CS4102 Algorithms
Spring 2019

Warm Up

Today's Keywords

- Graphs
- MaxFlow/MinCut
- Ford-Fulkerson
- Edmunds-Karp

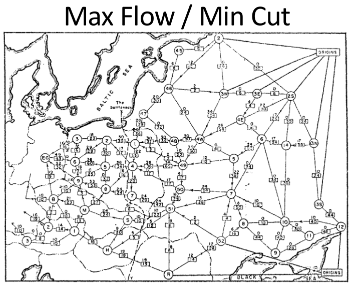
CLRS Readings

- Chapter 25
- Chapter 26

Homeworks

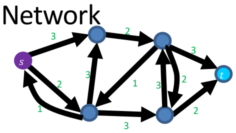
- HW8 due Tuesday 4/23 at 11pm
  - Python or Java
  - Tiling Dino

Max Flow / Min Cut



Railway map of Western USSR, 1955

Flow Network



Graph $G = (V, E)$
Source node $s \in V$
Sink node $t \in V$
Edge Capacities $c(e) \in$ Positive Real numbers

Max flow intuition: If s is a faucet, t is a drain, and s connects to t through a network of pipes with given capacities, what is the maximum amount of water which can flow from the faucet to the drain?

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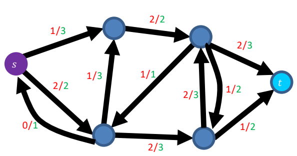
Flow



- Assignment of values to edges
  - $f(e) = n$
  - Amount of water going through that pipe
- Capacity constraint
  - $f(e) \le c(e)$
  - Flow cannot exceed capacity
- Flow constraint
  - $\forall v \in V - \{s, t\}, inflow(v) = outflow(v)$
  - $inflow(v) = \sum_{x \in V} f(x, v)$
  - $outflow(v) = \sum_{x \in V} f(v, x)$
  - Water going in must match water coming out
- Flow of G: $|f| = outflow(s) - inflow(s)$
  - Net outflow of s

3 in example above

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Max Flow

- Of all valid flows through the graph, find the one which maximizes:
  - $|f| = outflow(s) - inflow(s)$

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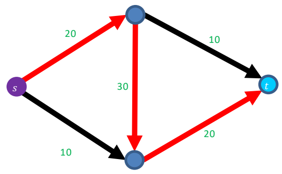
Greedy doesn't work

Saturate Highest Capacity Path First



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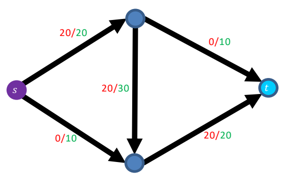
Greedy doesn't work

Saturate Highest Capacity Path First



Overall Flow: $|f| = 20$

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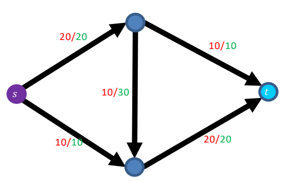
Greedy doesn't work

Better Solution



Overall Flow: $|f| = 30$

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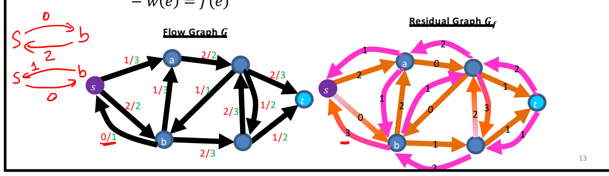
Residual Graph G_f

- Keep track of net available flow along each edge
- "Forward edges": weight is equal to available flow along that edge in the flow graph
  - $w(e) = c(e) - f(e)$
- "Back edges": weight is equal to flow along that edge in the flow graph
  - $w(e) = f(e)$

Flow I *could* add

Flow I *could* remove



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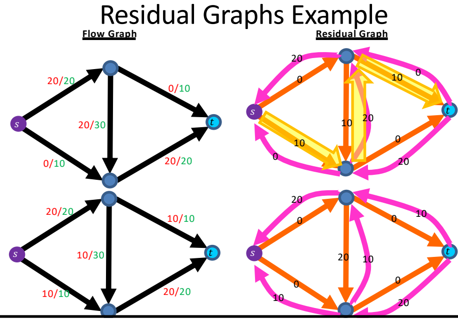
Residual Graphs Example



