CS4102 Algorithms

Spring 2019

No time for a warm-up today!

Today’s Keywords

- Reductions
- Bipartite Matching
- Vertex Cover
- Independent Set
- NP-Completeness

CLRS Readings

- Chapter 34

Homeworks, etc

- HW9 due ~~Monday 4/29~~ Tuesday 4/30 at 11pm
  - Written (use LaTeX)
  - Reductions
- Final Exam: Saturday, May 4, 2-5pm
  - Heavily from material since midterm
    - May ask for runtime of an algorithm, some knowledge of D&C
    - Won’t directly ask you to solve recurrences
  - Practice final online by tomorrow
  - Review session?

Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

MacGyver's Reduction

Problem we don't know how to solve

A

Opening a door

Solution for $\boldsymbol{A}$

Keg cannon battering ram

Aim duct at door, insert keg

Put fire under the Keg

Reduction

Problem we do know how to solve

B

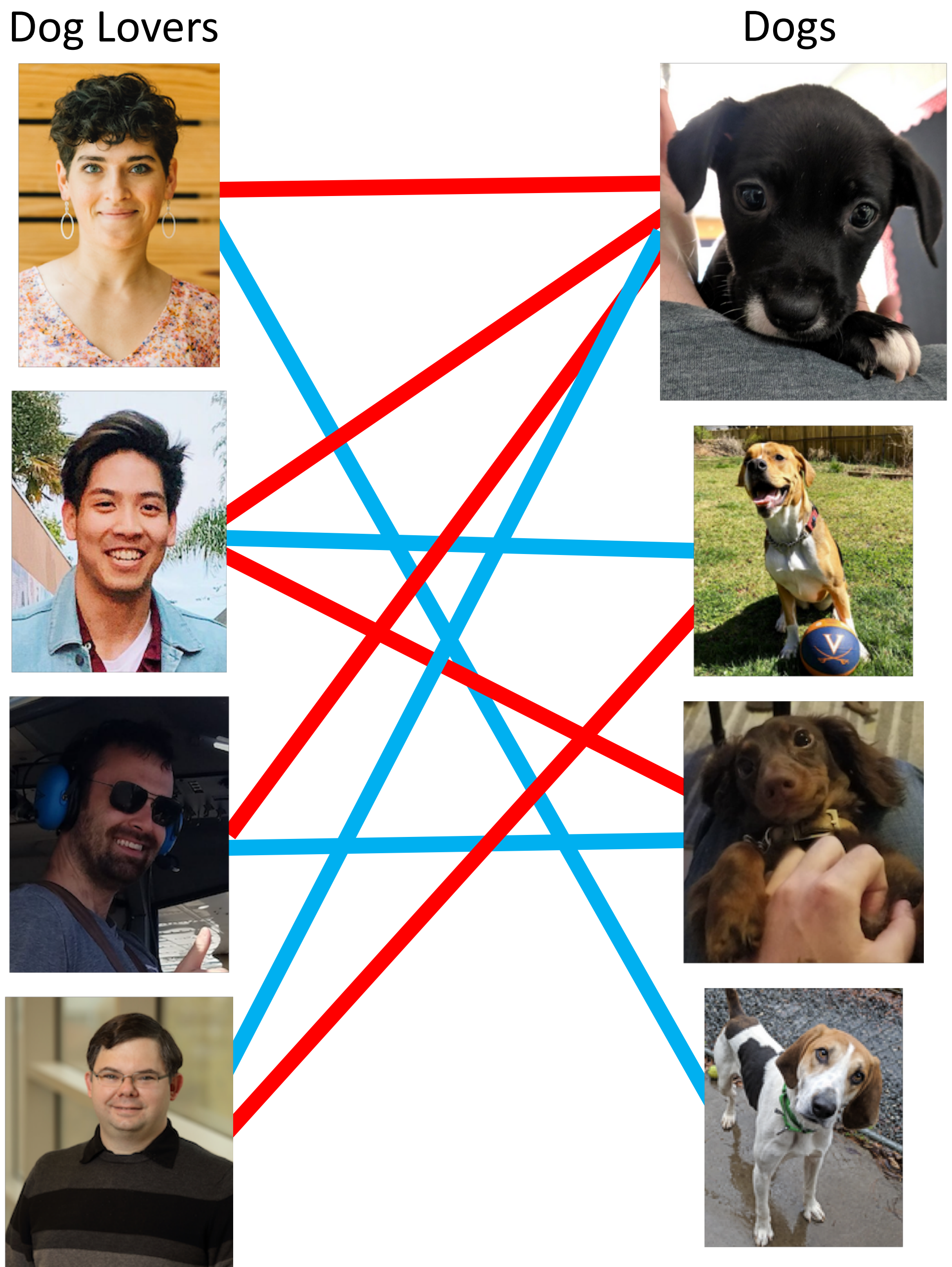
Lighting a fire

How?

Solution for $\boldsymbol{B}$

Alcohol, wood, matches

Maximum Bipartite Matching



Maximum Bipartite Matching Using Max Flow

Make $G = (L, R, E)$ a flow network $G' = (V', E')$ by:

- Adding in a source and sink to the set of nodes:
  - $V' = L \cup R \cup \{s, t\}$
- Adding an edge from source to L and from R to sink:
  - $E' = E \cup \{u \in L \mid (s, u)\} \cup \{v \in r \mid (v, t)\}$
- Make each edge capacity 1:
  - $\forall e \in E', c(e) = 1$

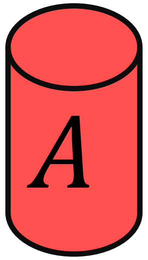
Remember: need to show

1. How to map instance of MBM to MF (and back) - construction
2. A valid solution to MF instance is a valid solution to MBM instance

Bipartite Matching Reduction

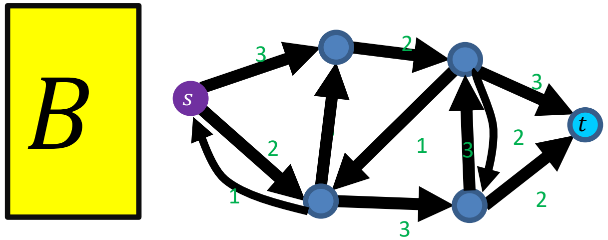
Problem we don’t know how to solve

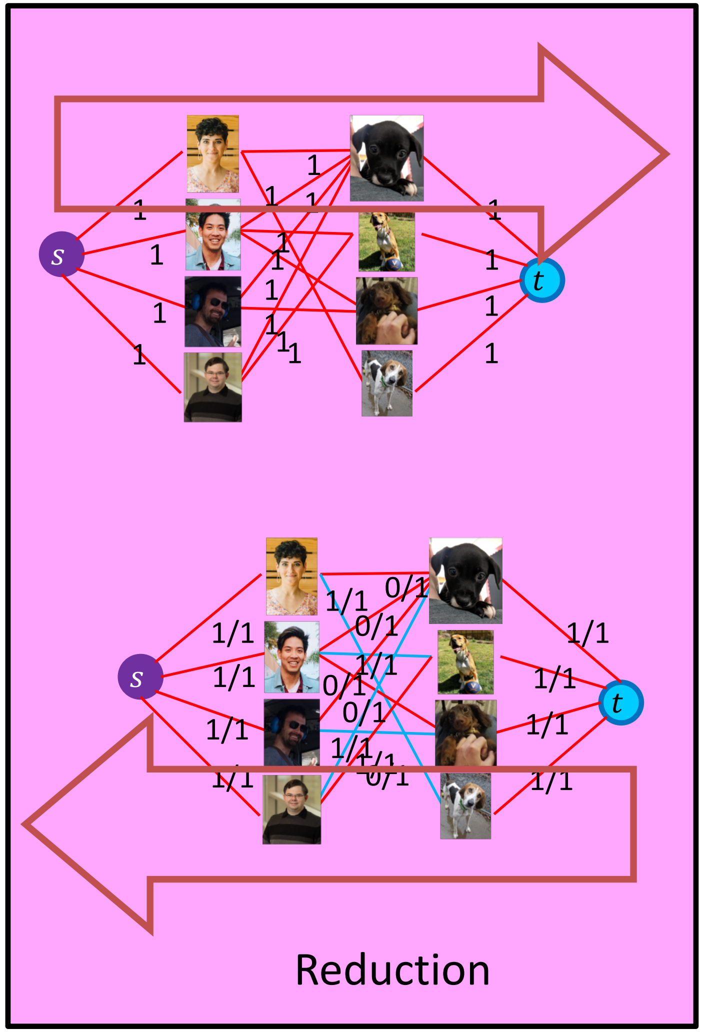
Bipartite Matching

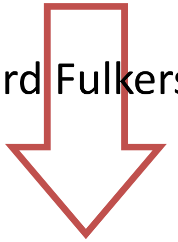


Problem we do know how to solve

Max Flow

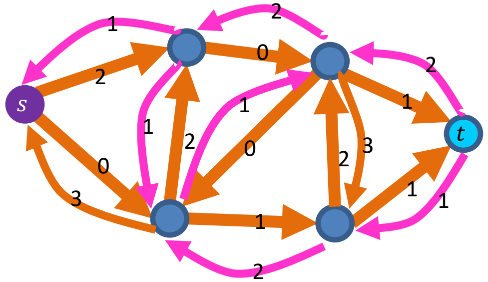




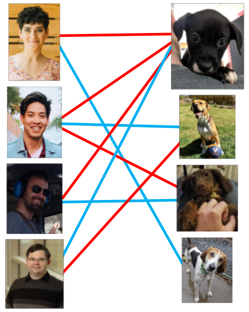


Ford Fulkerson

Solution for $\boldsymbol{B}$

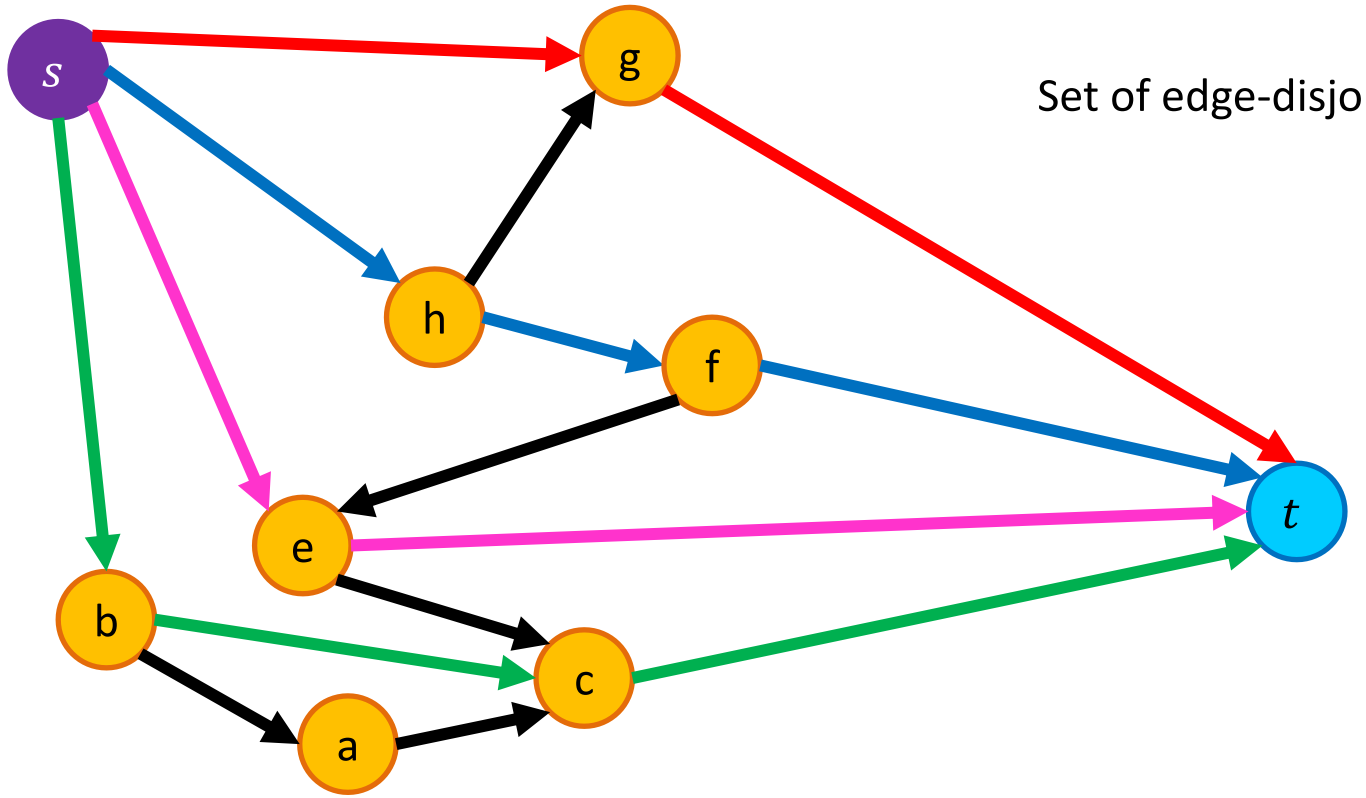


Solution for $\boldsymbol{A}$



Edge-Disjoint Paths

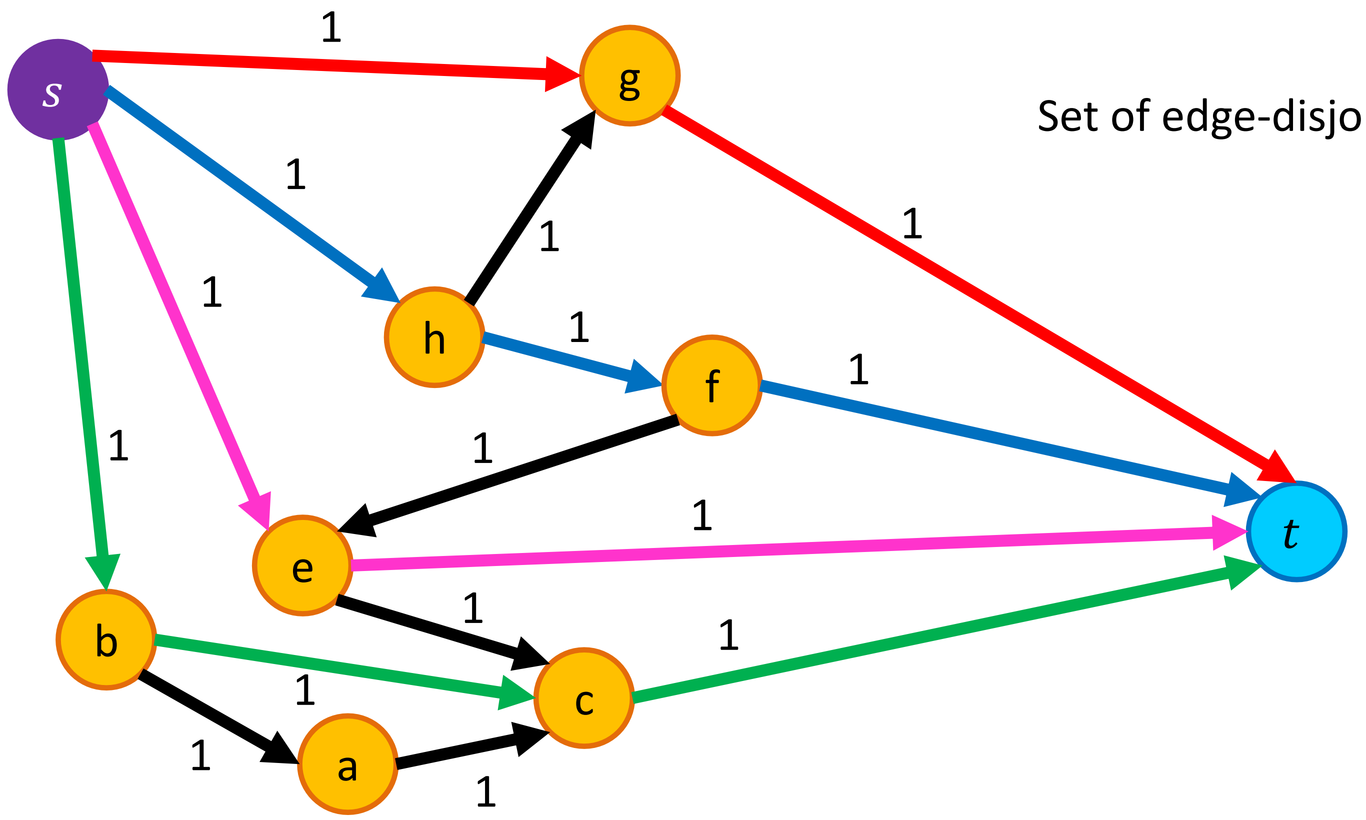
Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no edges



