CS4102 Algorithms
Spring 2019

Warm up:
Show that the sum of degrees of all nodes in any undirected graph is even

Show that for any graph $G = (V, E)$, $\sum_{v \in V} \deg(v)$ is even

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$\sum_{v \in V} \deg(v)$ is always even

- $\deg(v)$ counts the number of edges incident v
- Consider any edge e $\in E$
- This edge is incident 2 vertices (on each end)
- This means $2 \cdot |E| = \sum_{v \in V} \deg(v)$
- Therefore $\sum_{v \in V} \deg(v)$ is even

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Today's Keywords

- Greedy Algorithms
- Choice Function
- Graphs
- Minimum Spanning Tree
- Kruskal's Algorithm
- Prim's Algorithm
- Cut Theorem

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CLRS Readings

- Chapter 22
- Chapter 23

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Homeworks

- HW7 Due **Tuesday** April 16 @11pm
  - Written (use latex)
  - Graphs
  - Released tonight

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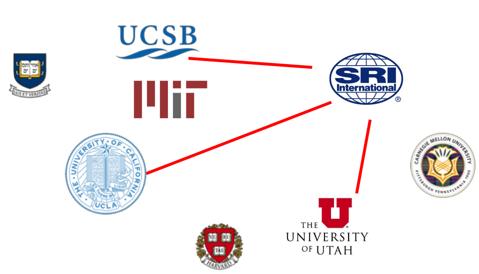
ARPANET



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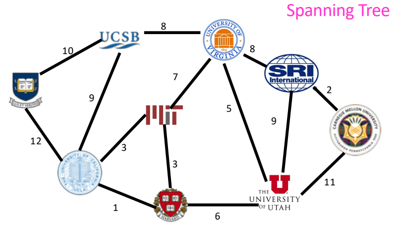
Problem

Find a
Minimum
Spanning Tree



We need to connect together all these places into a network
We have feasible wires to run, plus the cost of each wire
Find the cheapest set of wires to run to connect all places

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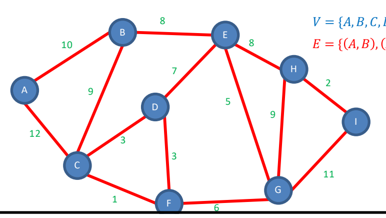
Graphs

Definition: $G = (V, E)$ Vertices/Nodes Edges

$w(e)$ = weight of edge e

$V = \{A, B, C, D, E, F, G, H, I\}$
$E = \{(A, B), (A, C), (B, C), \ldots\}$



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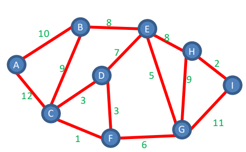
Adjacency List Representation



A	B	C		
B	A	C	E	
C	A	B	D	F
D	C	E	F	
E	B	D	G	H
F	C	D	G	
G	E	F	H	I
H	E	G	I	
I	G	H		

<u>Tradeoffs</u>
Space: $V + E$
Time to list neighbors: $Degree(A)$
Time to check edge (A, B): $Degree(A)$

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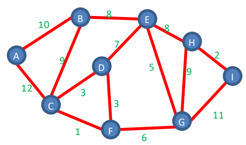
Adjacency Matrix Representation



	A	B	C	D	E	F	G	H	I
A		1	1						
B	1		1		1				
C	1	1		1		1			
D			1		1	1			
E		1		1			1	1	
F			1	1			1		
G					1	1		1	1
H					1		1		1
I							1	1	

Tradeoffs
Space: V^2
Time to list neighbors: V
Time to check edge (A, B): $O(1)$

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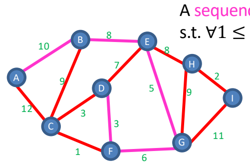
Definition: Path

A sequence of nodes $(v_1, v_2, \dots, v_k)$
s.t. $\forall 1 \le i \le k-1$, $(v_i, v_{i+1}) \in E$



Simple Path:
A path in which each node appears at most once

Cycle:
A path of >2 nodes in which $v_1 = v_k$

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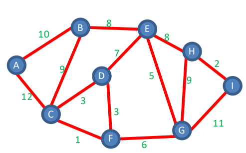
Definition: Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2



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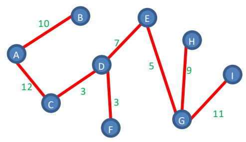
Definition: Tree

A connected graph with no cycles



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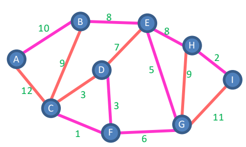
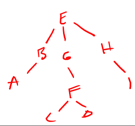
Definition: Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$



How many edges does T have?
$V - 1$



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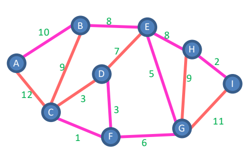
Definition: Minimum Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

$$Cost(T) = \sum_{e \in E_T} w(e)$$



How many edges does T have?
$V - 1$

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Greedy Algorithms

- Require Optimal Substructure
  - Solution to larger problem contains the solution to a smaller one
  - Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

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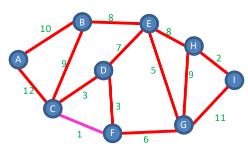
Kruskal's Algorithm

Start with an empty tree *A*
Add to *A* the lowest-weight edge that does not create a cycle



