CS4102 Algorithms

Spring 2019

<u>Warm up</u>

Given access to unlimited quantities of pennies, nickels dimes, and quarters, (worth value 1, 5, 10, 25 respectively), provide an algorithm which gives change for a given value x using the fewest number of coins.

Change Making

43 cents

Change Making Algorithm

- Given: target value x, list of coins $C = [c_1, \dots, c_n]$
  (in this case $C = [1,5,10,25]$)
- Repeatedly select the largest coin less than the remaining target value:

while($x > 0$)
  let $c = \max(c_i \in \{c_1, \dots, c_n\} \mid c_i \leq x)$
  print c
  $x = x - c$

Why does this always work?

- If $x < 5$, then pennies only
  - 5 pennies can be exchanged for a nickel

  Only case Greedy uses pennies!

- If $5 \leq x < 10$ we must have a nickel
  - 2 nickels can be exchanged for a dime

  Only case Greedy uses nickels!

- If $10 \leq x < 25$ we must have at least 1 dime
  - 3 dimes can be exchanged for a quarter and a nickel

  Only case Greedy uses dimes!

- If $x \geq 25$ we must have at least 1 quarter

  Only case Greedy uses quarters!

Today's Keywords

- Dynamic Programming
- Gerrymandering
- Greedy Algorithms
- Choice Function
- Change Making

CLRS Readings

- Chapter 15
- Chapter 16

Homeworks

- Homework 5 due Wednesday March 27 at 11pm
  – Seam Carving!
  – Dynamic Programming (implementation)
  – Java or Python
- Homework 6 out tonight, due Wednesday April 3 at 11pm
  – Dynamic Programming and Greedy Algorithms
  – Written (using Latex!)

Dynamic Programming

- Requires Optimal Substructure
  - Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the “last thing” done?
  2. Select a good order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest
  3. Save solution to each subproblem in memory

Generic Top-Down Dynamic Programming Soln

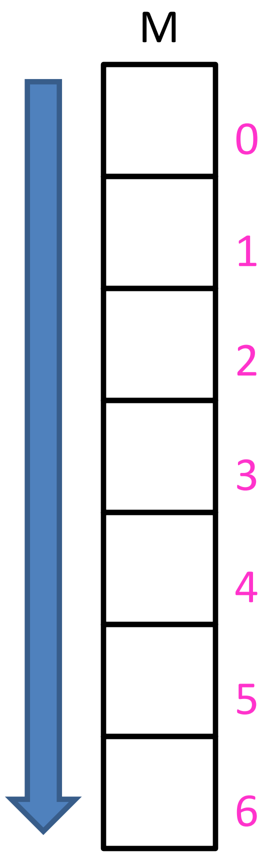
```
mem = {}
def myDPalgo(problem):
        if mem[problem] not blank:
                return mem[problem]
        if baseCase(problem):
                solution = solve(problem)
                mem[problem] = solution
                return solution
        for subproblem of problem:
                subsolutions.append(myDPalgo(subproblem))
        solution = OptimalSubstructure(subsolutions)
        mem[problem] = solution
        return solution
```

DP Algorithms so far

- $2 \times n$ domino tiling (Fibonacci)
- Log cutting
- Matrix Chaining
- Longest Common Subsequence
- Seam Carving

Domino Tiling

```
Tile(n):
    Initialize Memory M
    M[0] = 0
    M[1] = 0
    for i = 0 to n:
        M[i] = M[i-1] + M[i-2]
    return M[n]
```