Set of edge-disjoint paths of size 4

Edge-Disjoint Paths Algorithm

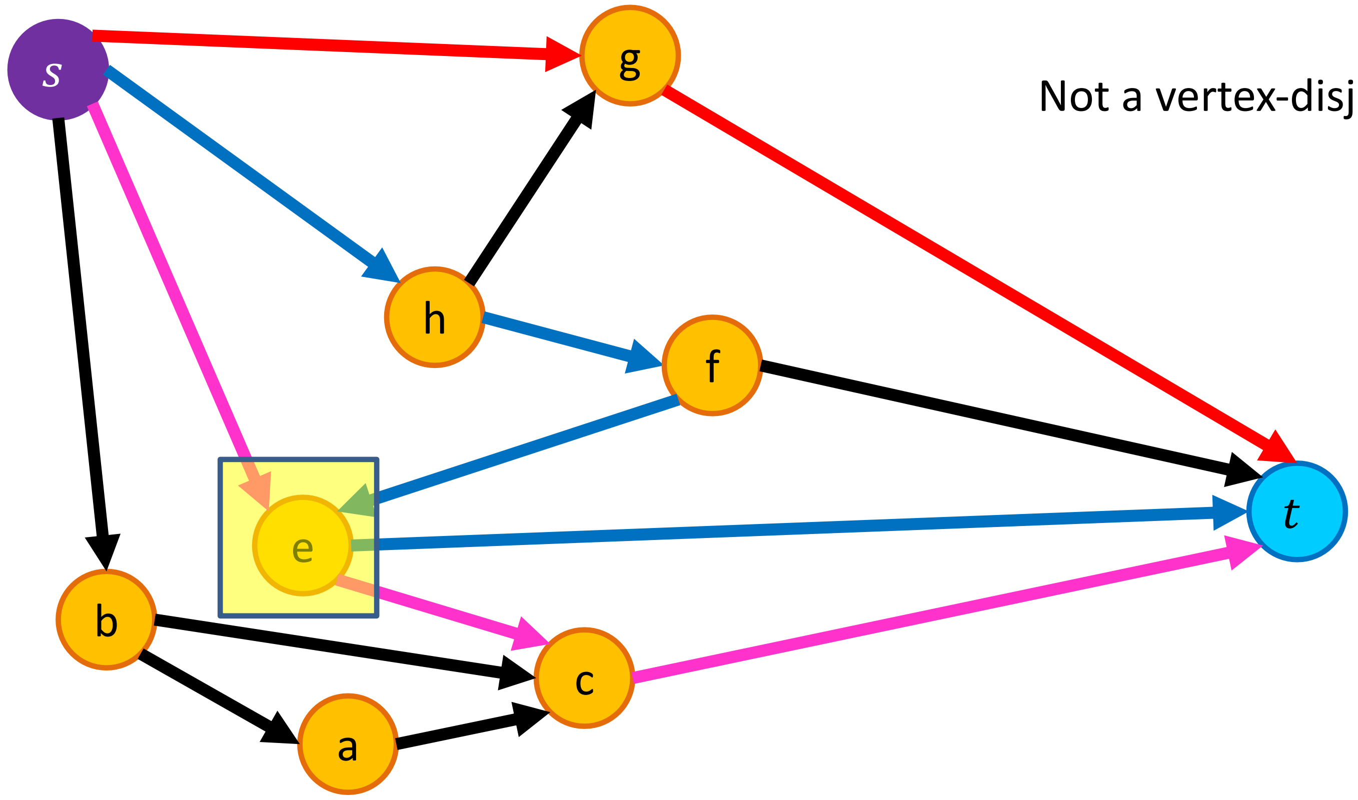
Make s and t the source and sink, give each edge capacity 1, find the max flow.



Set of edge-disjoint paths of size 4

Vertex-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no vertices



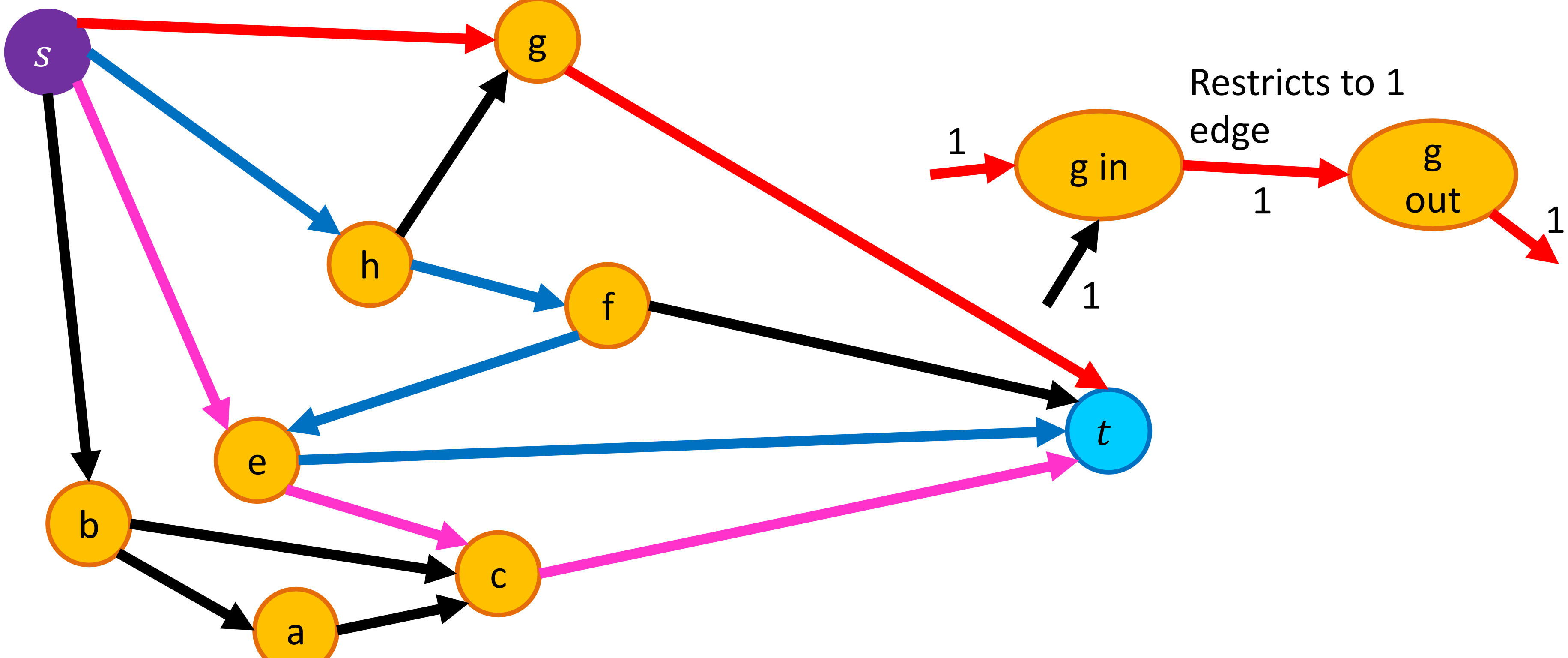
Not a vertex-disjoint path!

Vertex-Disjoint Paths Algorithm

Idea: Convert an instance of the vertex-disjoint paths problem into an instance of edge-disjoint paths

Make two copies of each node, one connected to incoming edges, the other to outgoing edges

Compute Edge-Disjoint paths on new graph



In General: Reduction

Problem we don't know how to solve

A

Map Instances of problem $\boldsymbol{A}$ to Instances of $\boldsymbol{B}$

Problem we do know how to solve

B

Using any Algorithm for $\boldsymbol{B}$

Remember: need to show

1. How to map instance of A to B (and back)
2. Why solution to B was a valid solution to A

Map Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$

Solution for $\boldsymbol{A}$

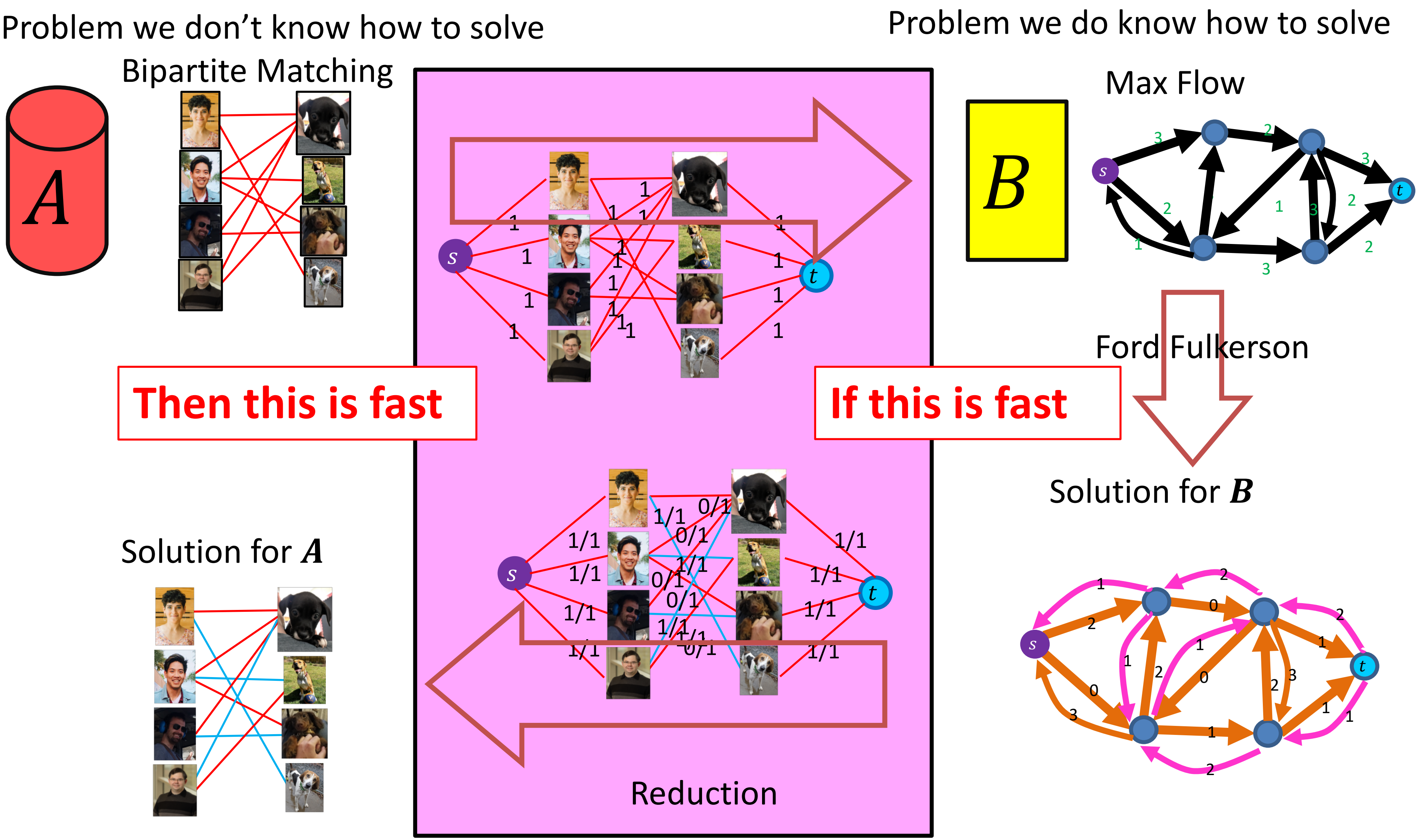
X

Solution for $\boldsymbol{B}$

Y

Reduction

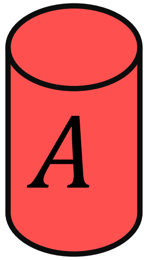
Bipartite Matching Reduction

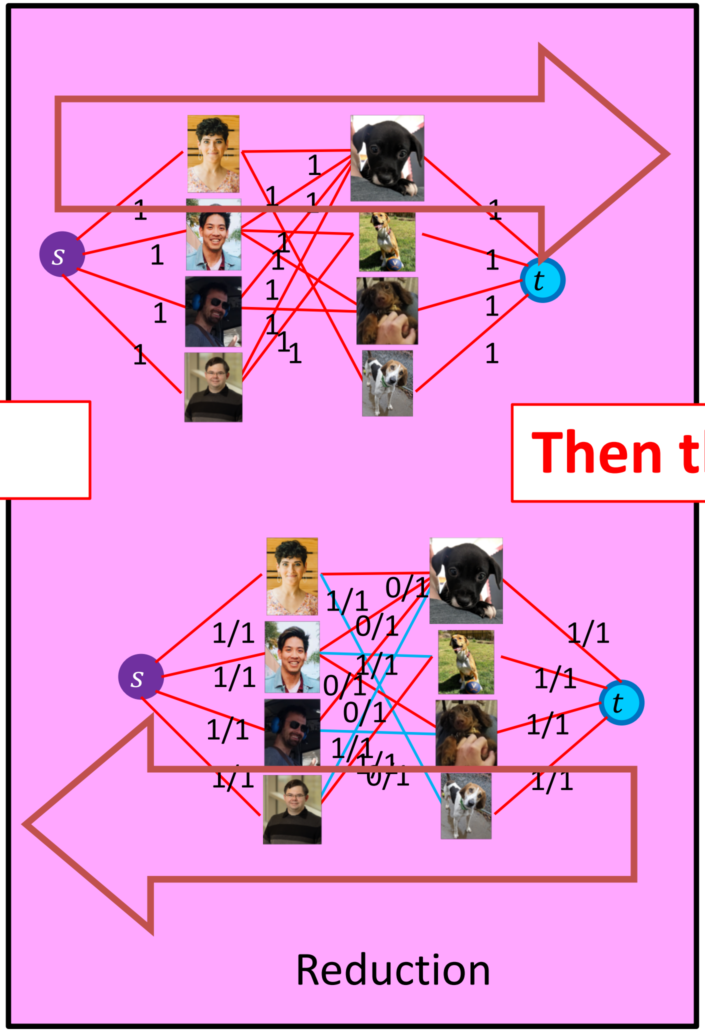


Bipartite Matching Reduction

Problem we don't know how to solve

Bipartite Matching

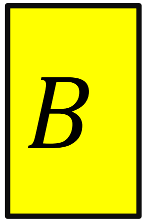


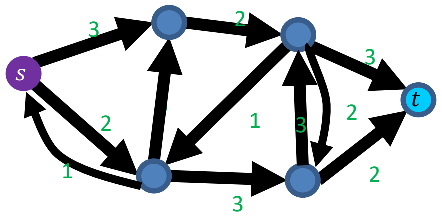


If this is slow

Problem we do know how to solve

Max Flow

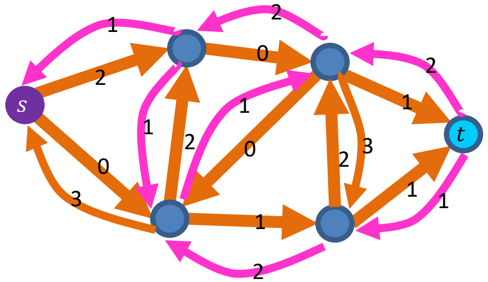




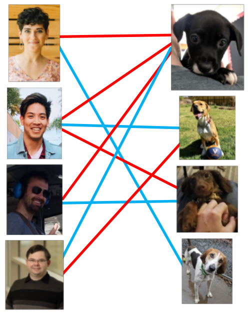
Ford Fulkerson

Then this is slow

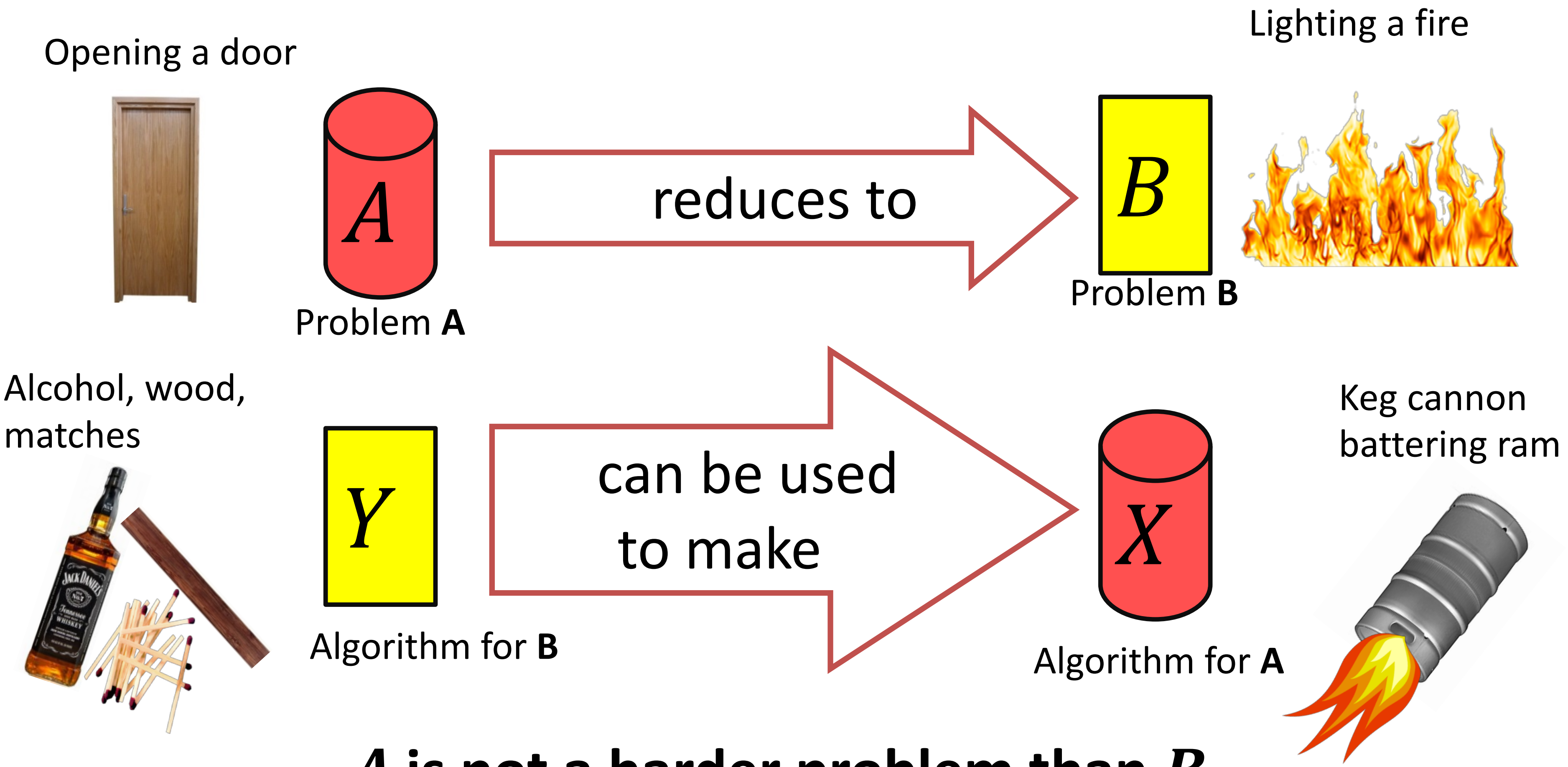
Solution for $\boldsymbol{B}$



Solution for $\boldsymbol{A}$



Worst-case lower-bound Proofs



A is not a harder problem than B

$$A \leq B$$

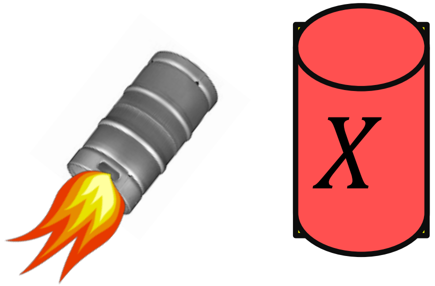
The name “reduces” is confusing: it is in the *opposite* direction of the making

