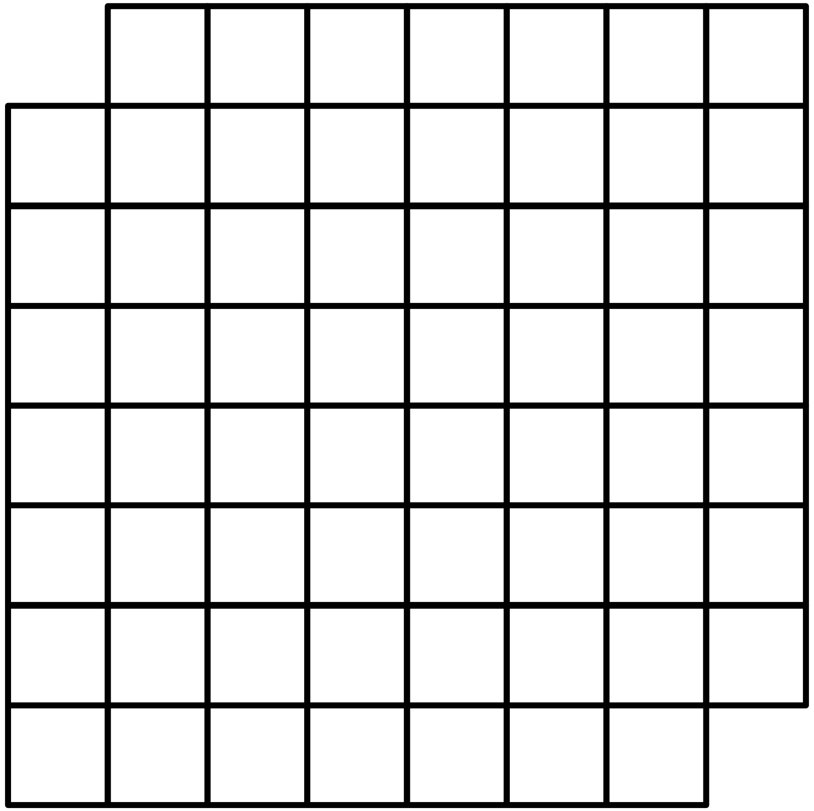
CS4102 Algorithms

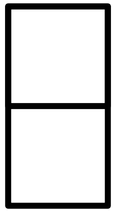
Spring 2019

Can you fill a 8×8 board with the corners missing using dominoes?

Can you tile this?



With these?



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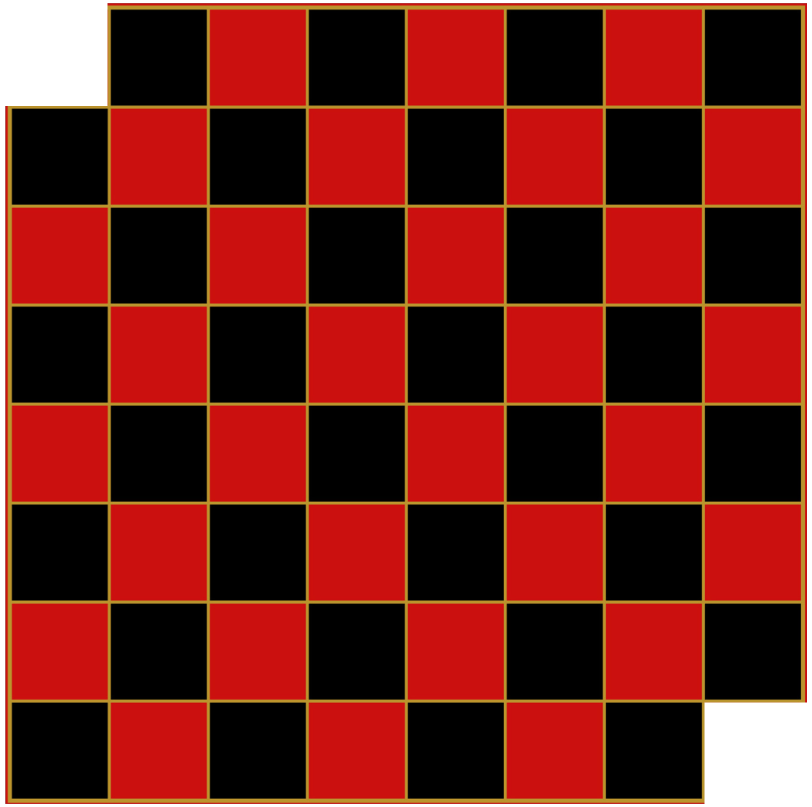