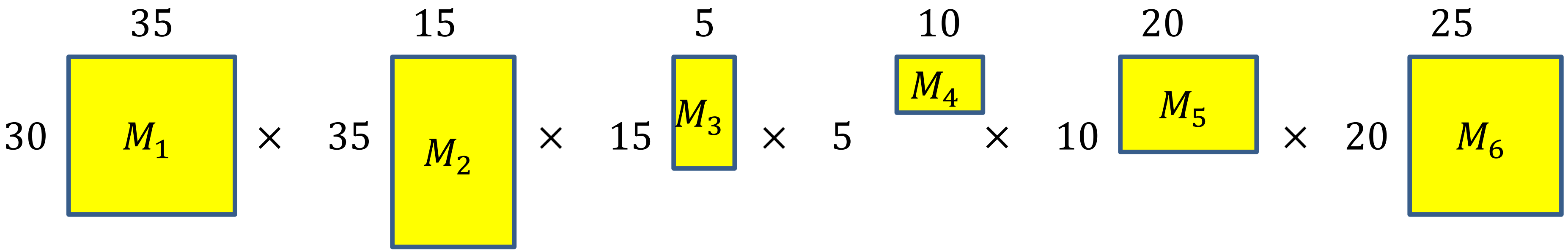
Log Cutting

Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

Matrix Chaining



$$Best(i,j) = \min_{k=1}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

i \ $j =$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Longest Common Subsequence

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

To fill in cell (i,j) we need cells $(i-1, j-1), (i-1, j), (i, j-1)$

Fill from Top->Bottom, Left->Right (with any preference)