Proof of Lower Bound by Reduction

<u>To Show: Y is slow</u>

1. We know X is slow
(e.g., X = some way to open the door)

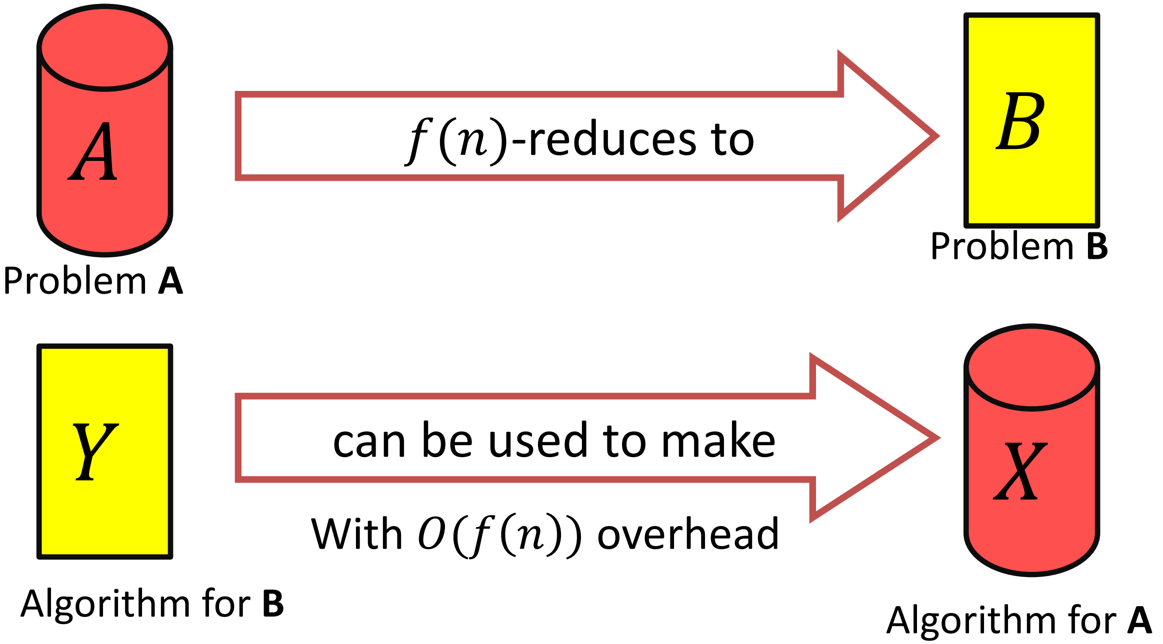
2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)



3. Show how to use Y to perform X quickly

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick

Reduction Proof Notation



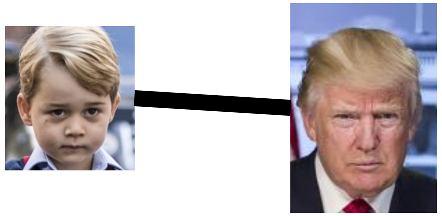
A is not a harder problem than B

$$A \le B$$

If A requires time $\Omega(f(n))$ time then B also requires $\Omega(f(n))$ time

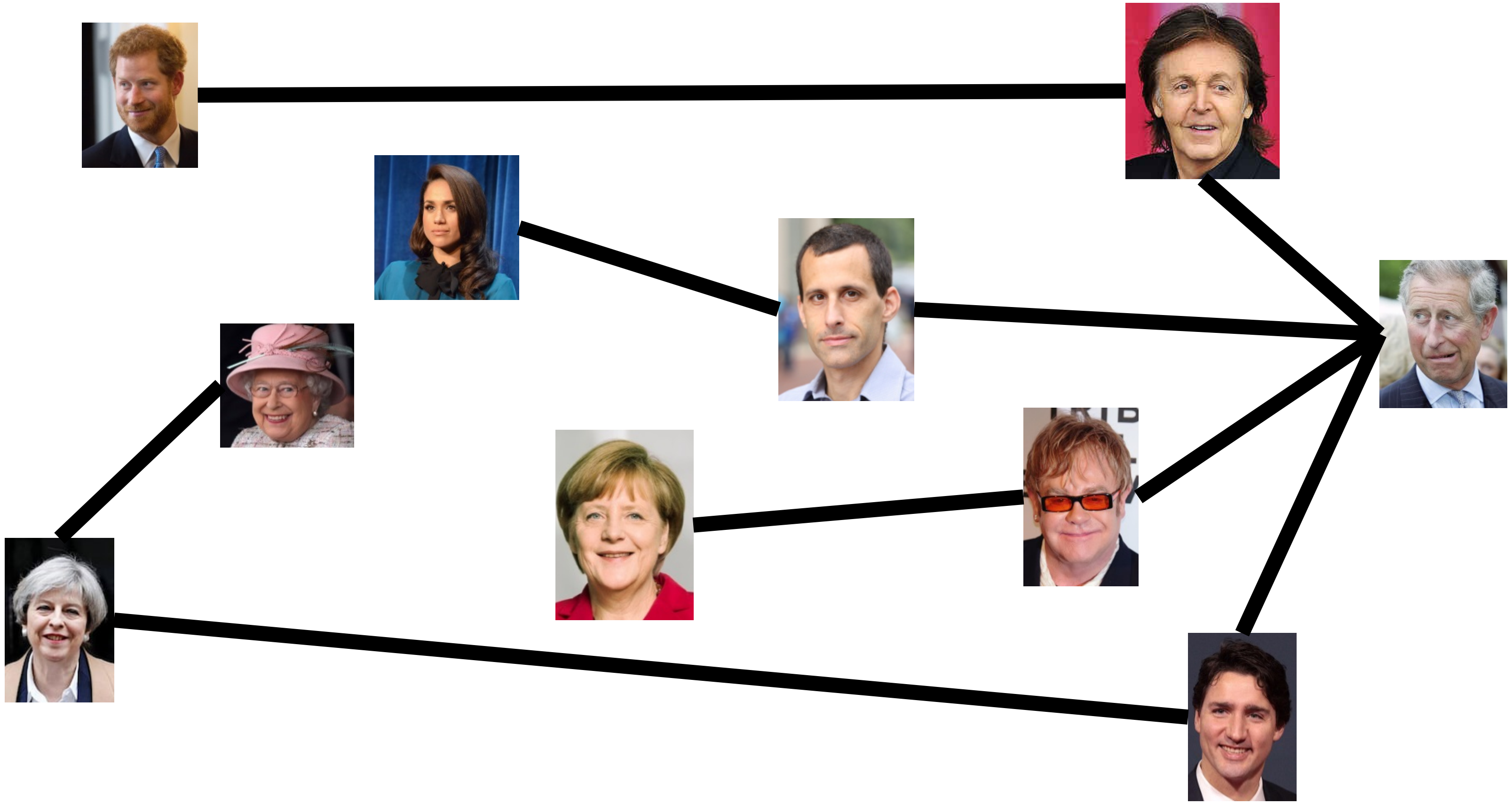
$$A \le_{f(n)} B$$

Party Problem



Draw Edges between people who don’t get along

Find the maximum number of people who get along

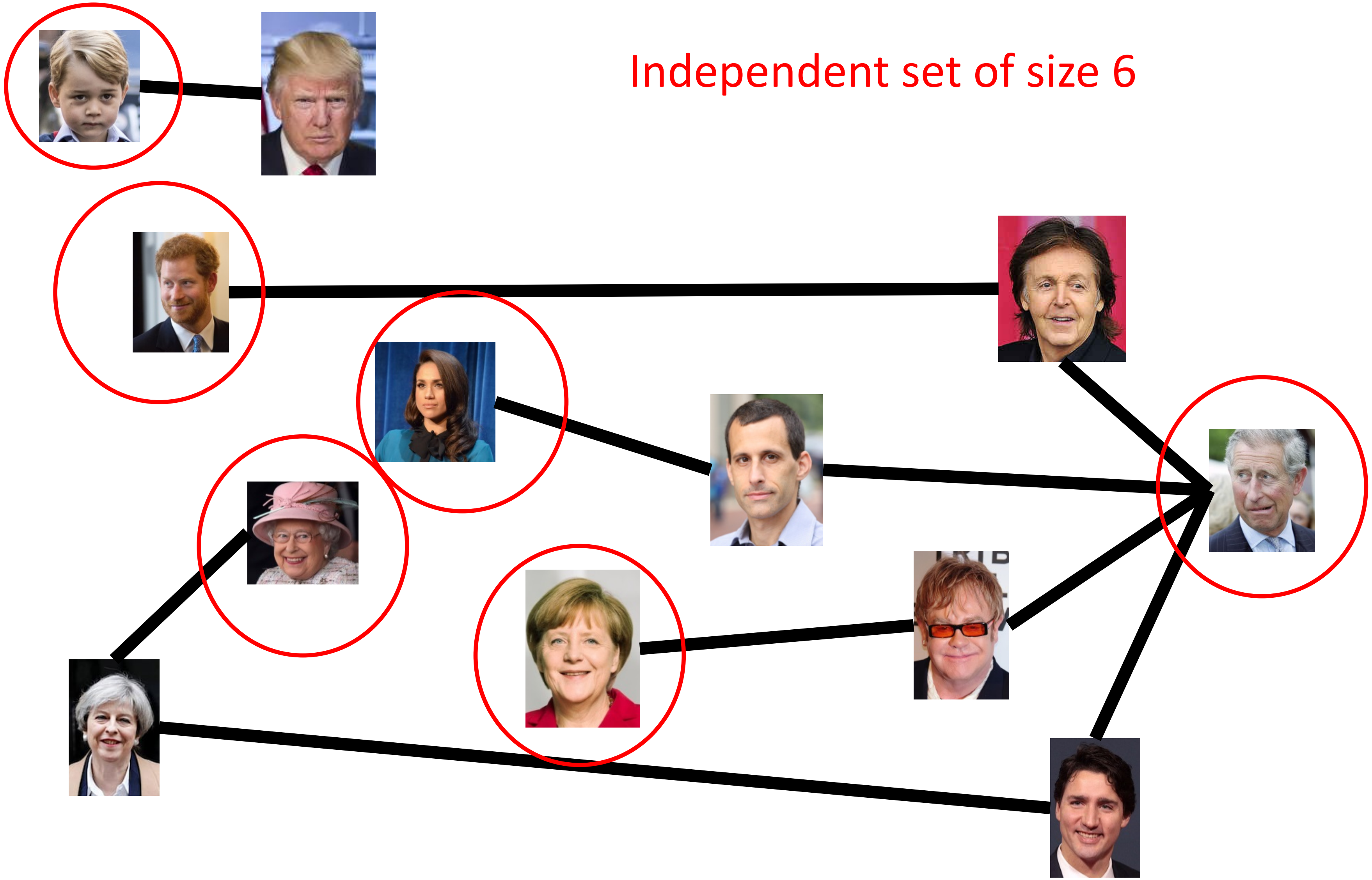


Maximum Independent Set

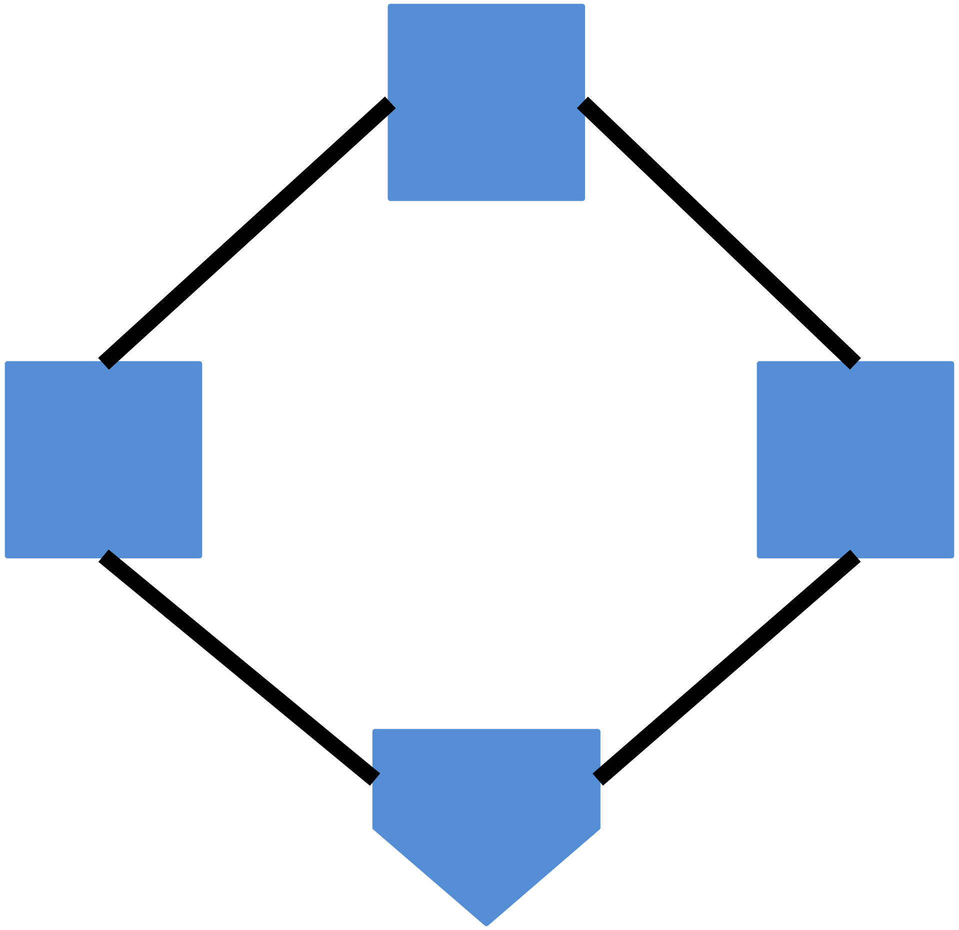
- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

