework

- Hw4 Due Tonight at 11pm
  - Sorting
  - Written

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Midterm

- Wednesday March 6 in class
  - Covers all content through last Monday
  - We will have a review session
    - Tonight! 7pm, Olsson 120
    - Will be recorded, so you'll have it if you can't make it

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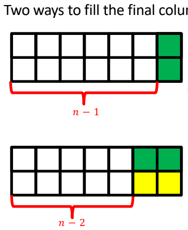
How many ways are there to tile a $2 \times n$ board with dominoes?

Two ways to fill the final column:



$Tile(n) = Tile(n-1) + Tile(n-2)$

$Tile(0) = Tile(1) = 1$

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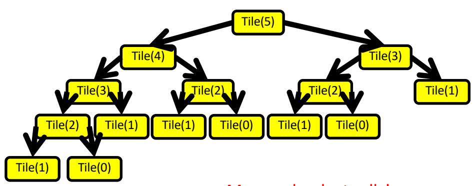
How to compute $Tile(n)$?

```
Tile(n):
    if n < 2:
        return 1
    return Tile(n-1)+Tile(n-2)
```

Problem?

Recursion Tree



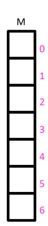
Many redundant calls!

Run time: $\Omega(2^n)$

Better way: Use Memory!

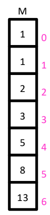
Computing $Tile(n)$ with Memory

```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```



Computing $Tile(n)$ with Memory "Top Down"

```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```



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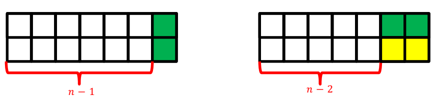
Dynamic Programming

- Requires Optimal Substructure
  - Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the "last thing" done?



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Generic Divide and Conquer Solution

```
def myDCalgo(problem):

    if baseCase(problem):
        solution = solve(problem)

        return solution
    for subproblem of problem:    # After dividing
        subsolutions.append(myDCalgo(subproblem))
    solution = Combine(subsolutions)

    return solution
```

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Generic Top-Down Dynamic Programming Soln

```
mem = {}
def myDPalgo(problem):
    if mem[problem] not blank:
        return mem[problem]
    if baseCase(problem):
        solution = solve(problem)
        mem[problem] = solution
        return solution
    for subproblem of problem:
        subsolutions.append(myDPalgo(subproblem))
    solution = OptimalSubstructure(subsolutions)
    mem[problem] = solution
    return solution
```

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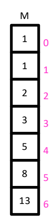
Computing $Tile(n)$ with Memory "Top Down"

```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```



Recursive calls happen in a predictable order

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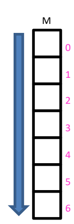
Better $Tile(n)$ with Memory "Bottom Up"

```
Tile(n):
    Initialize Memory M
    M[0] = 1
    M[1] = 1
    for i = 2 to n:
        M[i] = M[i-1] + M[i-2]
    return M[n]
```



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Dynamic Programming

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  - Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the "last thing" done?
  2. Select a good order for solving subproblems
     - Usually smallest problem first
     - "Bottom up"

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Log Cutting

Given a log of length n
A list (of length n) of prices P ($P[i]$ is the price of a cut of size i)
Find the best way to cut the log

Length:	1	2	3	4	5	6	7	8	9	10
Price:	1	5	8	9	10	17	17	20	24	30

Select a list of lengths $\ell_1, \dots, \ell_k$ such that:
$\sum \ell_i = n$
to maximize $\sum P[\ell_i]$

Brute Force: $O(2^n)$

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Greedy won't work

- Greedy algorithms (next unit) build a solution by picking the best option "right now"
  - Select the most profitable cut first

Length:	1	2	3	4	5	6
Price:	1	18	24	36	50	50

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

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Greedy won't work

- Greedy algorithms (next unit) build a solution by picking the best option "right now"
  - Select the "most bang for your buck"
    - (best price / length ratio)

Price:	1	18	24	36	50	50
Length:	1	2	3	4	5	6

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

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Dynamic Programming

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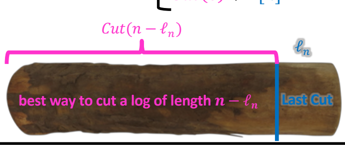
1. Identify Recursive Structure

$P[i]$ = value of a cut of length i
$Cut(n)$ = value of best way to cut a log of length n

$$Cut(n) = \max \begin{cases} Cut(n-1) + P[1] \\ Cut(n-2) + P[2] \\ \dots \\ Cut(0) + P[n] \end{cases}$$



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Dynamic Programming

- Idea:
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2. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$Cut(0) = 0$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

0

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2. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$Cut(1) = Cut(0) + P[1]$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

1

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2. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(2) = \max \begin{cases} Cut(1) + P[1] \\ Cut(0) + P[2] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

2

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2. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(3) = \max \begin{cases} Cut(2) + P[1] \\ Cut(1) + P[2] \\ Cut(0) + P[3] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

3

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2. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

4

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Log Cutting Pseudocode

```
Initialize Memory C
Cut(n):
    C[0] = 0
    for i=1 to n:
        best = 0
        for j = 1 to i:
            best = max(best, C[i-j] + P[j])
        C[i] = best
    return C[n]
```

Run Time: $O(n^2)$

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How to find the cuts?

- This procedure told us the profit, but not the cuts themselves
- Idea: remember the choice that you made, then backtrack

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Remember the choice made

```
Initialize Memory C, Choices
Cut(n):
    C[0] = 0
    for i=1 to n:
        best = 0
        for j = 1 to i:
            if best < C[i-j] + P[j]:
                best = C[i-j] + P[j]
                Choices[i]=j
        C[i] = best
    return C[n]
```

Gives the size of the last cut

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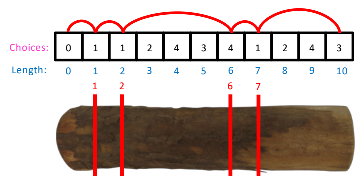
Reconstruct the Cuts

- Backtrack through the choices



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Backtracking Pseudocode

```
i = n
while i>0:
    print Choices[i]
    i = i – Choices[i]
```

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