Supreme Court Associate Justice Anthony Kennedy gave no sign that he has abandoned his view that extreme partisan gerrymandering might violate the Constitution. I Eric Thayer/Getty Images

Supreme Court eyes partisan gerrymandering

Anthony Kennedy is seen as the swing vote that could blunt GOP's map-drawing successes.

Gerrymandering

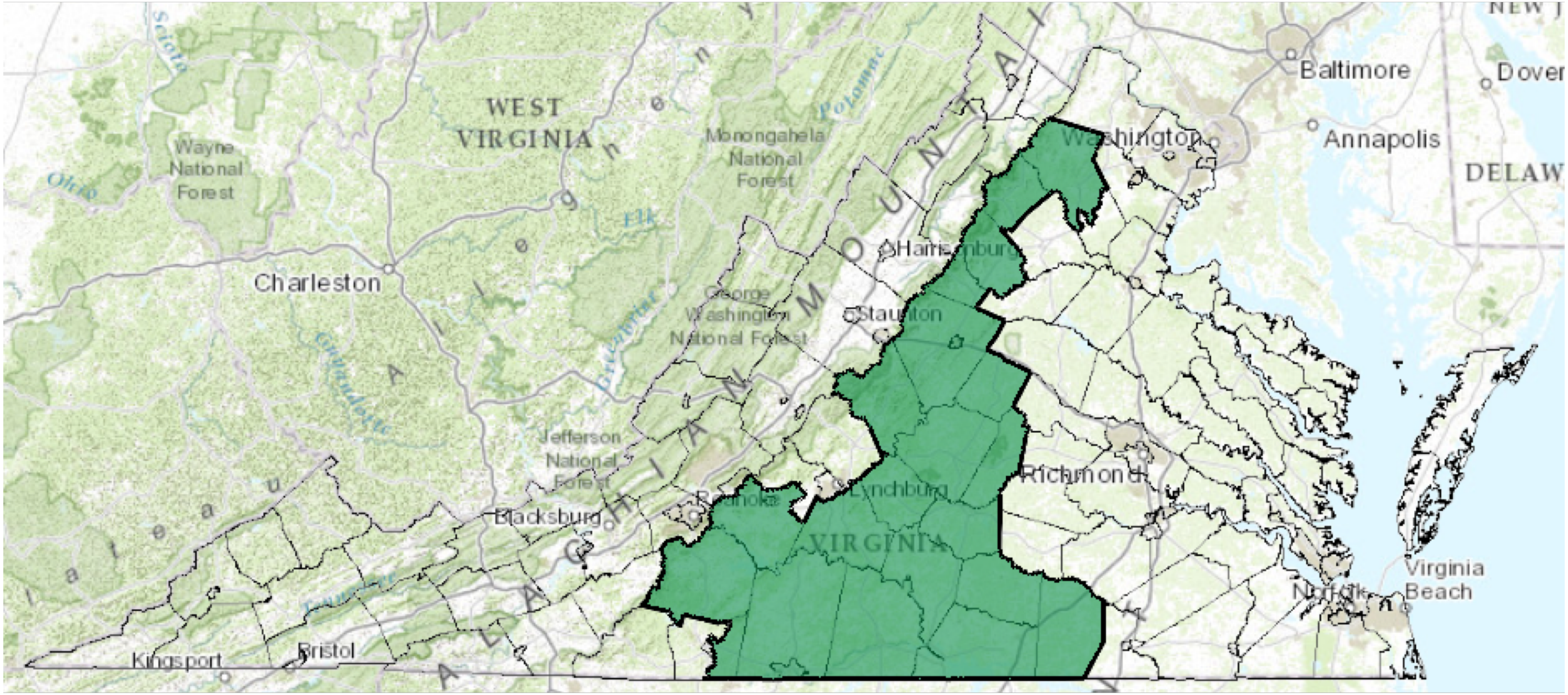
- Manipulating electoral district boundaries to favor one political party over others
- Coined in an 1812 Political cartoon
- Governor Gerry signed a bill that redistricted Massachusetts to benefit his Democratic-Republican Party