Example

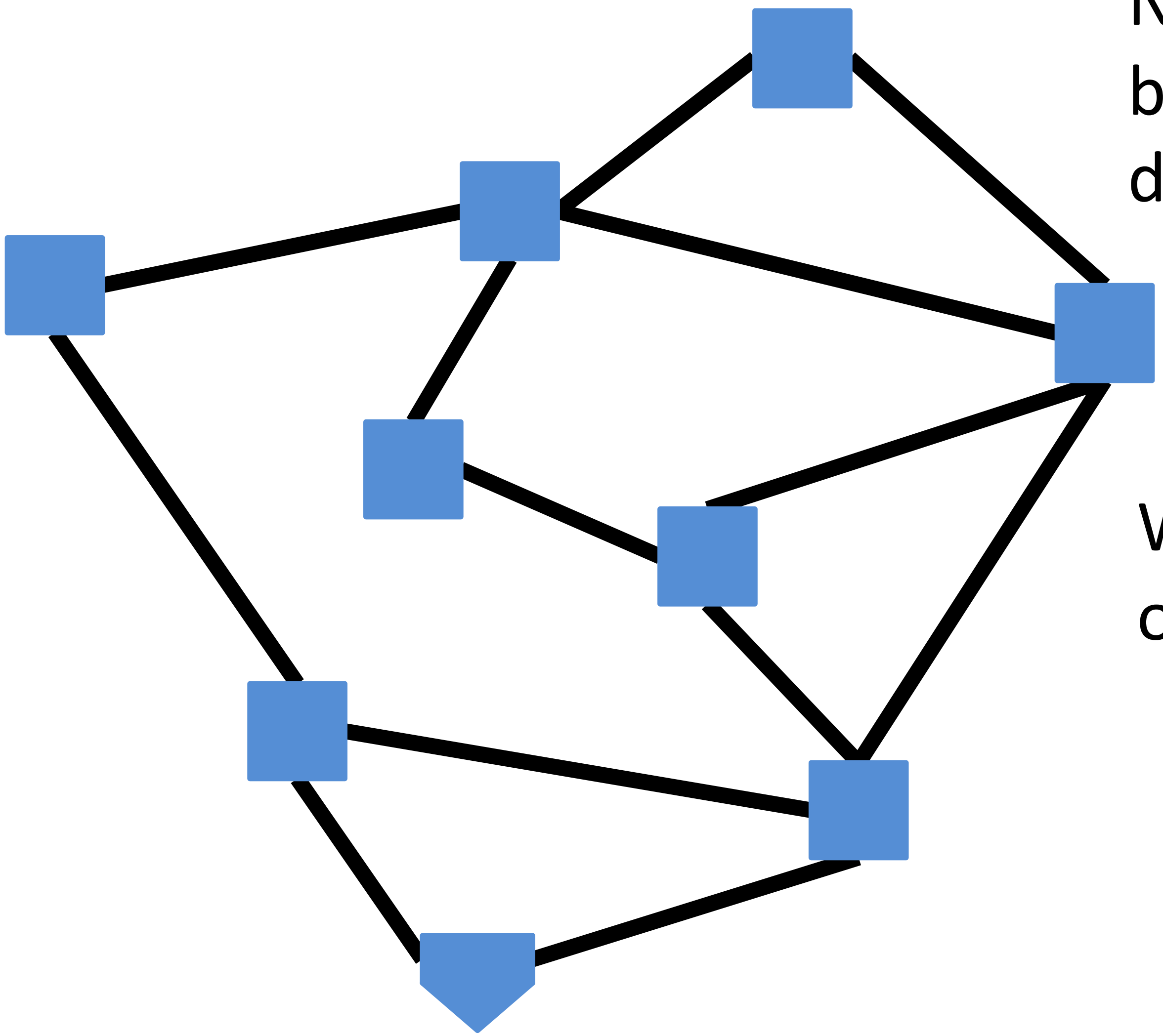
Independent set of size 6



Generalized Baseball



Generalized Baseball



Need to place defenders on bases such that every edge is defended

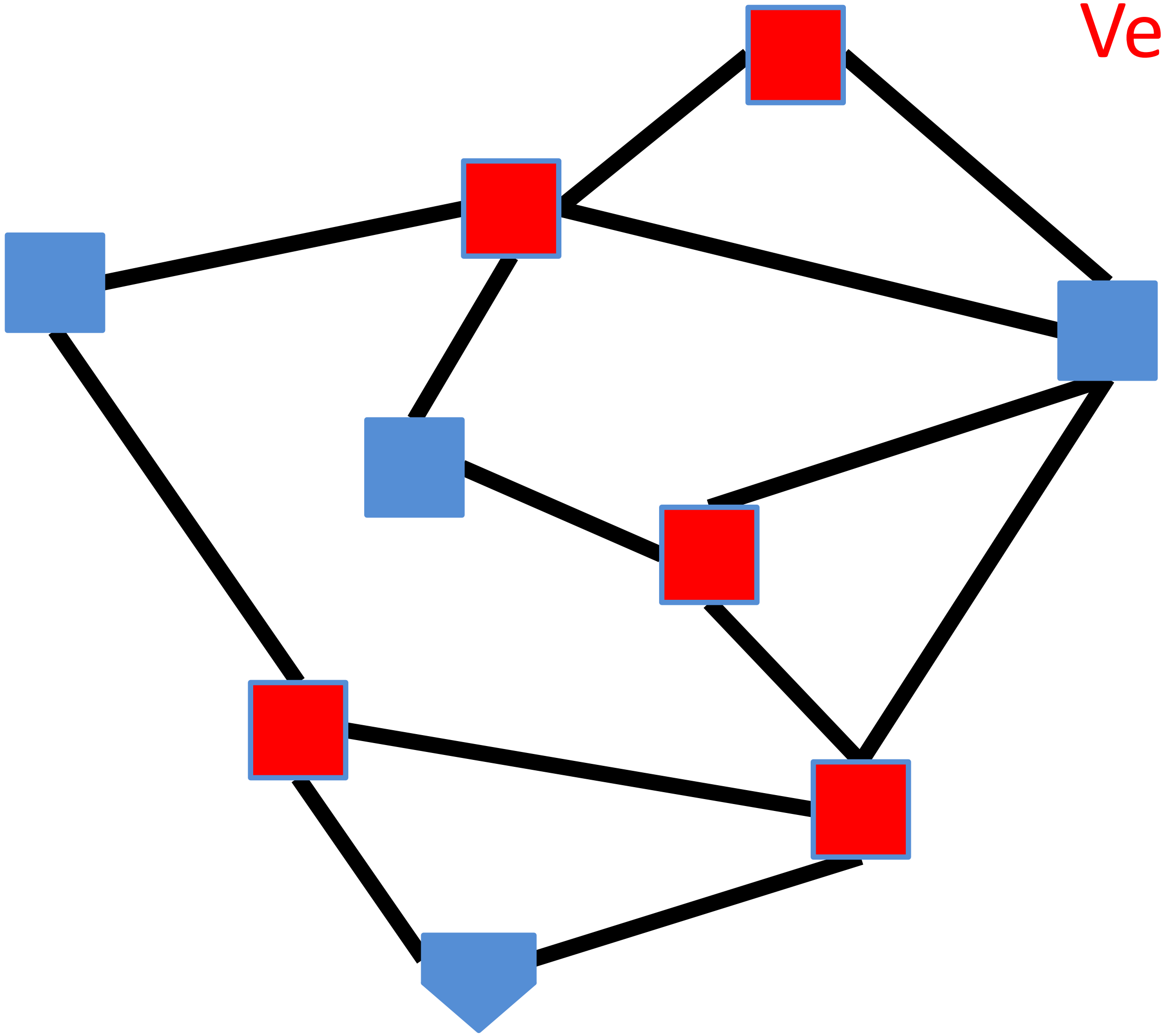
What’s the fewest number of defenders needed?

Minimum Vertex Cover

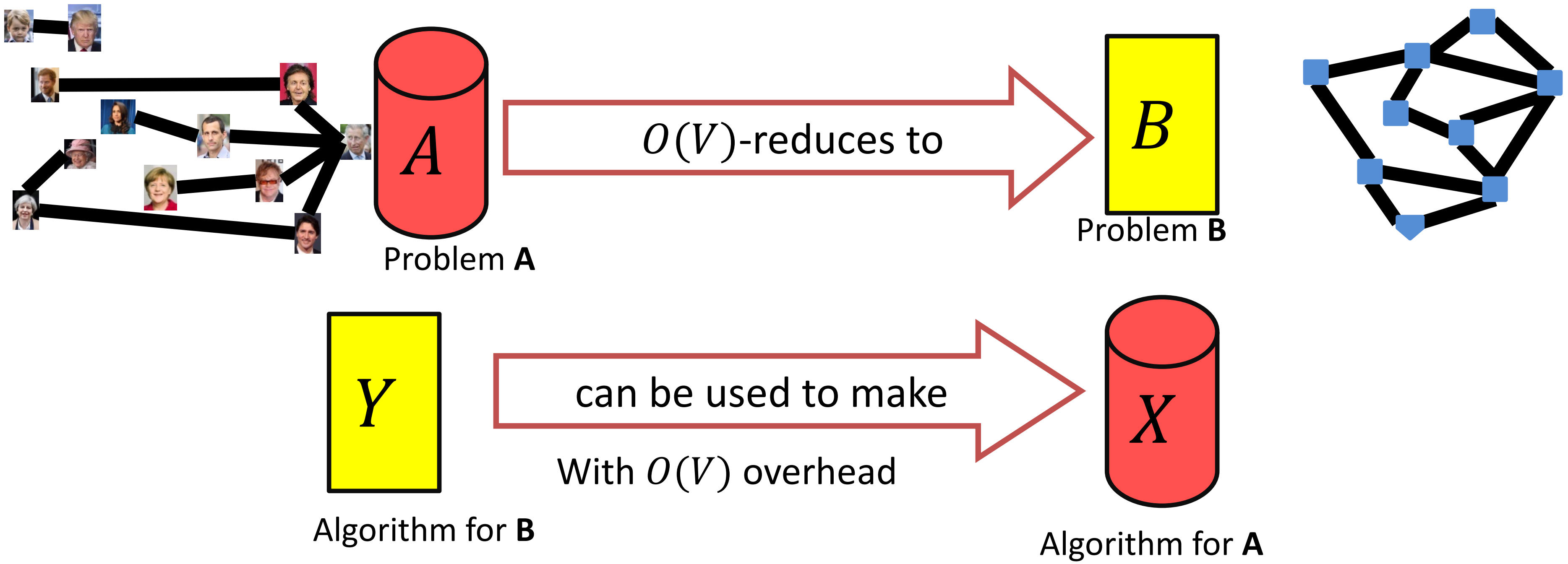
- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover: Given a graph $G = (V, E)$ find the minimum vertex cover C

