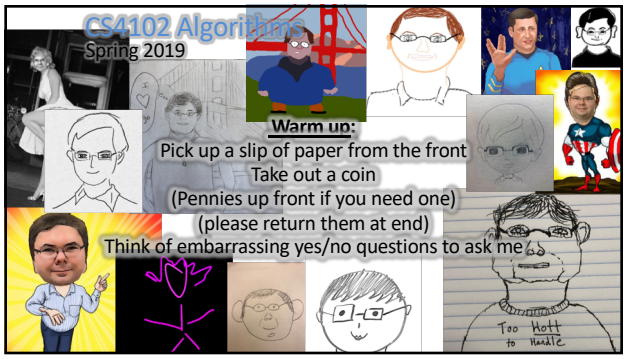


Today's Keywords

- Reductions
- NP-Completeness
- Vertex Cover
- Independent Set
- 3-SAT
- Clique
- Differential Privacy

2

CLRS Readings

- Chapter 34

3

Homeworks, etc

- HW9 due tomorrow at 11pm
  - Written (use LaTeX)
  - Reductions
- Final Exam: Saturday, May 4, 2-5pm
  - Heavily from material since midterm
    - May ask for runtime of an algorithm, some knowledge of D&C
    - Won't directly ask you to solve recurrences
  - Practice final online by tomorrow
  - Review session: Wednesday, 4pm, MEC 205
- Office Hours: Tomorrow 11am-1pm, Wed 11am-12pm

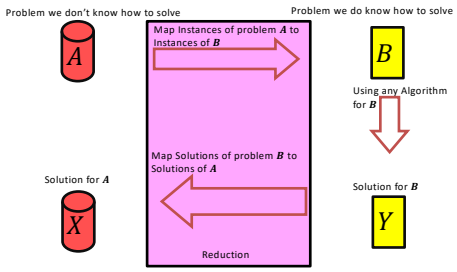
4

Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

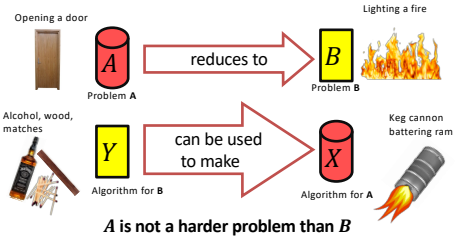
5

In General: Reduction



6

Worst-case lower-bound Proofs



A is not a harder problem than B
$A \leq B$

The name "reduces" is confusing: it is in the *opposite* direction of the making

Proof of Lower Bound by Reduction

To Show: Y is slow

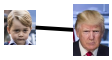
1. We know X is slow
(e.g., X = some way to open the door)

2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)

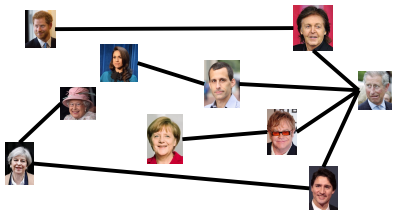
3. Show how to use Y to perform X quickly

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick

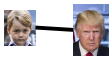
Max Independent Set



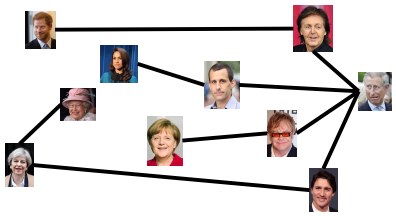
Find the largest set of non-adjacent nodes



k Independent Set



Is there a set of non-adjacent nodes of size k?



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Maximum Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

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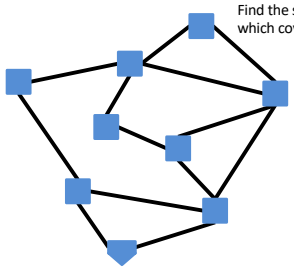
k Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- k Independent Set Problem: Given a graph $G = (V, E)$ and a number k, **determine whether there is an independent set S of size k**

12

Min Vertex Cover

Find the smallest set of nodes which covers every edge



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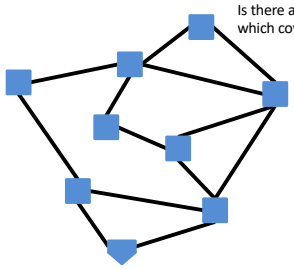
k Vertex Cover

Is there a set of nodes of size k which covers every edge?



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Minimum Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover: Given a graph $G = (V, E)$ find the minimum vertex cover C

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k Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- k Vertex Cover: Given a graph $G = (V, E)$ and a number k, **determine whether there is a vertex cover C of size k**

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Problem Types

- Decision Problems: If we can solve this
  - Is there a solution?
    - Output is True/False
  - Is there a vertex cover of size k?
- Search Problems: Then we can solve this
  - Find a solution
    - Output is complex
  - Give a vertex cover of size k
- Verification Problems:
  - Given a potential solution, is it valid?
    - Output is True/False
  - Is **this** a vertex cover of size k**?**

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Using a k-VertexCover decider to build a searcher

- Set $i = k - 1$
- Remove nodes (and incident edges) one at a time
- Check if there is a vertex cover of size i
  - If so, then that removed node was part of the k vertex cover, set $i = i - 1$
  - Else, it wasn't

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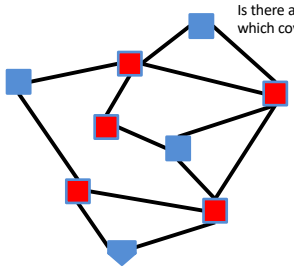
5 Vertex Cover (Decision)

Is there a set of nodes of size 5 which covers every edge?

Yes!