The Gerrymander

According to the Supreme Court

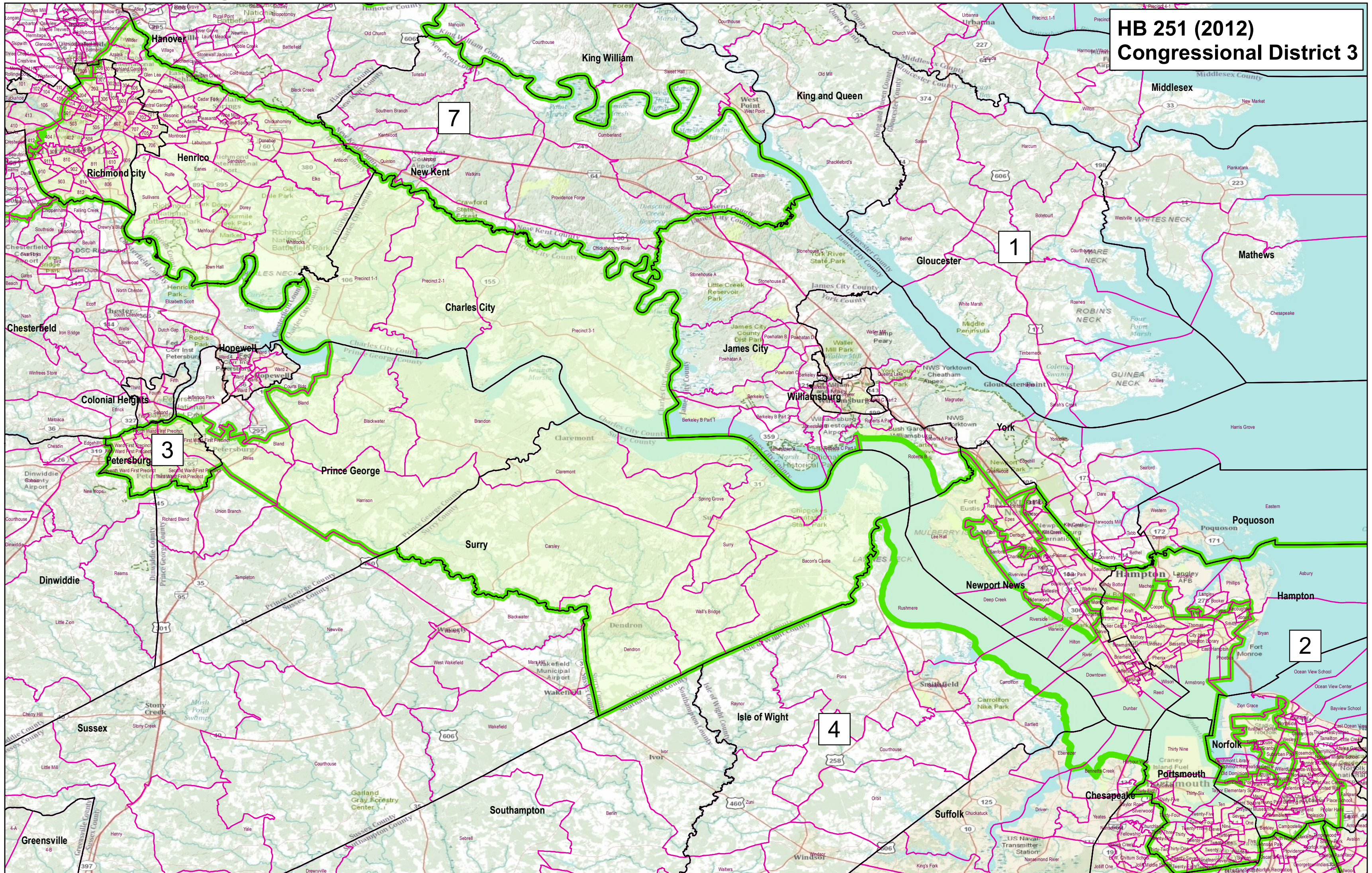
- Gerrymandering cannot be used to:
  - Disadvantage racial/ethnic/religious groups
- It can be used to:
  - Disadvantage political parties