Example

Vertex cover of size 5



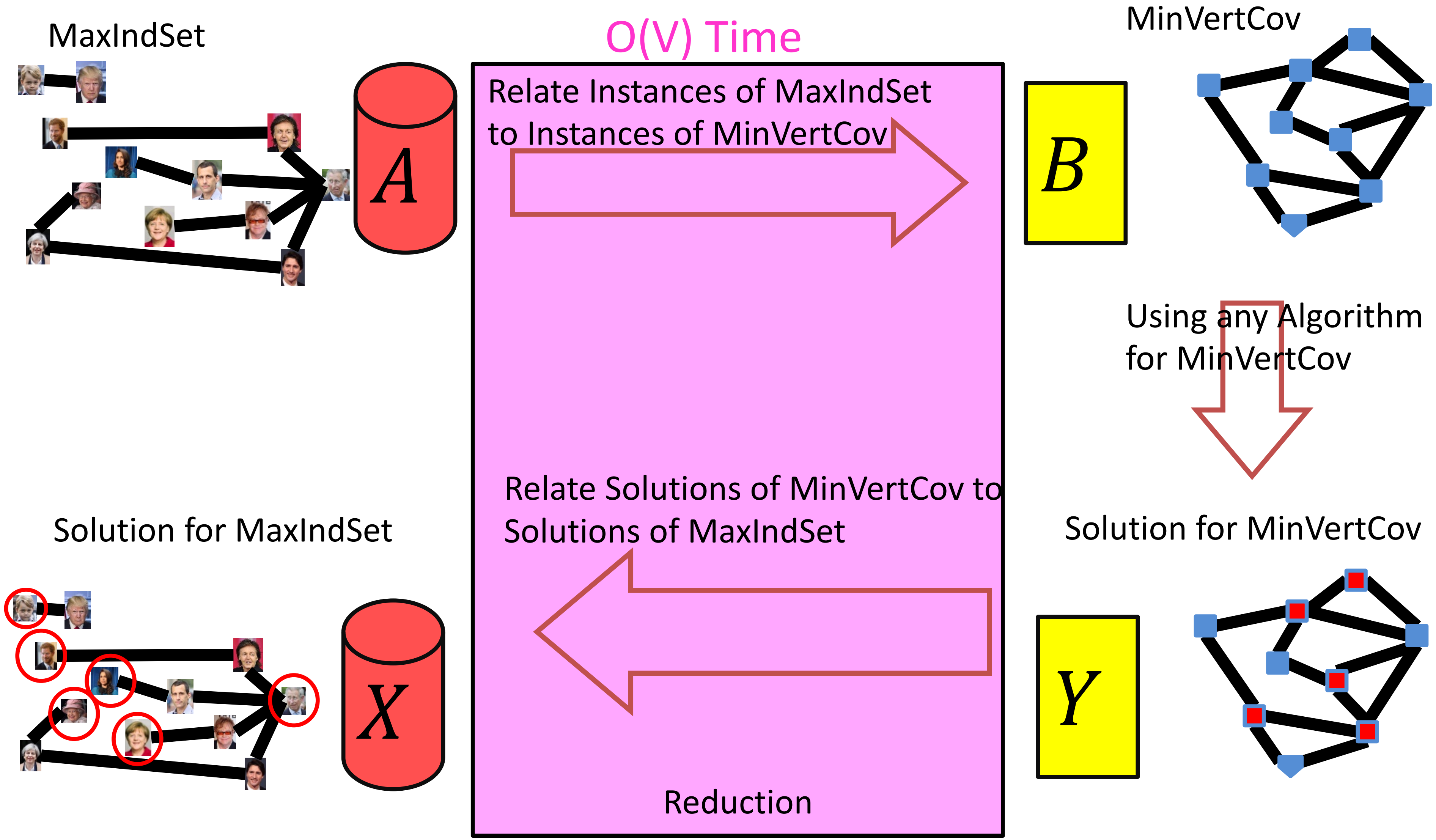
MaxIndSet$\leq_V$MinVertCov



If A requires time $\Omega(f(n))$ time then B also requires $\Omega(f(n))$ time

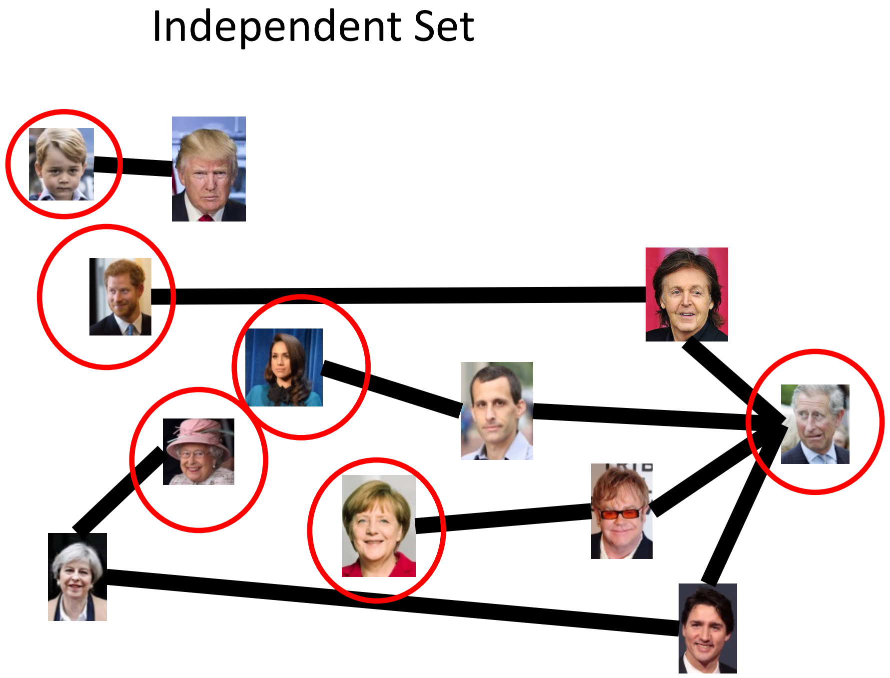
$$\boldsymbol{A \leq_V B}$$

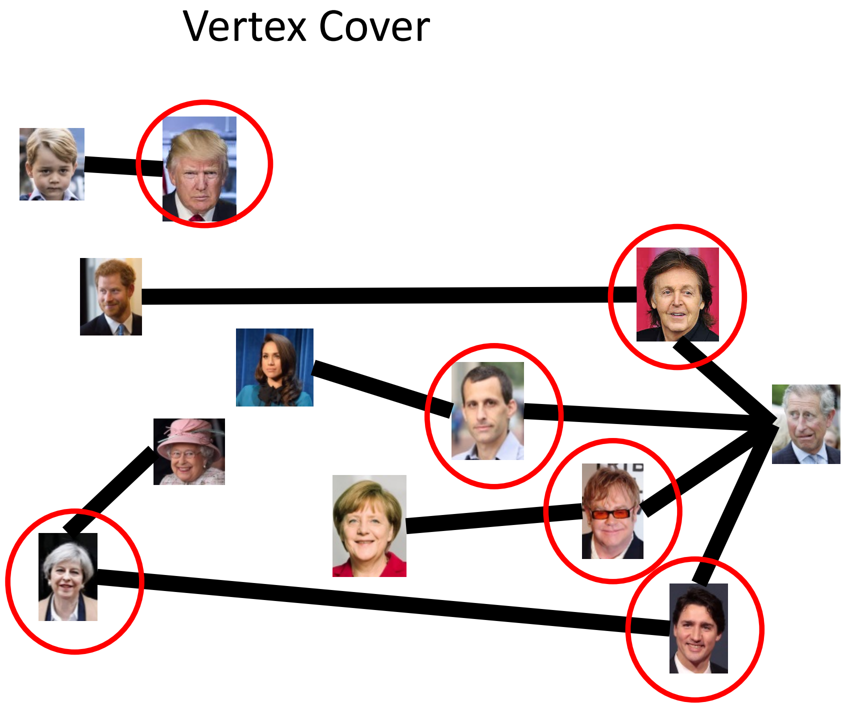
We need to build this Reduction



Reduction Idea

S is an independent set of G iff $V - S$ is a vertex cover of G

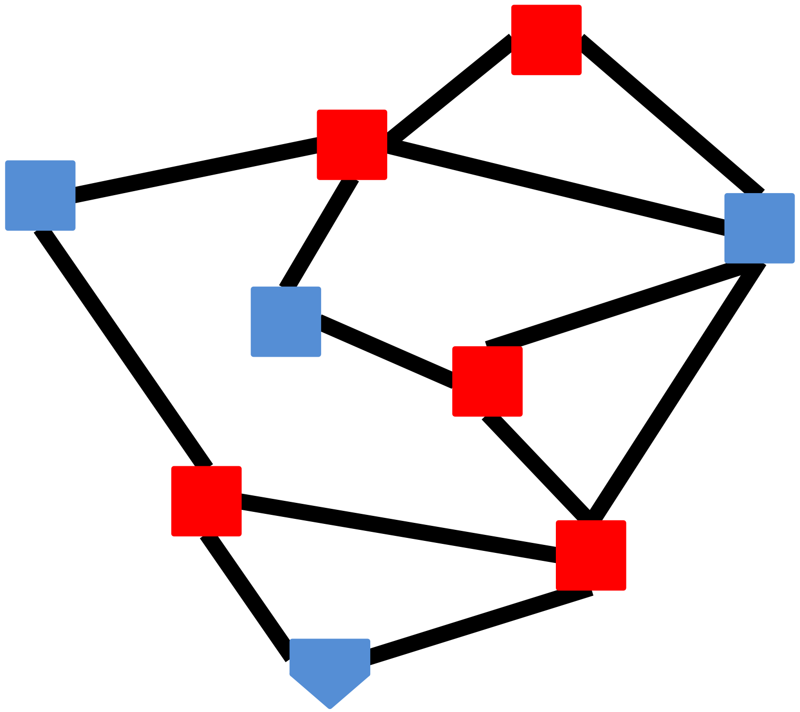




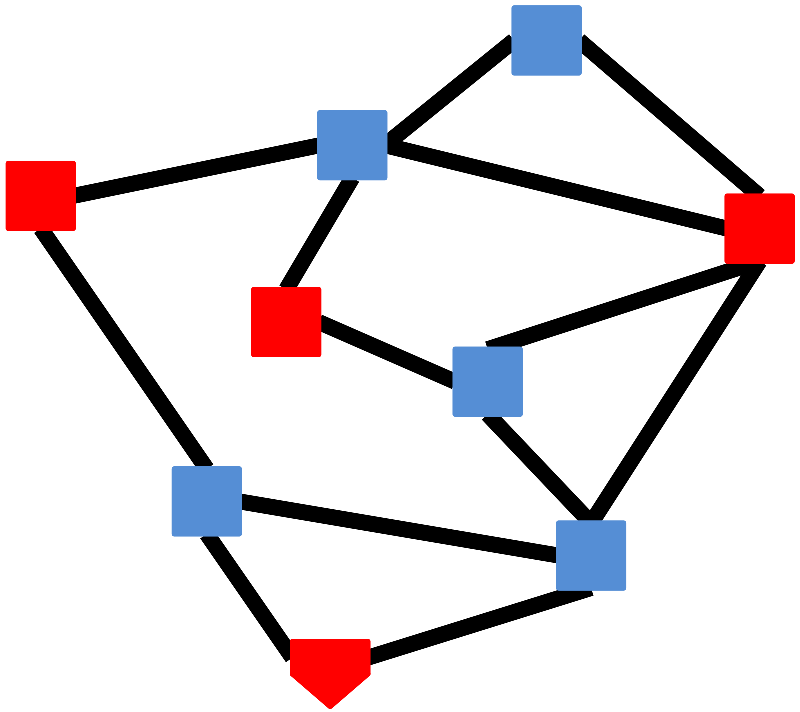
Reduction Idea

S is an independent set of G iff $V - S$ is a vertex cover of G

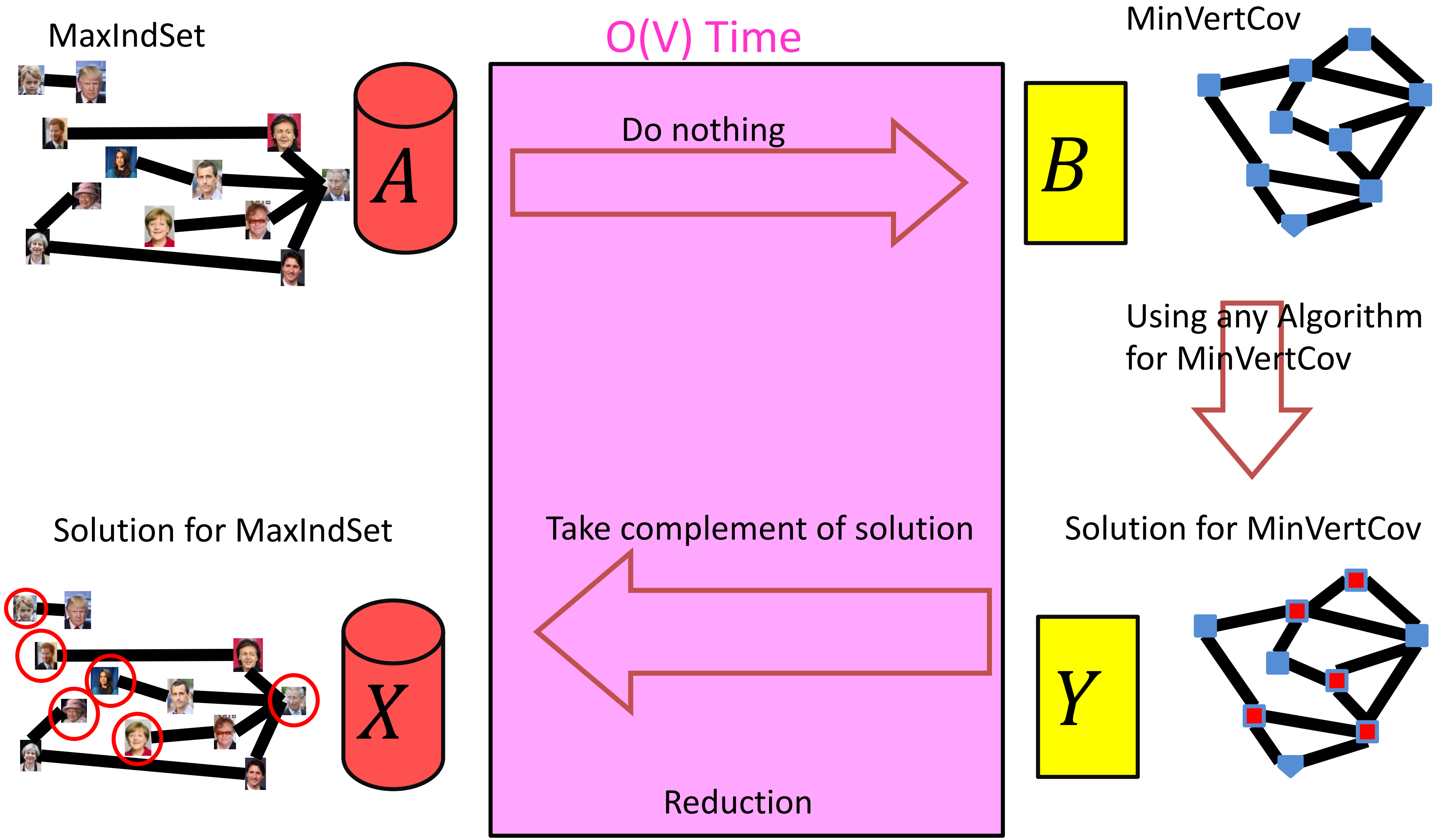
Vertex Cover



Independent Set



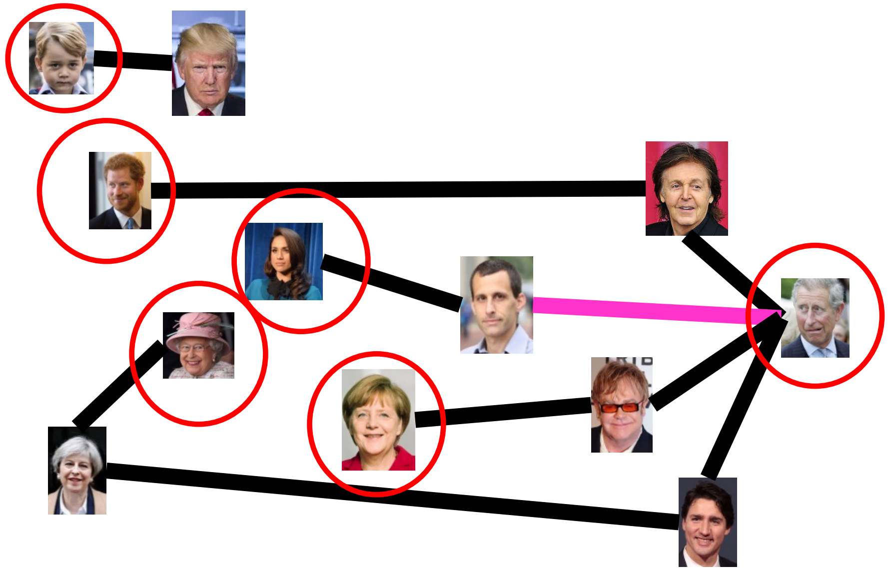
MaxVertCov V-Time Reducible to MinIndSet



Proof: $\Rightarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let S be an independent set



Consider any edge $(x, y) \in E$

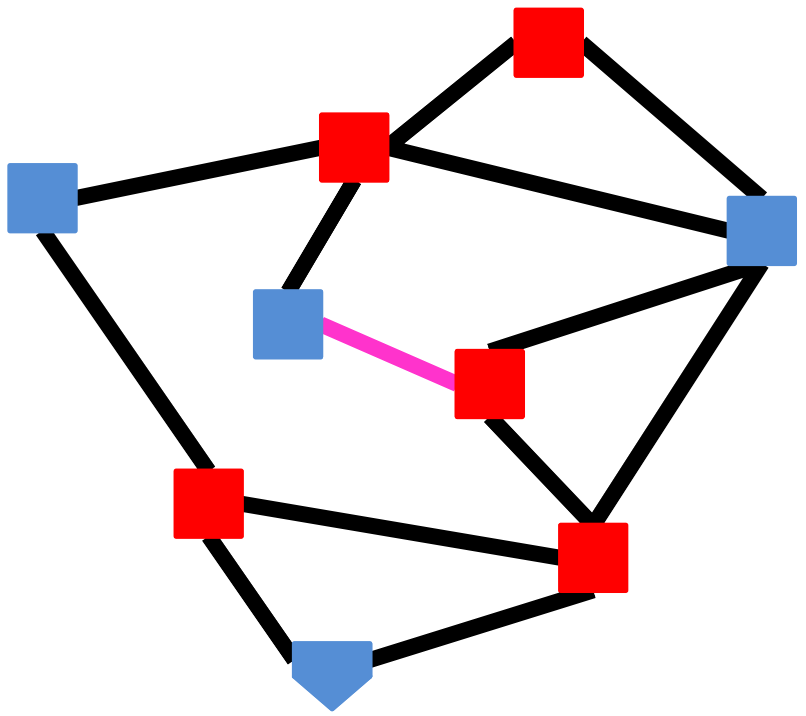
If $x \in S$ then $y \notin S$, because otherwise S would not be an independent set

Therefore $y \in V - S$, so edge (x, y) is covered by $V - S$

Proof: $\Leftarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let $\mathrm{V} - S$ be a vertex cover



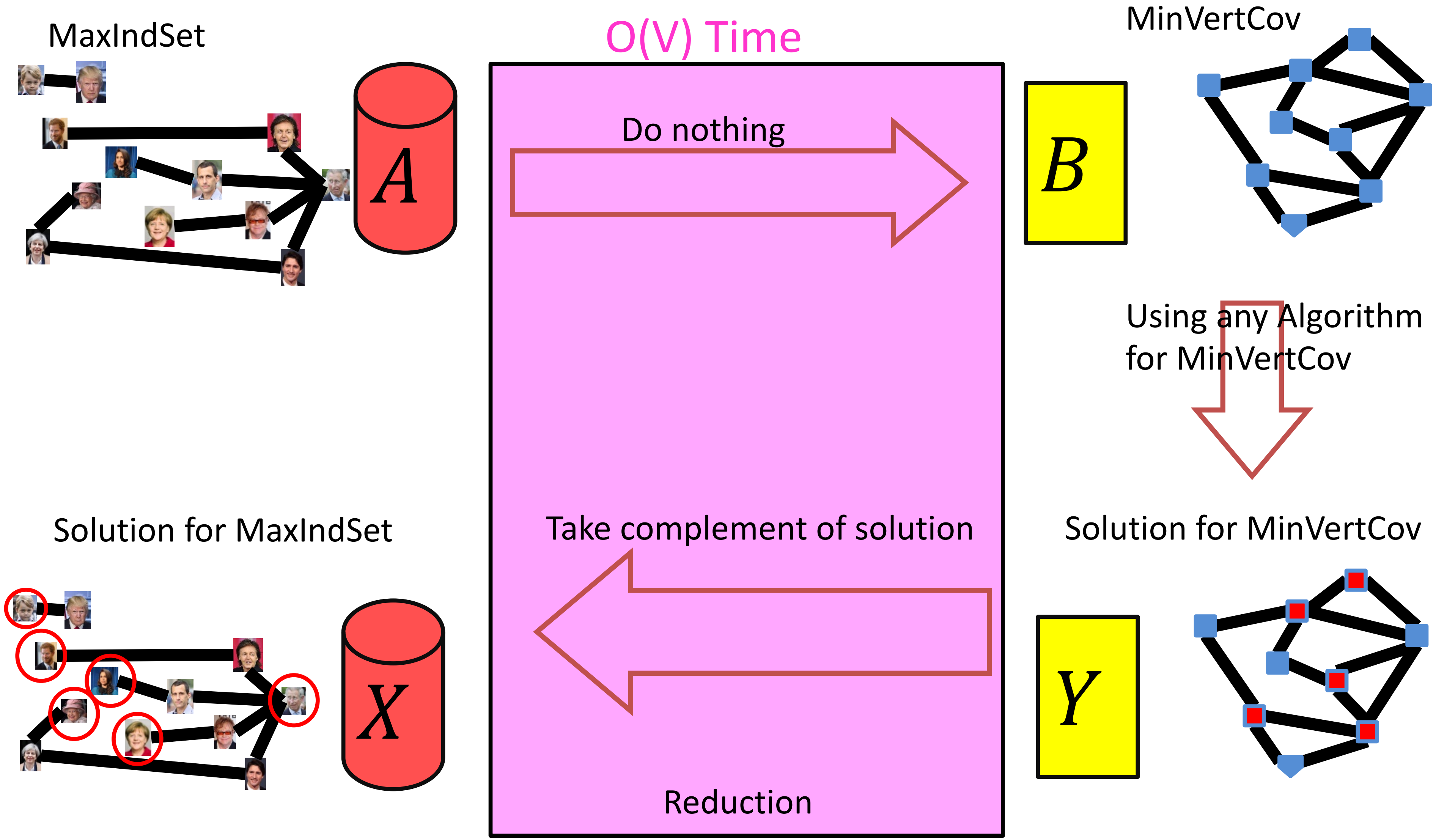
Consider any edge $(x, y) \in E$

At least one of x and y belong to $V - S$, because $\mathrm{V} - S$ is a vertex cover