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Kruskal's Algorithm

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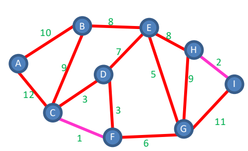

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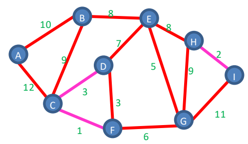

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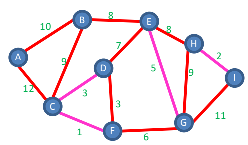

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Kruskal's Algorithm

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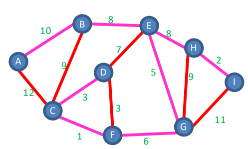

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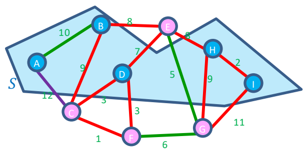
Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

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Exchange argument

- Shows correctness of a greedy algorithm
- Idea:
  - Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
  - How to show my sandwich is at least as good as yours:
    - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

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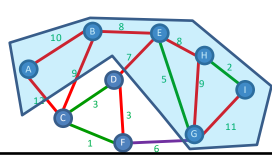
Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.



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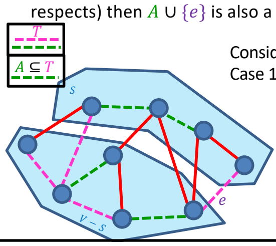
Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V-S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,
Case 1: (the easy case)
If $e \in T$ Then claim holds

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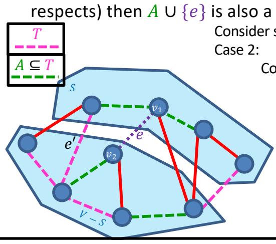
Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V-S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,
Case 2:
Consider if $e = (v_1, v_2) \notin T$

Since T is a MST, there is some path from v_1 to v_2.

Let e' be the first edge on this path which crosses the cut

Build tree T' by exchanging e' for e

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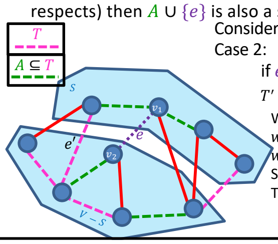
Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V-S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,
Case 2:
if $e = (v_1, v_2) \notin T$
$T' = T$ with edge e instead of e'
We assumed $w(e) \leq w(e')$
$w(T') = w(T) - w(e') + w(e)$
$w(T') \leq w(T)$
So T' is also a MST!
Thus the claim holds

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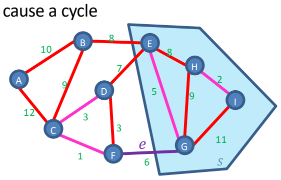
Kruskal's Algorithm

Start with an empty tree A
Repeat $V-1$ times:
    Add the min-weight edge that doesn't cause a cycle

Keep edges in a Disjoint-set data structure (very fancy)
$O(E \log V)$



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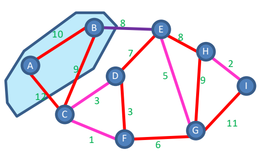
General MST Algorithm

Start with an empty tree A
Repeat $V-1$ times:
    Pick a cut $(S, V-S)$ which A respects
    Add the min-weight edge which crosses $(S, V-S)$



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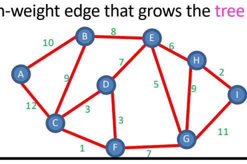
Prim's Algorithm

Start with an empty tree A
Repeat $V-1$ times:
    Pick a cut $(S, V-S)$ which A respects
    Add the min-weight edge which crosses $(S, V-S)$

S is all endpoint of edges in A
e is the min-weight edge that grows the tree



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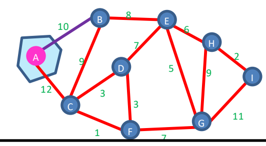
Prim's Algorithm

Start with an empty tree A
Pick a start node
Repeat $V-1$ times:
Add the min-weight edge which connects to node in A with a node not in A



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Prim's Algorithm

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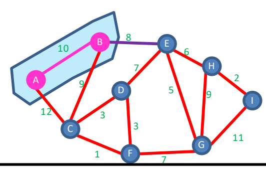

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Prim's Algorithm

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Pick a start node
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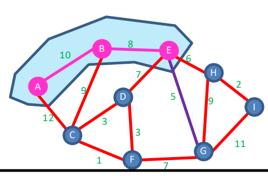

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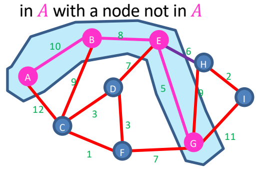

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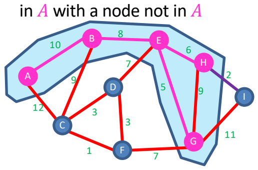
Prim's Algorithm

Start with an empty tree A
Pick a start node
Repeat $V - 1$ times:
Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap
$O(E \log V)$



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Summary of MST results

- Fredman-Tarjan '84: $\Theta(E + V \log V)$
- Gabow et al '86: $\Theta(E \log \log^* V)$
- Chazelle '00: $\Theta(E\alpha(V))$
- Pettie-Ramachandran '02: $\Theta(?)$ (optimal)
- Karger-Klein-Tarjan '95: $\Theta(E)$ (randomized)

- [read and summarize any/all for EC]

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