VA 5th District



Gerrymandering Today

- Computers make it really effective



Gerrymandering Today

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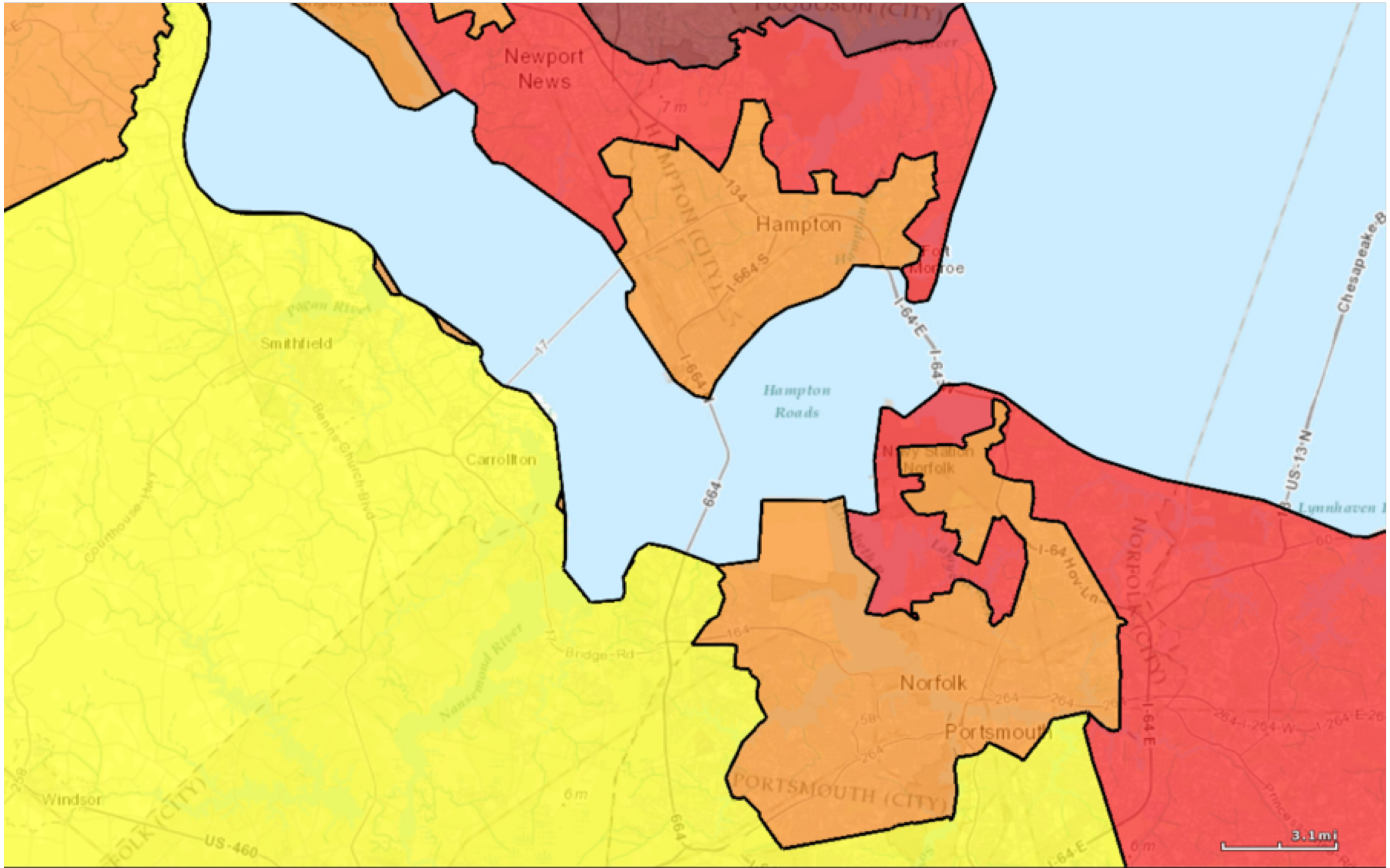