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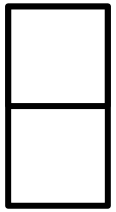

Today’s Keywords

- Midterm Review
- Dynamic Programming
- Longest Common Subsequence

CLRS Readings

- Chapter 15

Administrativa

- HW5 due March 27 at 11pm
  - Seam Carving!
  - Dynamic Programming (implementation)
  - Java or Python
- Grades Released
  - HW3
  - Midterm

Dynamic Programming

- Requires Optimal Substructure
  – Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the “last thing” done?
  2. Select a good order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest
  3. Save solution to each subproblem in memory

Generic Top-Down Dynamic Programming Soln

```
mem = {}
def myDPalgo(problem):
    if mem[problem] not blank:
        return mem[problem]
    if baseCase(problem):
        solution = solve(problem)
        mem[problem] = solution
        return solution
    for subproblem of problem:
        subsolutions.append(myDPalgo(subproblem))
    solution = OptimalSubstructure(subsolutions)
    mem[problem] = solution
    return solution
```

Longest Common Subsequence

Given two sequences X and Y, find the length of their longest common subsequence

Example:

$X = AT\underline{C}\underline{T}GAT$

$Y = TGCATA$

$LCS = TCTA$

Brute force: Compare every subsequence of X with Y

$\Omega(2^n)$

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1. Identify Recursive Structure

Let $LCS(i,j)$ = length of the LCS for the first i characters of X, first j character of Y

Find $LCS(i,j)$:

Case 1: $X[i] = Y[j]$

$X = ATCTGCGT$
$Y = TGCATAT$
$LCS(i,j) = LCS(i-1, j-1) + 1$

Case 2: $X[i] \neq Y[j]$

$X = ATCTGCGA$
$Y = TGCATAT$
$LCS(i,j) = LCS(i, j-1)$

$X = ATCTGCGT$
$Y = TGCATAC$
$LCS(i,j) = LCS(i-1, j)$

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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$$Y = TGCATAC$$
$$LCS(i, j) = LCS(i-1, j)$$

Read from M[I,j] if present

$$LCS(i, j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

Save to M[I,j]

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2. Solve in a Good Order

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$X =$		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
$Y =$	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

To fill in cell (i, j) we need cells $(i-1, j-1), (i-1, j), (i, j-1)$

Fill from Top->Bottom, Left->Right (with any preference)

Run Time?

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

Run Time: $\Theta(n \cdot m)$ (for $|X| = n$, $|Y| = m$)

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$	0	$\boxed{A}$ 1	T 2	C 3	$\boxed{T}$ 4	$\boxed{G}$ 5	$\boxed{A}$ 6	$\boxed{T}$ 7
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$\boxed{T}$ 1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
$\boxed{G}$ 2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C 3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
$\boxed{A}$ 4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
$\boxed{T}$ 5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A 6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

Start from bottom right,

if symbols matched, print that symbol then go diagonally

else go to largest adjacent

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$	$X =$		A	$\boxed{T}$	$\boxed{C}$	T	G	$\boxed{A}$	$\boxed{T}$
		0	1	2	3	4	5	6	7
	0	0	0	0	0	0	0	0	0
$\boxed{T}$	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
$\boxed{C}$	3	0	0	1	2	2	2	2	2
$\boxed{A}$	4	0	1	1	2	2	2	3	3
$\boxed{T}$	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

Start from bottom right,
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$Y =$ \ $X =$		0	A 1	$\boxed{T}$ 2	$\boxed{C}$ 3	$\boxed{T}$ 4	G 5	$\boxed{A}$ 6	T 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$\boxed{T}$	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
$\boxed{C}$	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
$\boxed{T}$	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
$\boxed{A}$	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

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