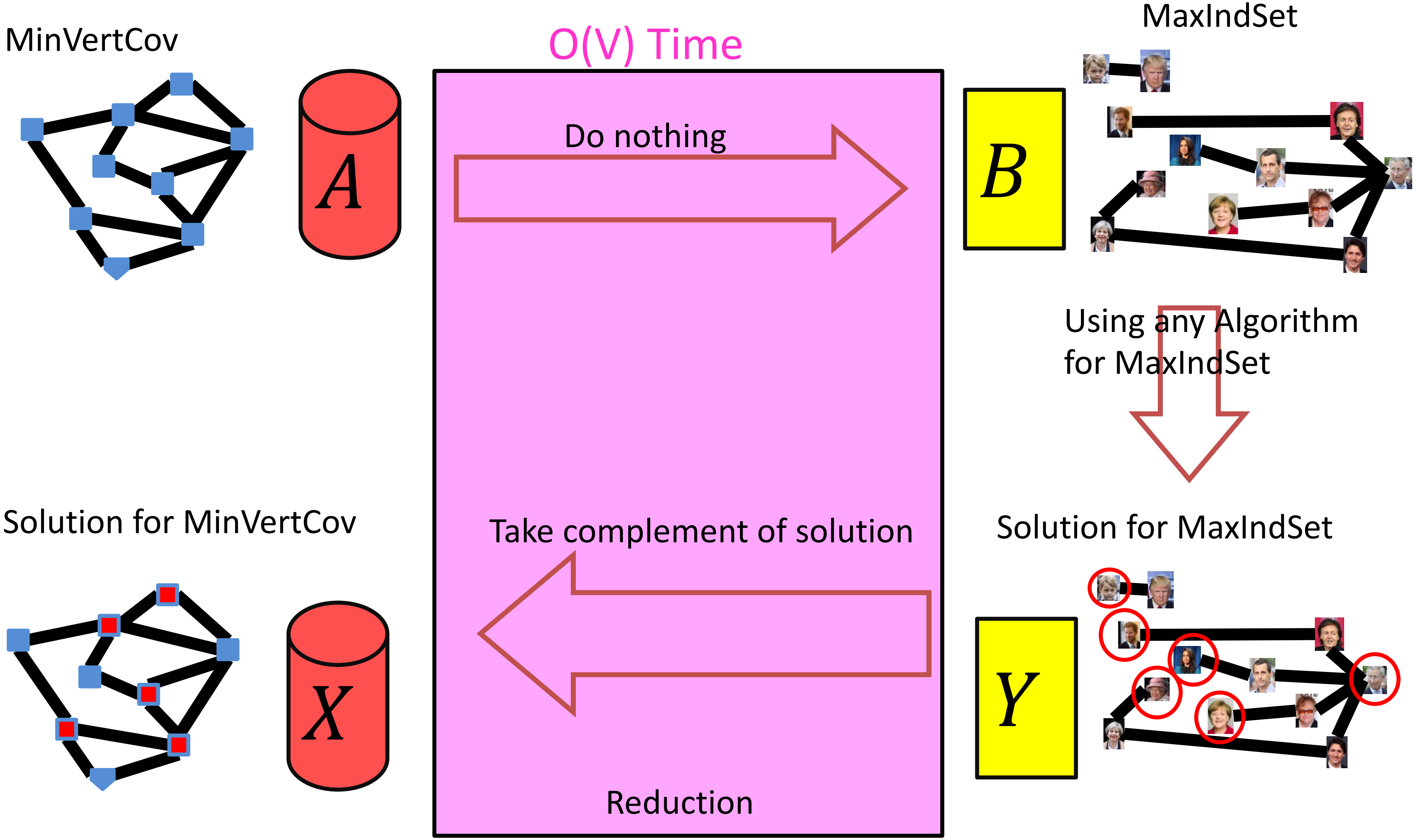
Therefore x and y are not both in S,

No edge has both end-nodes in S, thus S is an independent set

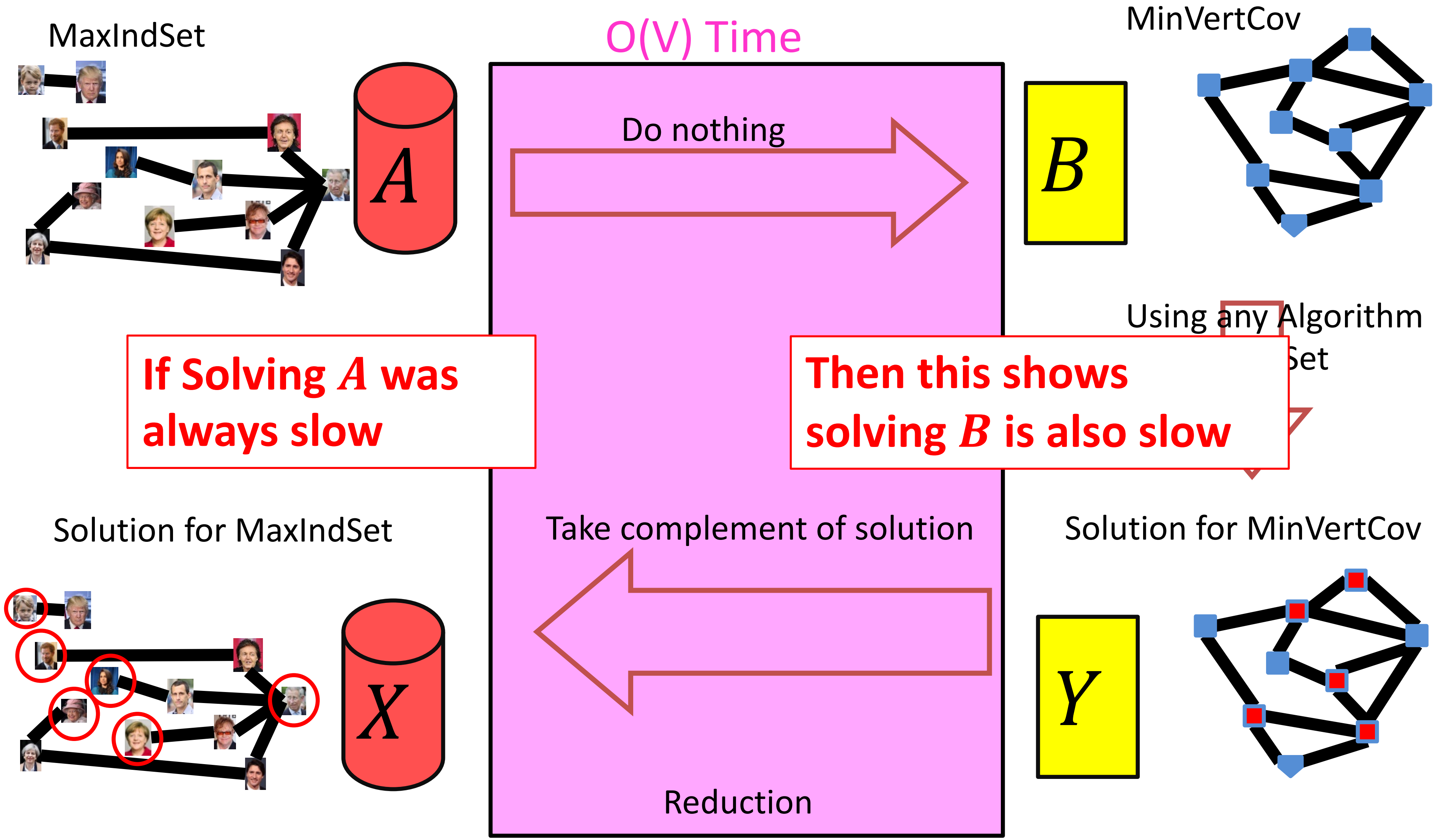
MaxVertCov V-Time Reducible to MinIndSet



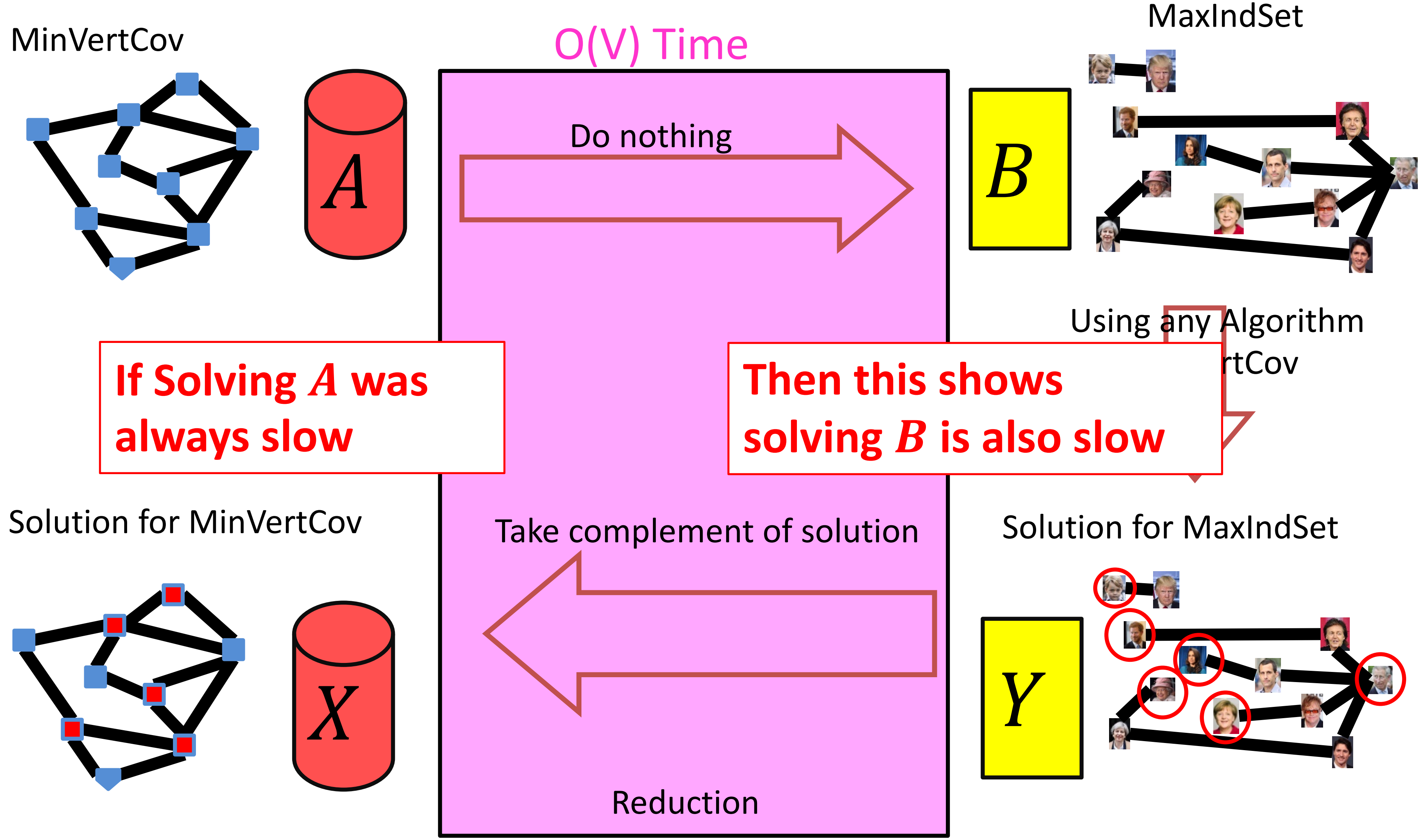
MaxIndSet V-Time Reducible to MinVertCov



Corollary


MaxIndSet
O(V) Time
MinVertCov
A
Do nothing
B
Using any Algorithm
If Solving A was always slow
Then this shows solving B is also slow
Solution for MaxIndSet
Take complement of solution
Solution for MinVertCov
X
Y
Reduction

Corollary



If Solving A was always slow

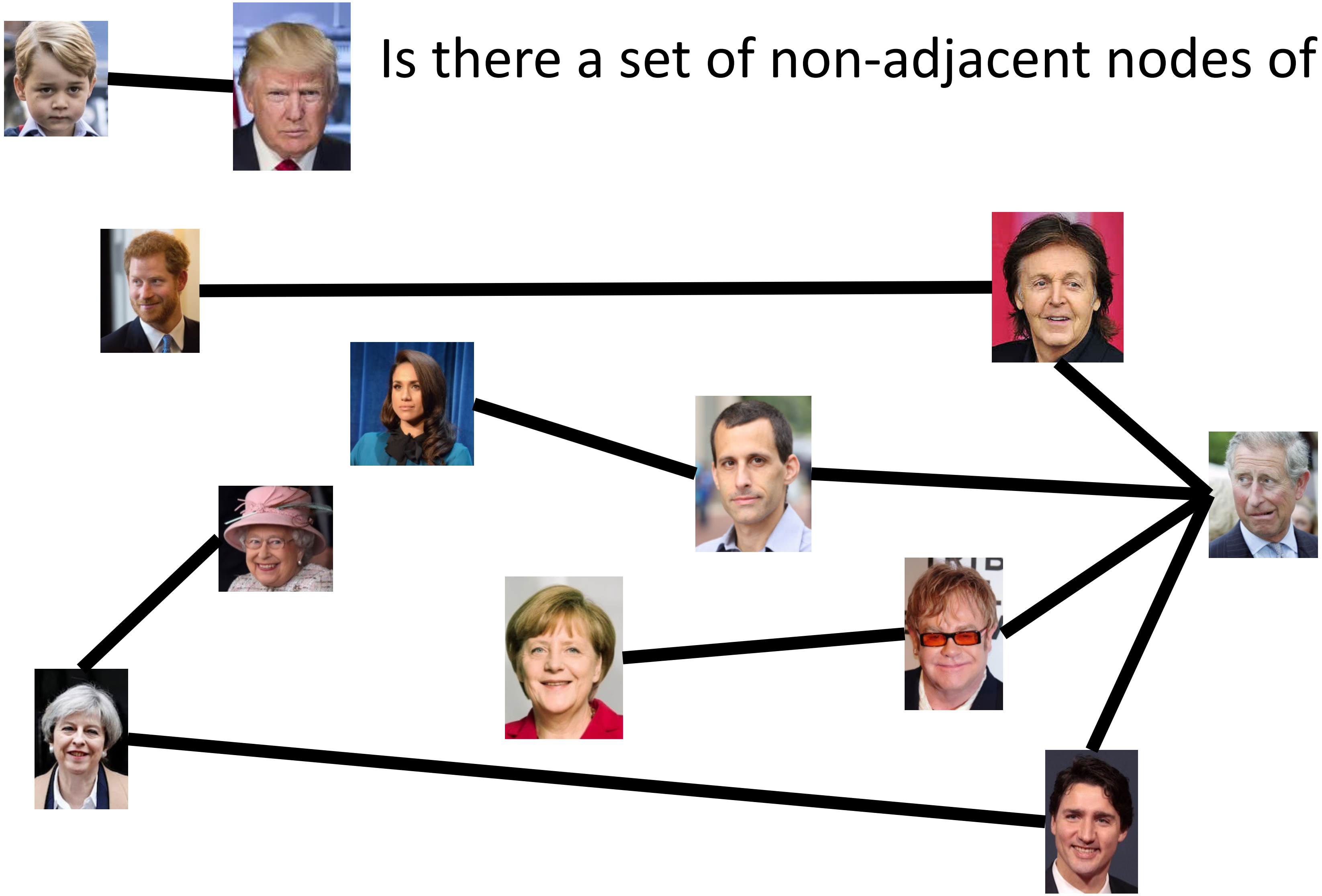
Then this shows solving B is also slow

Conclusion

- MaxIndSet and MinVertCov are either both fast, or both slow
  - Spoiler alert: We don’t know which!
    - (But we think they’re both slow)
  - Both problems are NP-Complete

k Independent Set

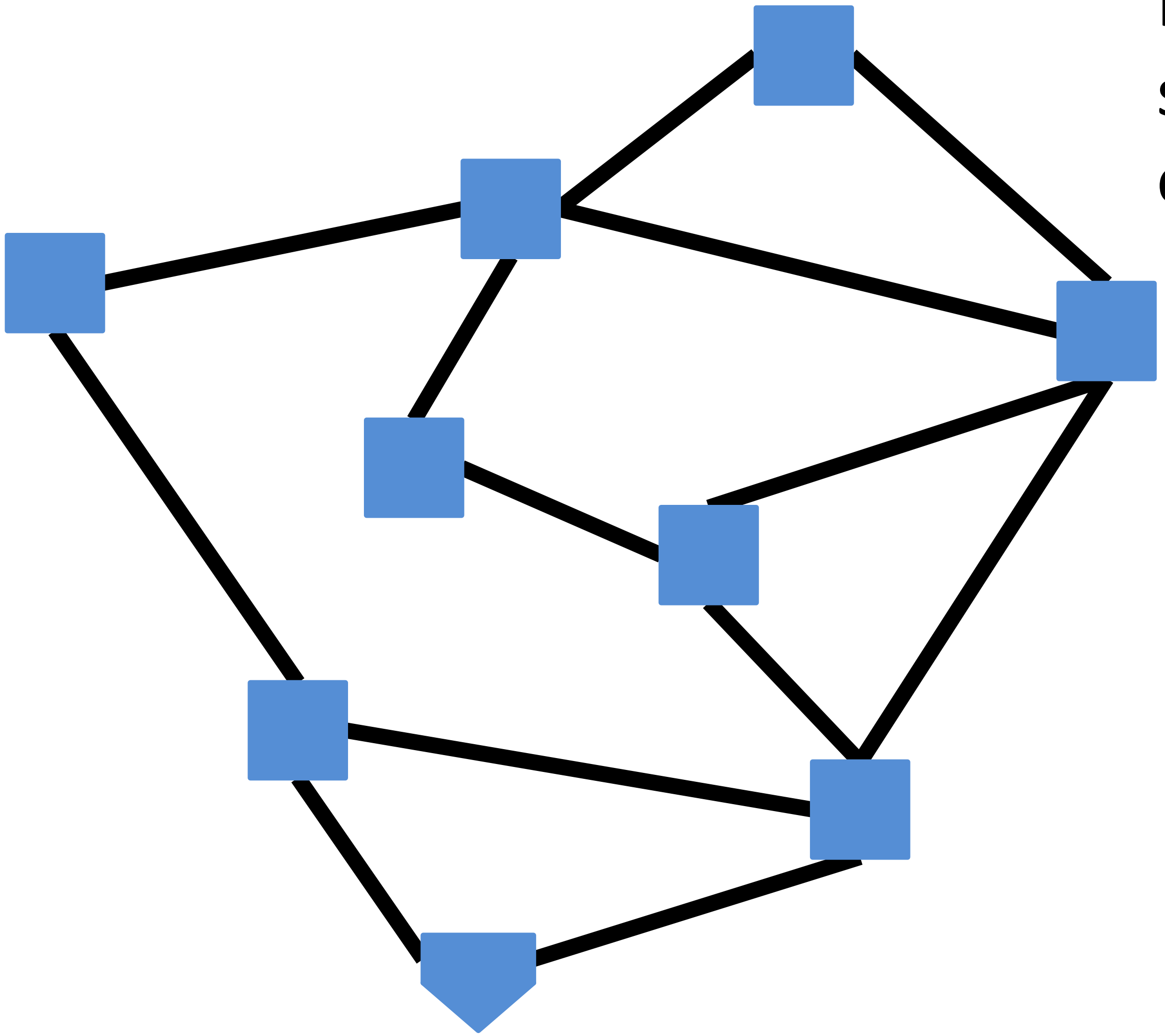
Is there a set of non-adjacent nodes of size k?



k Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- k Independent Set Problem: Given a graph $G = (V, E)$ and a number k, **determine whether there is an independent set $\boldsymbol{S}$ of size $\boldsymbol{k}$**

k Vertex Cover



Is there a set of nodes of size k which covers every edge?

k Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- k Vertex Cover: Given a graph $G = (V, E)$ and a number k, **determine whether there is a vertex cover $\boldsymbol{C}$ of size $\boldsymbol{k}$**