Gerrymandering Today

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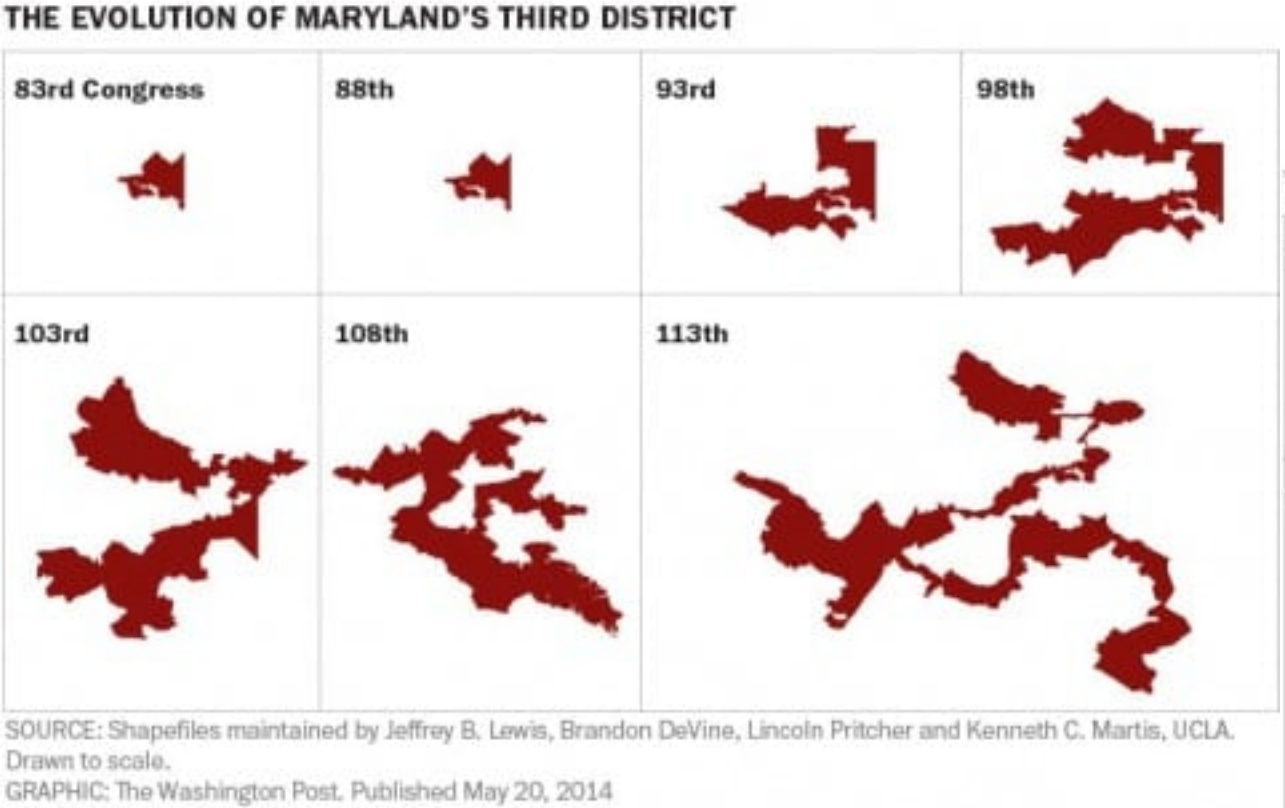