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Ford-Fulkerson

- Augmenting Path: a path of positive-weight edges from s to t in the residual graph
- Algorithm: Repeatedly add the flow of any augmenting path

$\forall(u,v) \in E$ Initialize $f(u,v) = 0$
While there is an augmenting path p in G_f
  let $f = \min_{u,v \in p} c_f(u,v)$
  add f to the flow of each edge in p

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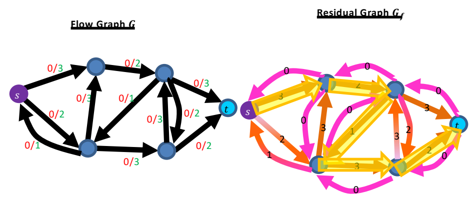
Ford Fulkerson: example



Add flow of 1 to this path

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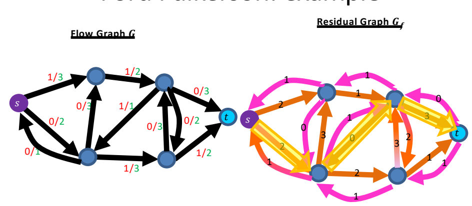
Ford Fulkerson: example



Add flow of 1 to this path

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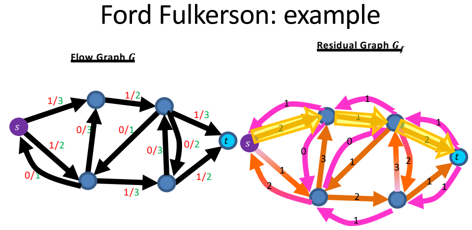
Ford Fulkerson: example



Add flow of 1 to this path

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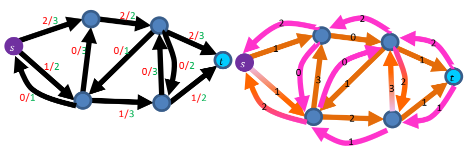
Ford Fulkerson: example

Flow Graph G

Residual Graph G_f



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Ford-Fulkerson: Run Time

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While there is an augmenting path p in G_f
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    add f to the flow of each edge in p

Time to find an augmenting path: BFS: $\Theta(V+E)$

Number of iterations of While loop: $|f|$

$\Theta(E \cdot |f|)$

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Why might we loop $|f|$ times?

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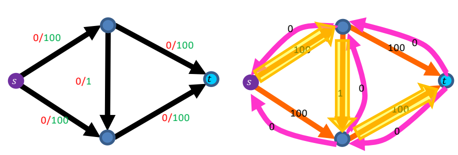

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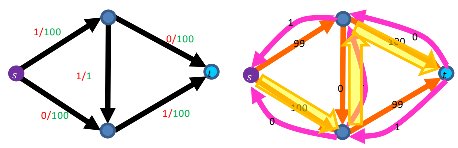

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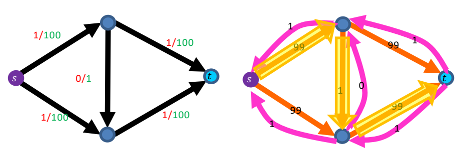

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Why might we loop $|f|$ times?

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While there is an augmenting path p in G_f
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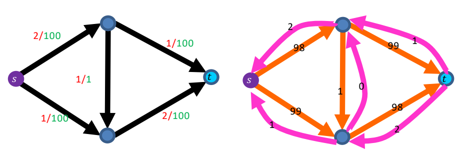
add f to the flow of each edge in p

Each time we increase flow by 1
Loop runs 200 times



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Can We Avoid this?