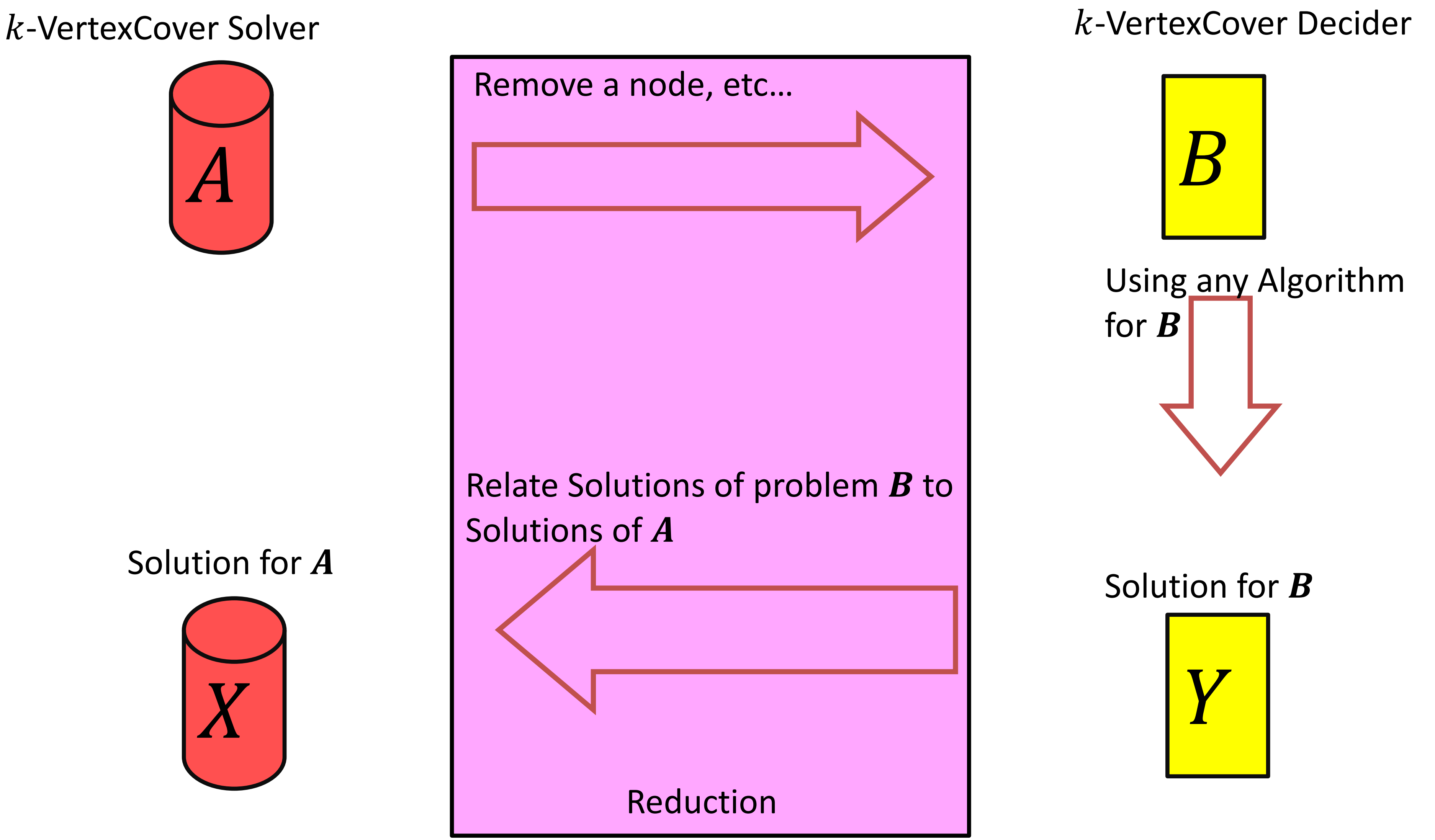
Problem Types

- Decision Problems:
  - Is there a solution?
    - Output is True/False
  - Is there a vertex cover of size k?
- Search Problems:
  - Find a solution
    - Output is complex
  - Give a vertex cover of size k
- Verification Problems:
  - Given a potential solution, is it valid?
    - Output is True/False
  - Is **this** a vertex cover of size k**?**

If we can solve this

Then we can solve this

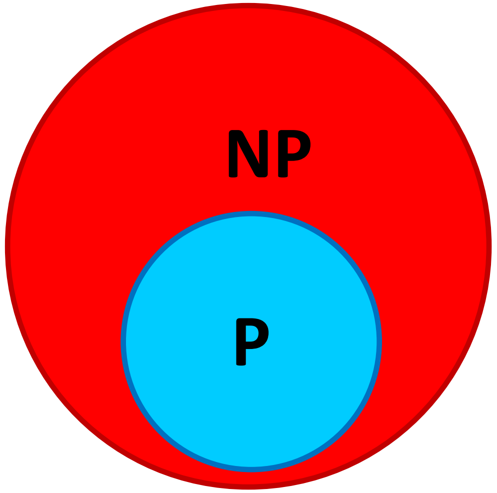
Reduction



k-VertexCover Solver
A
k-VertexCover Decider
Remove a node, etc…
B
Using any Algorithm for $\boldsymbol{B}$
Relate Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$
Solution for $\boldsymbol{A}$
X
Solution for $\boldsymbol{B}$
Y
Reduction

P vs NP

- P
  - Deterministic Polynomial Time
  - Problems solvable in polynomial time
    - $O(n^p)$ for some number p
- NP
  - Non-Deterministic Polynomial Time
  - Problems verifiable in polynomial time
    - $O(n^p)$ for some number p
- Open Problem: Does P=NP?
  - Certainly $P \subseteq NP$



k-Independent Set is NP

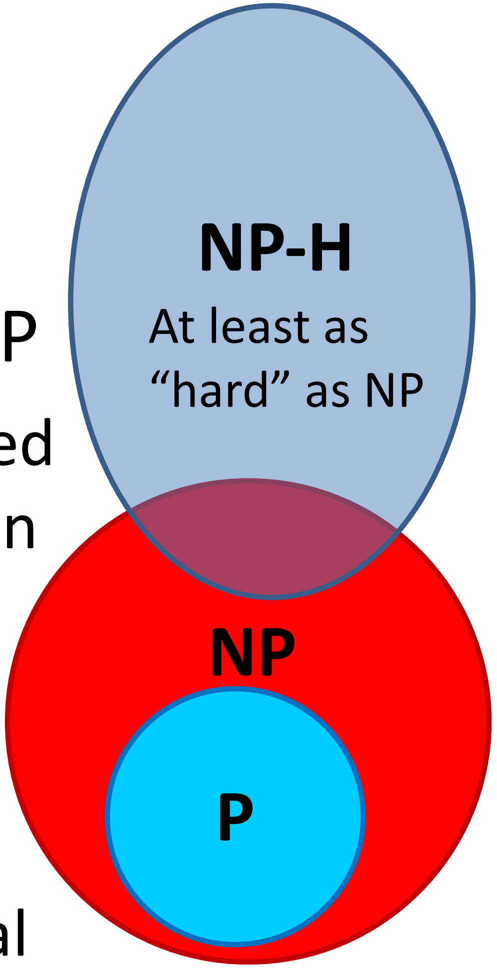
- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

How can we verify it?

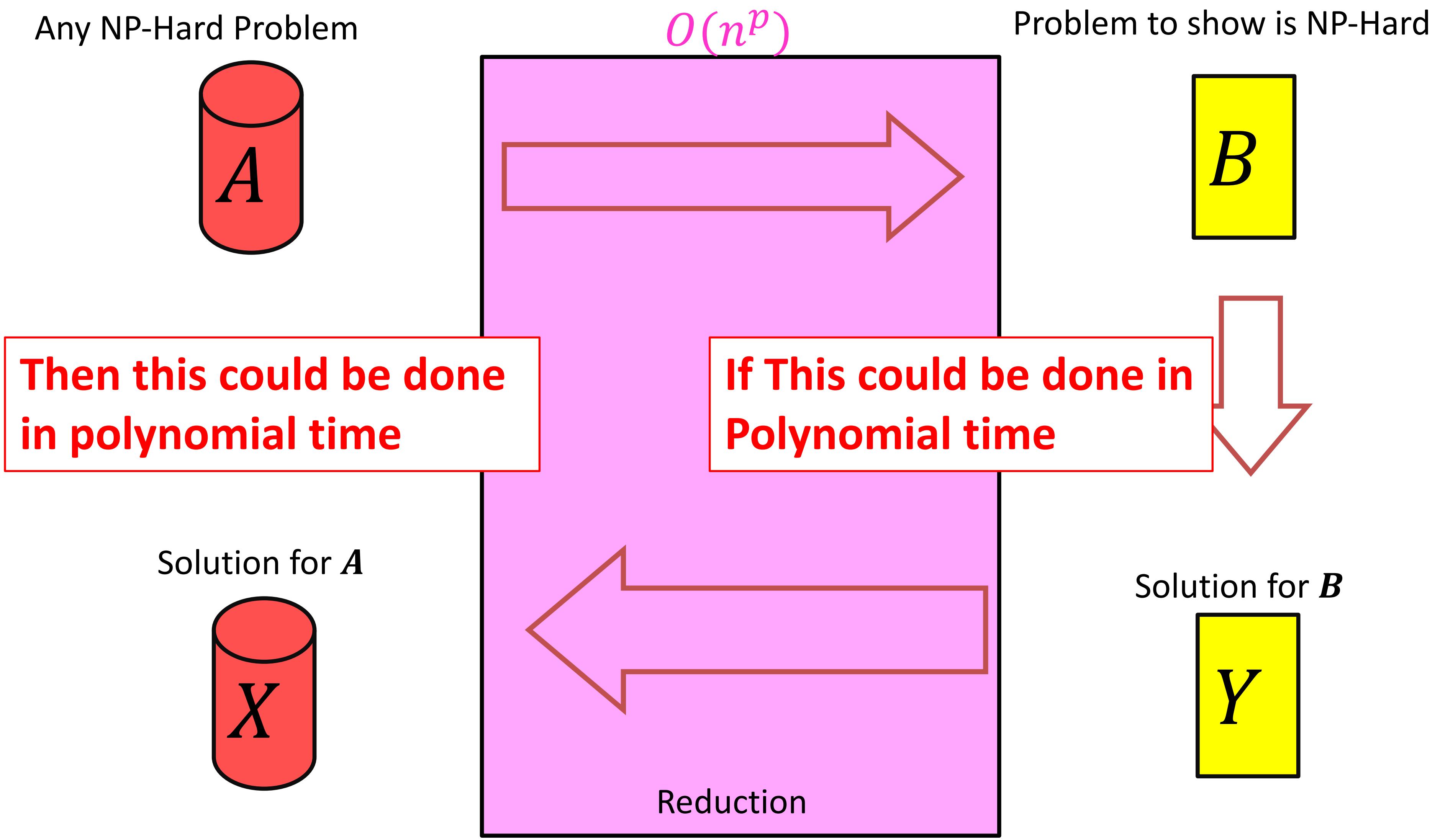
1. Check that it's of size k $O(V)$
2. Check that it's an independent set $O(V^2)$

NP-Hard

- How can we try to figure out if P=NP?
- Identify problems at least as “hard” as NP
  - If any of these “hard” problems can be solved in polynomial time, then all NP problems can be solved in polynomial time.
- Definition: NP-Hard:
  - B is NP-Hard if $\forall A \in NP, A \leq_p B$
  - $A \leq_p B$ means A reduces to B in polynomial time

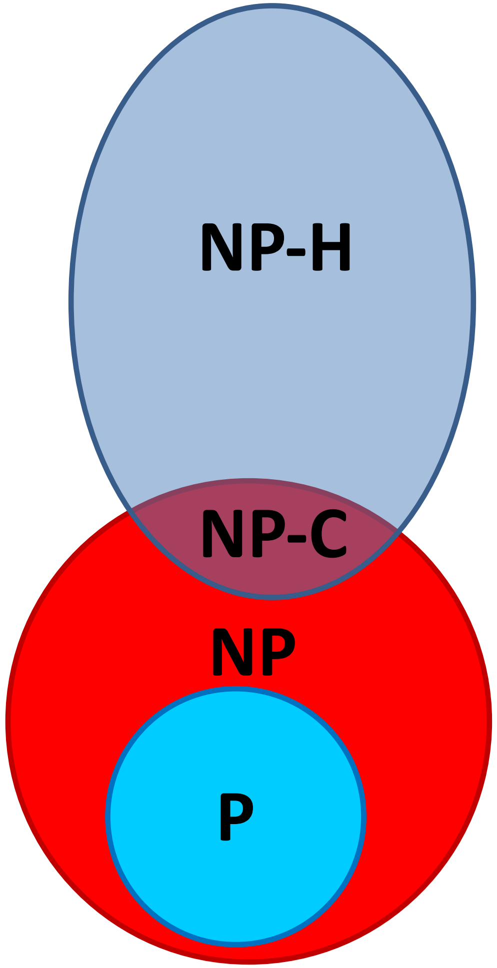


NP-Hardness Reduction



NP-Complete

- "Together they stand, together they fall"
- Problems solvable in polynomial time iff ALL NP problems are
- NP-Complete = NP ∩ NP-Hard
- **How to show a problem is NP-Complete?**
  - Show it belongs to NP
    - Give a polynomial time verifier
  - Show it is NP-Hard
    - Give a reduction from another NP-H problem

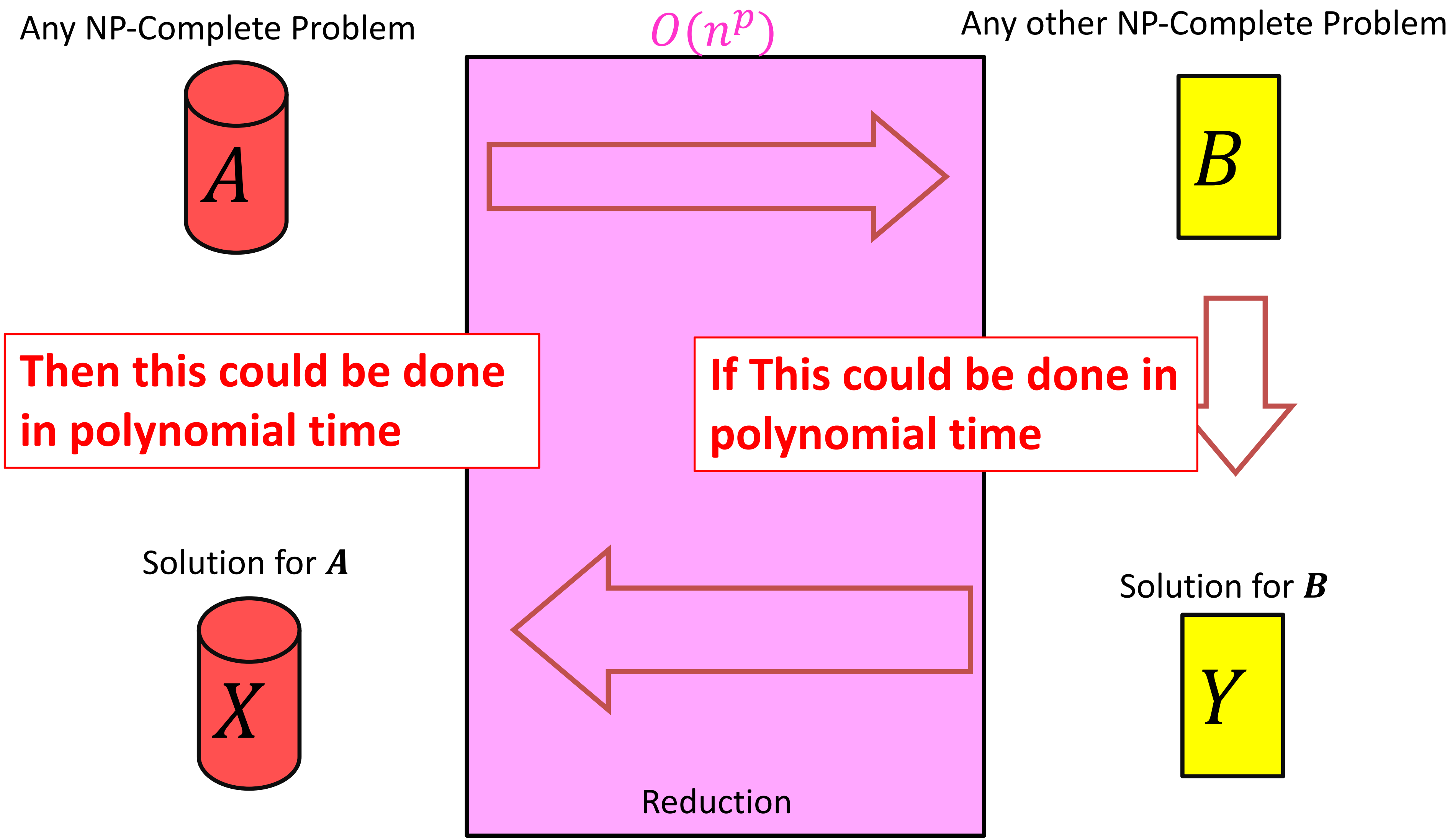


$A \leq_p H \leq_p B$

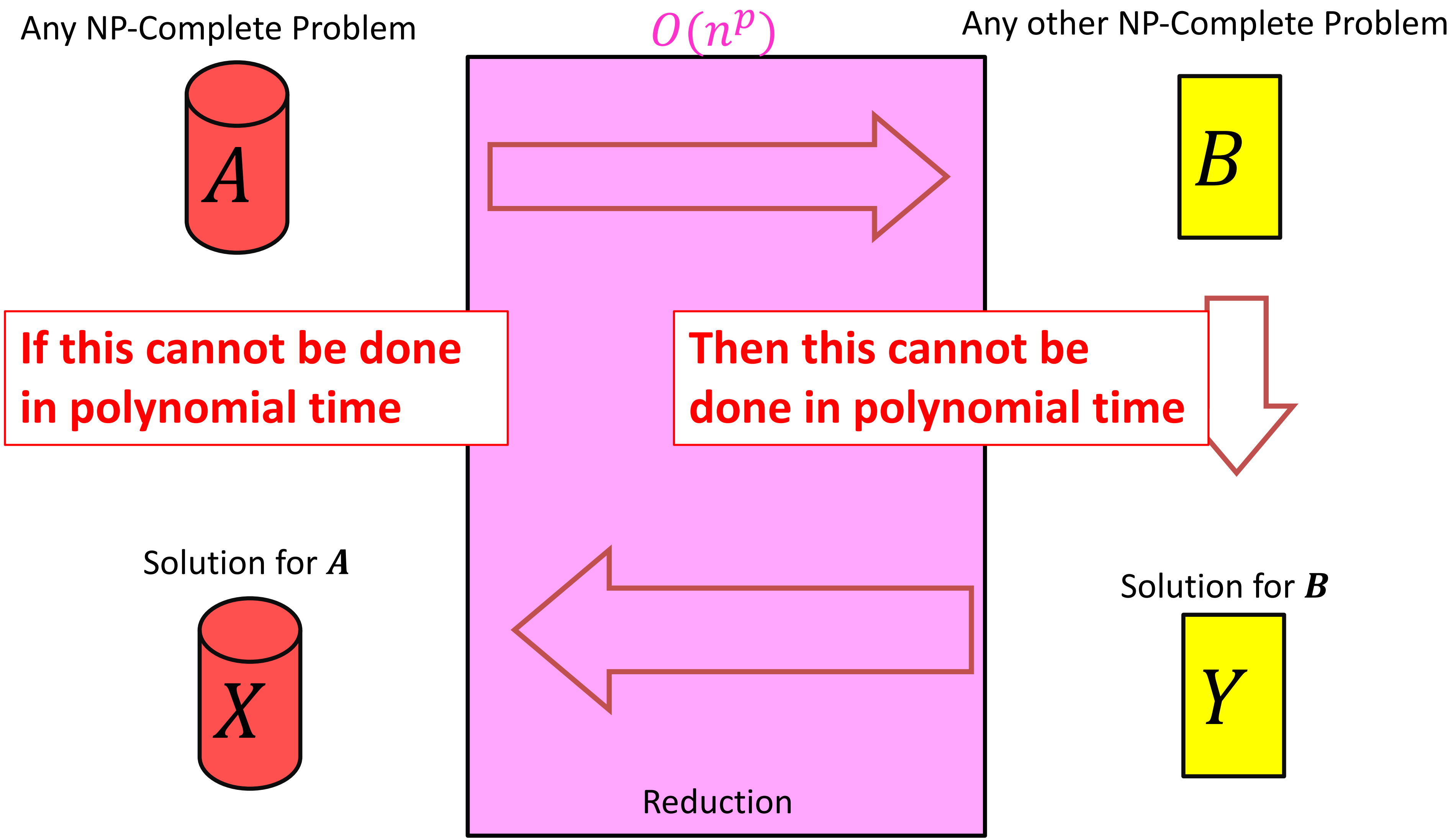
We now just need a FIRST NP-Hard problem