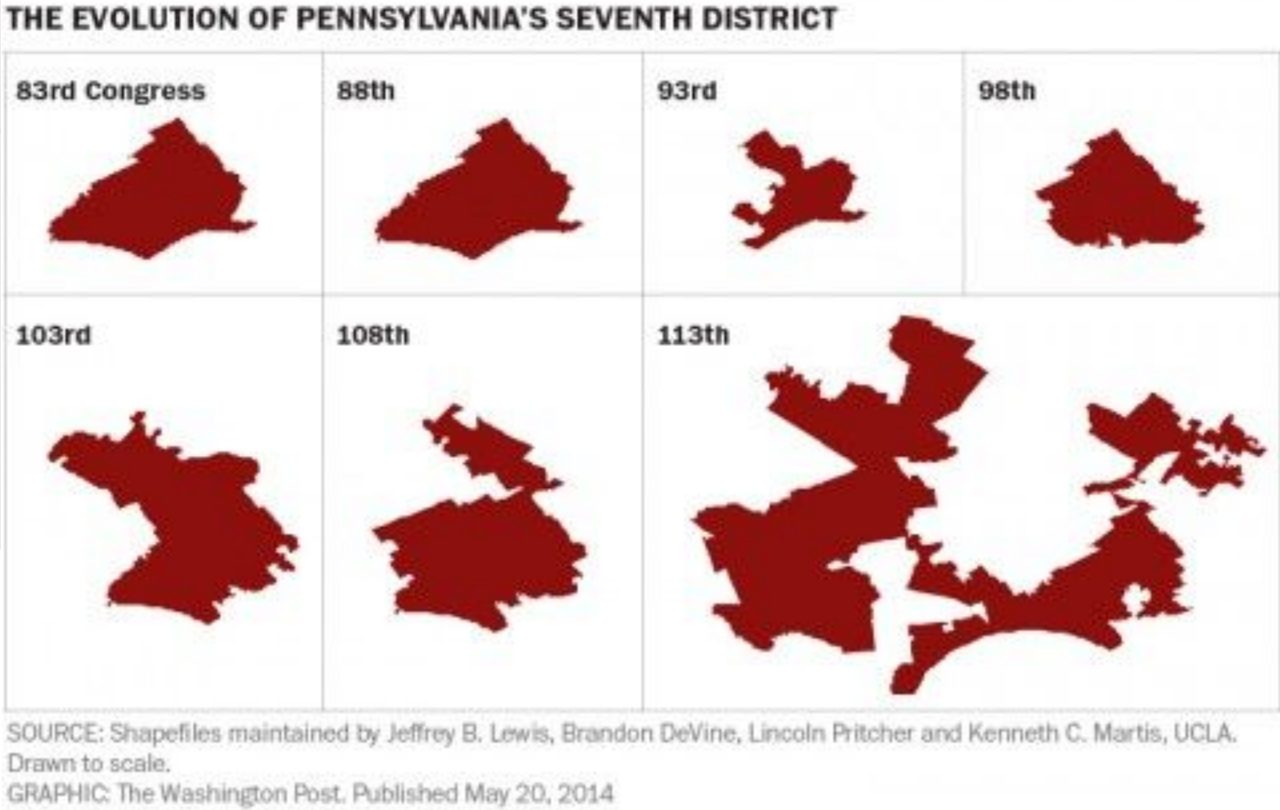
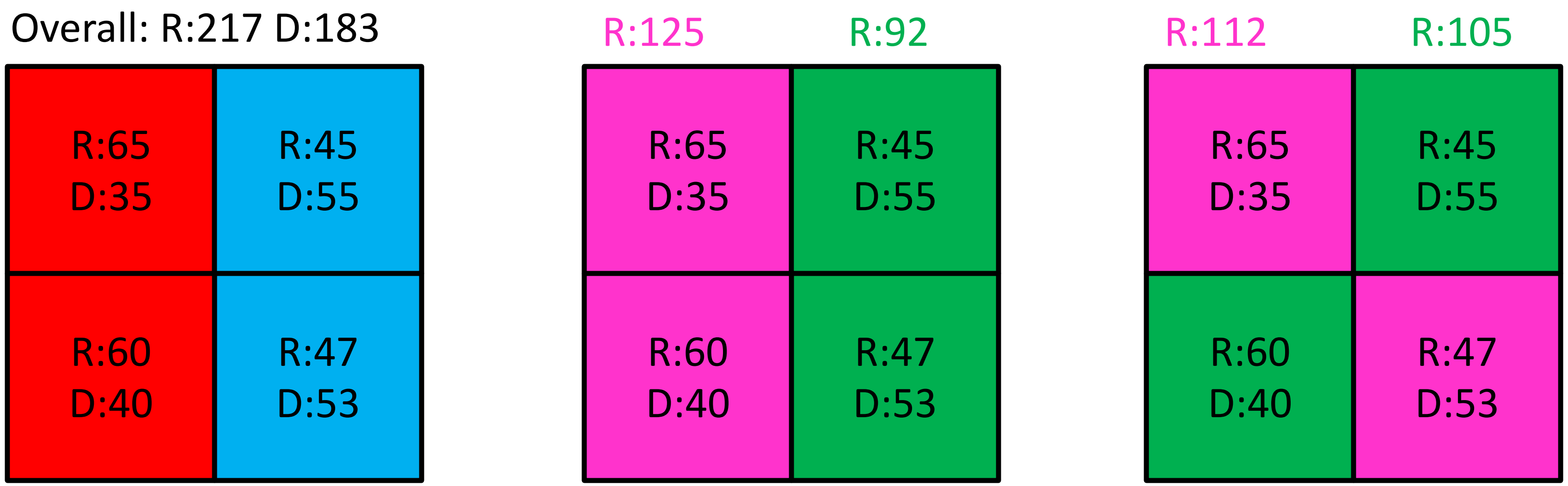