- Edmunds-Karp Algorithm
- $\Theta(\min(E|f|, VE^2))$
- Choose augmenting path with fewest edges

$\forall (u,v) \in E$ Initialize $f(u,v) = 0$
While there is an augmenting path in G_f
  let p be the shortest augmenting path
  let $f = \min_{u,v \in p} c_f(u,v)$
  add f to the flow of each edge in p

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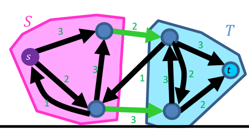
Showing Correctness of Ford-Fulkerson

- Consider cuts which separate s and t
  - Let $s \in S$, $t \in T$, s.t. $V = S \cup T$
- Cost of cut $(S, T) = ||S, T||$
  - Sum **capacities** of edges which go from S to T
  - This example: 5



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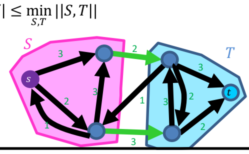
Maxflow≤MinCut

- Max flow upper bounded by any cut separating s and t
- Why? "Conservation of flow"
  - All flow exiting s must eventually get to t
  - To get from s to t, all "tanks" must cross the cut
- Conclusion: If we find the minimum-cost cut, we've found the maximum flow
  - $\max_f |f| \le \min_{S,T} ||S, T||$



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Maxflow/Mincut Theorem

- To show Ford-Fulkerson is correct:
  - Show that when there are no more augmenting paths, there is a cut with cost equal to the flow
- Conclusion: the maximum flow through a network matches the minimum-cost cut
  - $\max_f |f| = \min_{S,T} ||S,T||$
- Duality
  - When we've maximized max flow, we've minimized min cut (and vice-versa), so we can check when we've found one by finding the other

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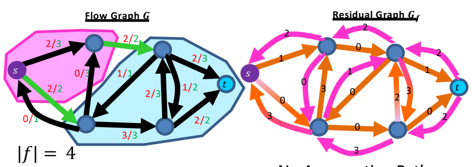
Example: Maxflow/Mincut



$|f| = 4$

No Augmenting Paths

$||S, T|| = 4$

Idea: When there are no more augmenting paths, there exists a cut in the graph with cost matching the flow

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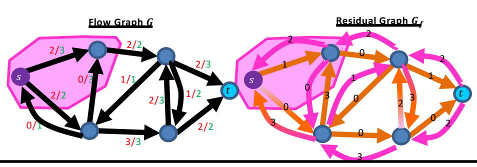
Proof: Maxflow/Mincut Theorem

- If $|f|$ is a max flow, then G_f has no augmenting path
  - Otherwise, use that augmenting path to "push" more flow
- Define S = nodes reachable from source node s by positive-weight edges in the residual graph
  - $T = V - S$
  - S separates s, t (otherwise there's an augmenting path)



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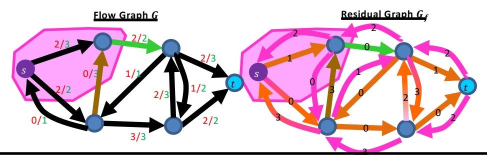
Proof: Maxflow/Mincut Theorem

- To show: $||S, T|| = |f|$
  - Weight of the cut matches the flow across the cut
- Consider edge (u, v) with $u \in S$, $v \in T$
  - $f(u, v) = c(u, v)$, because otherwise $w(u, v) > 0$ in G_f, which would mean $v \in S$
- Consider edge (y, x) with $y \in T$, $x \in S$
  - $f(y, x) = 0$, because otherwise the back edge $w(y, x) > 0$ in G_f, which would mean $x \in S$



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Proof Summary

1. The flow $|f|$ of G is upper-bounded by the sum of capacities of edges crossing any cut separating source s and sink t
2. When Ford-Fulkerson Terminates, there are no more augmenting paths in G_f
3. When there are no more augmenting paths in G_f then we can define a cut S = nodes reachable from source node s by positive-weight edges in the residual graph
4. The sum of edge capacities crossing this cut must match the flow of the graph
5. Therefore this flow is maximal

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Other Maxflow algorithms

- **Ford-Fulkerson**
  - $\boldsymbol{\Theta(E|f|)}$
- **Edmonds-Karp**
  - $\boldsymbol{\Theta(E^2 V)}$
- Push-Relabel (Tarjan)
  - $\Theta(EV^2)$
- Faster Push-Relabel (also Tarjan)
  - $\Theta(V^3)$

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