$\forall A \in NP$

NP-Completeness



NP-Completeness



3-SAT

- Shown to be NP-Hard by Cook and Levin (independently)
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), Is there an assignment of true/false to each variable to make the formula true?

$$\underbrace{(x \vee y \vee z)}_{\text{Clause}} \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

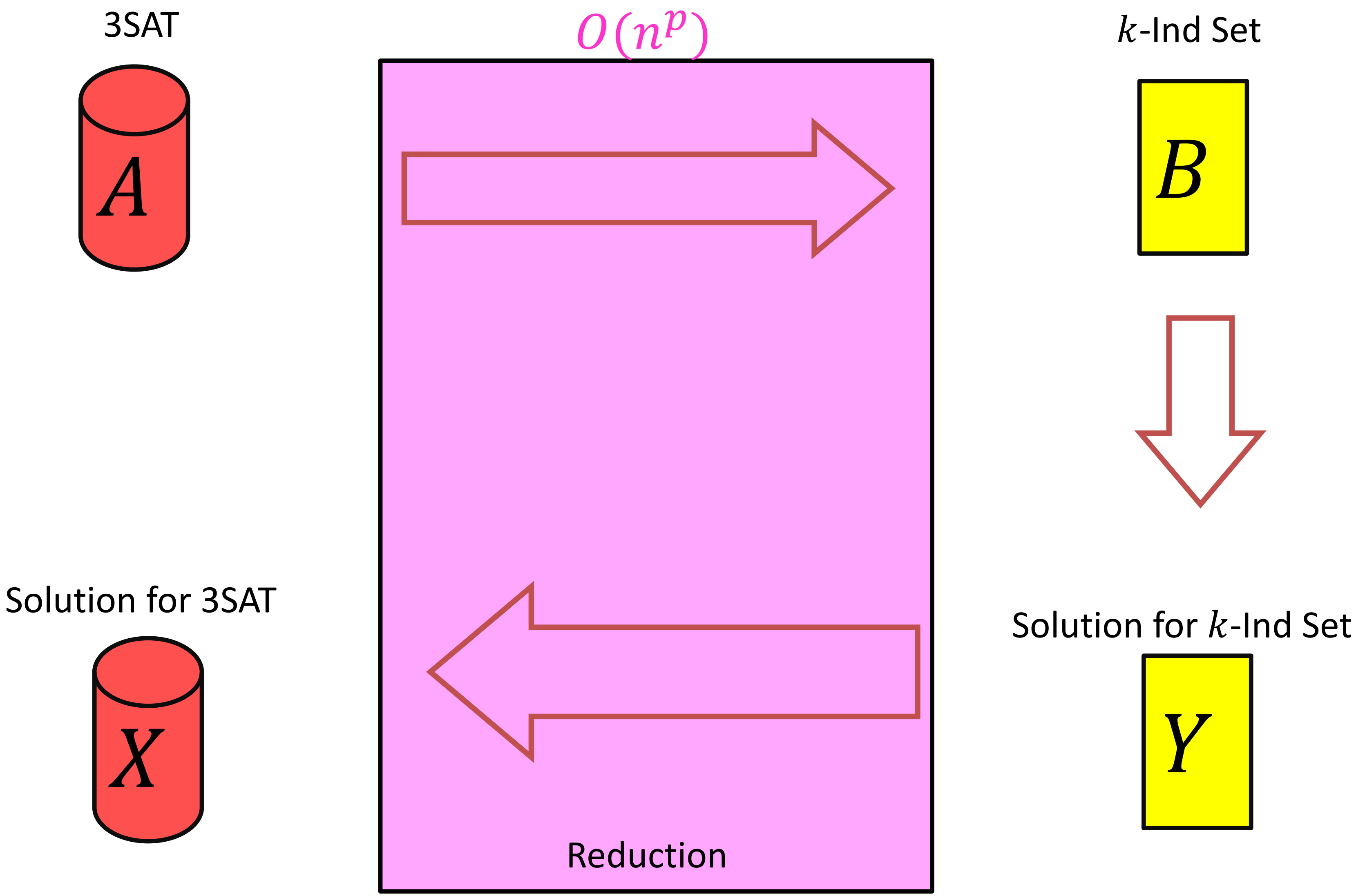
Variables

$$x = true$$
$$y = false$$
$$z = false$$
$$u = true$$

k-Independent Set is NP-Complete

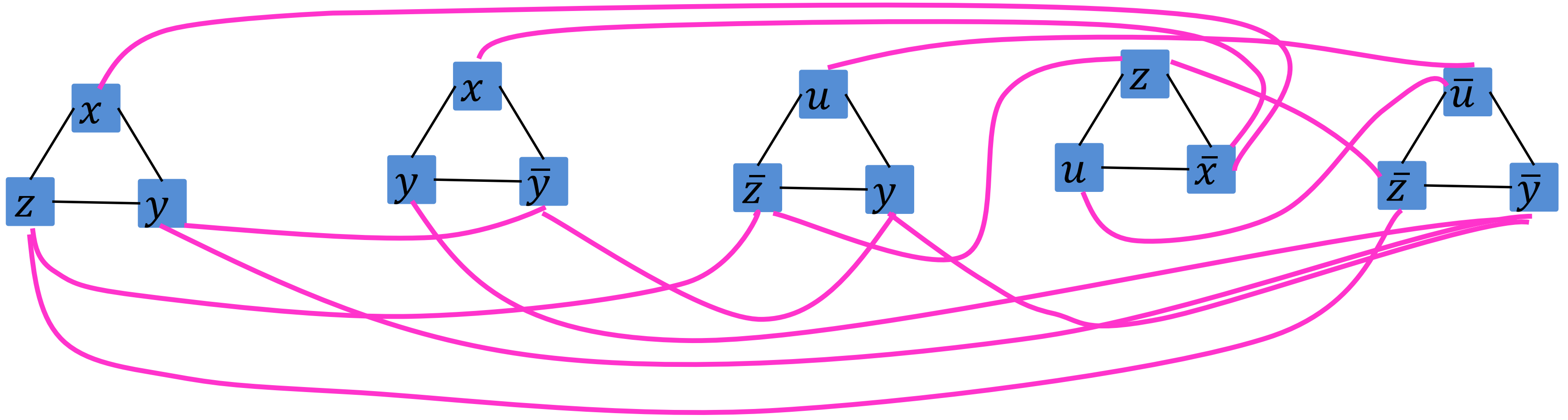
1. Show that it belongs to NP
   - Give a polynomial time verifier (slide 56)
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - Show $3SAT \leq_p kIndSet$

$3SAT \leq_p kIndSet$



Instance of 3SAT to Instance of kIndSet

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



For each clause, produce a triangle graph with its three variables as nodes

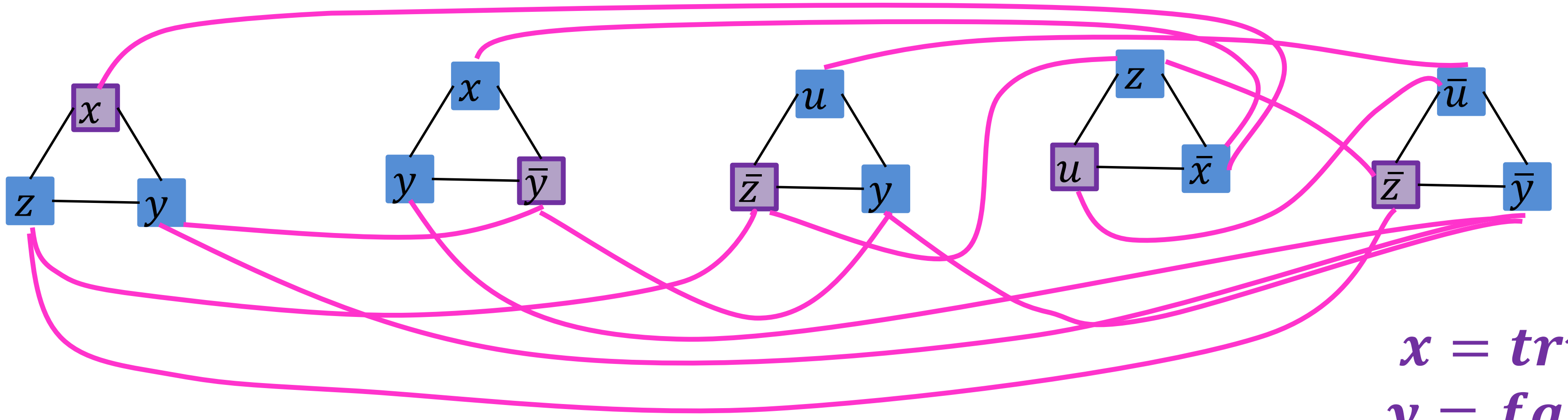
Connect each node to all of its opposites

Let k = number of clauses

There is a k-IndSet in this graph, iff there is a satisfying assignment

kIndSet ⇒ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

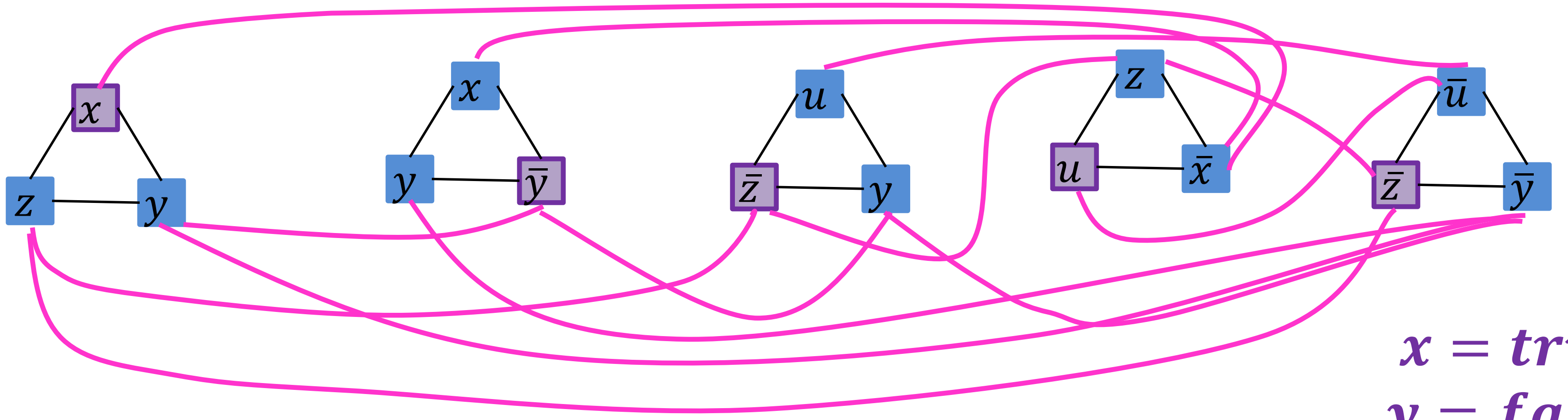
One node per triangle is in the Independent set:
because we can have exactly k total in the set,
and 2 in a triangle would be adjacent

If x is selected in some triangle, $\bar{x}$ is not selected in any triangle:
Because every x is adjacent to every $\bar{x}$

Set the variable which each included node represents to “true”

Satisfying Assignment $\Rightarrow$ kIndSet

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



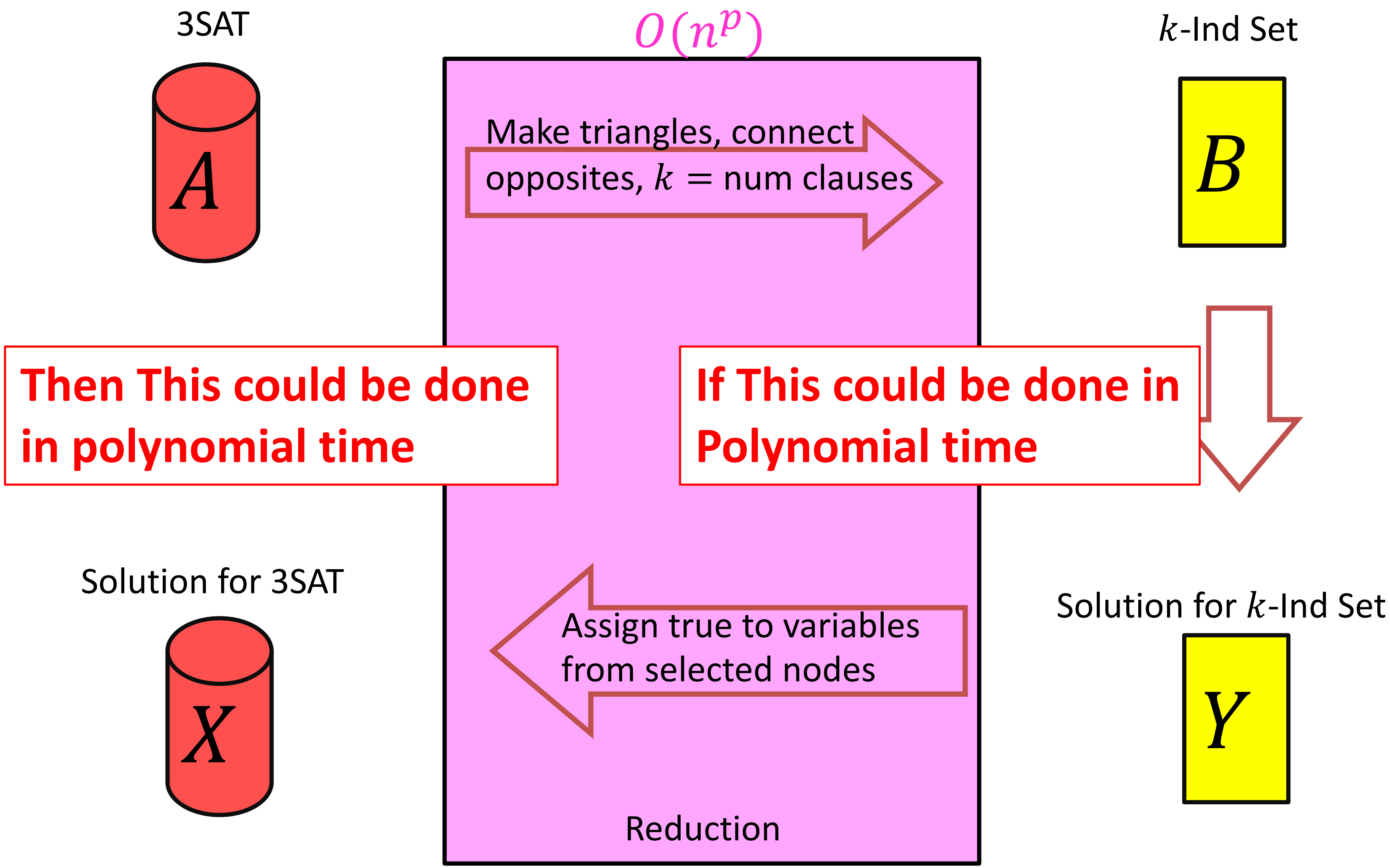
$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

Use one true variable from the assignment for each triangle

The independent set has k nodes, because there are k clauses

If any variable x is true then $\bar{x}$ cannot be true

$3SAT \leq_p kIndSet$



k-Vertex Cover is NP

- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's a Vertex Cover $O(E)$

k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier (slide 69)
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We showed $kIndSet \leq_p kVertCov$

$kIndSet \leq_p kVertCov$