How does it work?

- States are broken into precincts
- All precincts have the same size
- We know voting preferences of each precinct
- Group precincts into districts to maximize the number of districts won by my party



Gerrymandering Problem Statement

- Given:
  - A list of precincts: $p_1, p_2, \ldots, p_n$ — precincts, n of them
  - Each containing m voters
  - m – not the voters, but how many
- Output:
  - Districts $D_1, D_2 \subset \{p_1, p_2, \ldots, p_n\}$
  - Where $|D_1| = |D_2|$
  - $R(D_1), R(D_2) > \frac{mn}{4}$ $\quad \frac{n}{2} \cdot m \cdot \frac{1}{2}$
    - $R(D_i)$ gives number of "Regular Party" voters in D_i
    - $R(D_i) > \frac{\mathrm{mn}}{4}$ means D_i is majority "Regular Party"
  - "failure" if no such solution is possible

Dynamic Programming

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Consider the last precinct

D_1
k precincts
x voters for R

D_2
$n-k-1$ precincts
y voters for R

After assigning the first $n-1$ precincts

p_n

If we assign p_n to D_1

D_1
$k+1$ precincts
$x+R(p_n)$ voters for R

Valid gerrymandering if:
$k+1=\frac{n}{2}$,
$x+R(p_n), y>\frac{mn}{4}$

If we assign p_n to D_2

D_2
$n-k$ precincts
$y+R(p_n)$ voters for R

Valid gerrymandering if:
$\mathrm{n}-k=\frac{n}{2}$,
$x, y+R(p_n)>\frac{mn}{4}$

Define Recursive Structure

$S(j, k, x, y) =$ True if from among the first $\boldsymbol{j}$ precincts:

$\boldsymbol{k}$ are assigned to D_1

$n \times n \times mn \times mn$

exactly $\boldsymbol{x}$ vote for R in D_1

exactly $\boldsymbol{y}$ vote for R in D_2

4D Dynamic Programming!!!

Two ways to satisfy $S(j, k, x, y)$:

$S(j, k, x, y) =$ True if:
from among the first j precincts
k are assigned to D_1
exactly x vote for R in D_1
exactly y vote for R in D_2

D_1
$k - 1$ precincts
$x - R(p_j)$ voters for R

D_2
$j - k$ precincts
y voters for R

p_j

Then assign p_j to D_1

D_1
k precincts
x voters for R

D_2
$j - k$ precincts
y voters for R

OR

D_1
k precincts
x voters for R

D_2
$j - 1 - k$ precincts
$y - R(p_j)$ voters for R

p_j

Then assign p_j to D_2

$$S(j, k, x, y) = S(j-1, k-1, x-R(p_j), y) \vee S\left(j-1, k, x, y-R(p_j)\right)$$

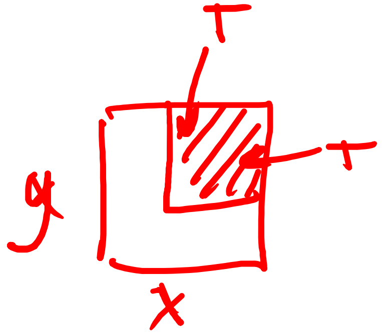
Final Algorithm

$$S(j,k,x,y) = S\left(j-1,k-1,x-R(p_j),y\right) \vee S\left(j-1,k,x,y-R(p_j)\right)$$

Initialize $S(0,0,0,0) =$ True
for $j = 1, \ldots, n$:
  for $k = 1, \ldots, \min(j, \frac{n}{2})$:
    for $x = 0, \ldots, jm$:
      for $y = 0, \ldots, jm$:
        $S(j,k,x,y) =$
          $S\left(j-1,k-1,x-R(p_j),y\right)$
          $\vee\ S\left(j-1,k,x,y-R(p_j)\right)$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$S(j,k,x,y) =$ True if:
from among the first j precincts
k are assigned to D_1
exactly x vote for R in D_1
exactly y vote for R in D_2



Run Time

$$S(j,k,x,y) = S(j-1,k-1,x-R(p_j),y) \vee S\left(j-1,k,x,y-R(p_j)\right)$$

Initialize $S(0,0,0,0) =$ True

n for $j = 1, \ldots, n$:

$\frac{n}{2}$ for $k = 1, \ldots, \min(j, \frac{n}{2})$:

nm for $x = 0, \ldots, jm$:

nm for $y = 0, \ldots, jm$:

$S(j,k,x,y) =$

$S(j-1,k-1,x-R(p_j),y)$

$\vee S\left(j-1,k,x,y-R(p_j)\right)$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$\Theta(n^4 m^2)$

$\Theta(n^4 m^2)$

- Runtime depends on the *value* of m, not *size* of m
- Run time is exponential in *size* of input
- Note: Gerrymandering is NP-Complete