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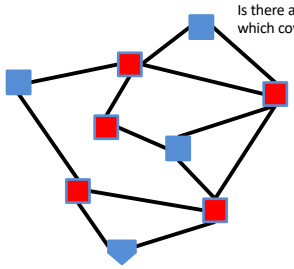
4 Vertex Cover (Decision)

Is there a set of nodes of size 4 which covers every edge?

No!

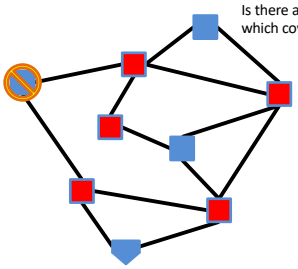


20

4 Vertex Cover (Decision)

Is there a set of nodes of size 4 which covers every edge?

No!

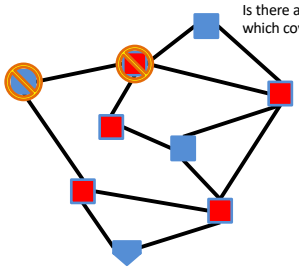


21

4 Vertex Cover (Decision)

Is there a set of nodes of size 4 which covers every edge?

Yes!



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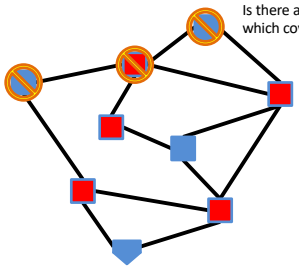
3 Vertex Cover (Decision)

Is there a set of nodes of size 3 which covers every edge?

No!



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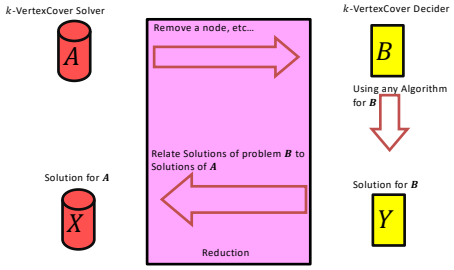
Reduction



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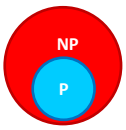
P vs NP

- P
  - Deterministic Polynomial Time
  - Problems solvable in polynomial time
    - $O(n^p)$ for some number p
- NP
  - Non-Deterministic Polynomial Time
  - Problems verifiable in polynomial time
    - $O(n^p)$ for some number p
- Open Problem: Does P=NP?
  - Certainly $P \subseteq NP$



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k-Independent Set is NP

- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's an independent set $O(V^2)$

26

k-Vertex Cover is NP

- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's a Vertex Cover $O(E)$

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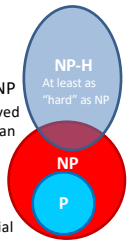
NP-Hard

- How can we try to figure out if P=NP?
- Identify problems at least as "hard" as NP
  - If any of these "hard" problems can be solved in polynomial time, then all NP problems can be solved in polynomial time.
- Definition: NP-Hard:
  - B is NP-Hard if $\forall A \in NP, A \leq_p B$
  - $A \leq_p B$ means A reduces to B in polynomial time



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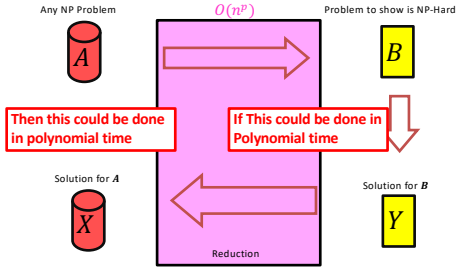
NP-Hardness Reduction



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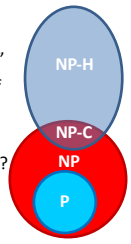
NP-Complete

- "Together they stand, together they fall"
- Problems solvable in polynomial time iff ALL NP problems are
- NP-Complete = NP ∩ NP-Hard
- How to show a problem is NP-Complete?
  - Show it belongs to NP
    - Give a polynomial time verifier
  - Show it is NP-Hard
    - Give a reduction from another NP-H problem



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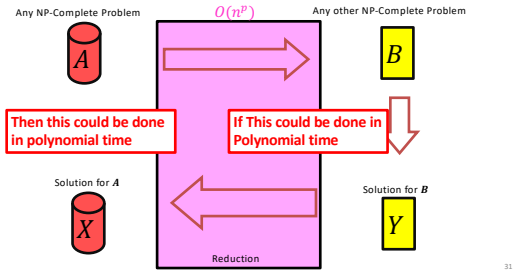
NP-Completeness



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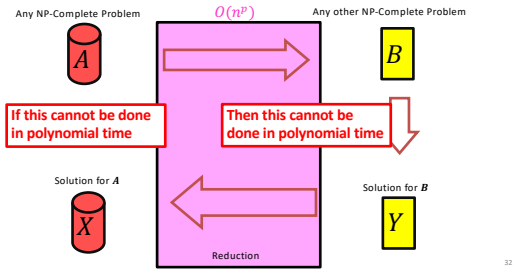
NP-Completeness

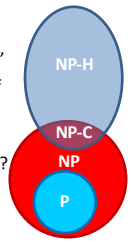


32

NP-Complete

- "Together they stand, together they fall"
- Problems solvable in polynomial time iff ALL NP problems are
- NP-Complete = NP ∩ NP-Hard
- How to show a problem is NP-Complete?
  - Show it belongs to NP
    - Give a polynomial time verifier
  - Show it is NP-Hard
    - Give a reduction from another NP-H problem

We now just need a FIRST NP-Hard problem



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3-SAT

- Shown to be NP-Hard by Cook and Levin (independently)
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), Is there an assignment of true/false to each variable to make the formula true?

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (x \vee y \vee z)$

Clause

Variables

$x = true$
$y = false$
$z = false$
$u = true$

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k-Independent Set is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier (see earlier slide)
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - Show $3SAT \leq_p kIndSet$

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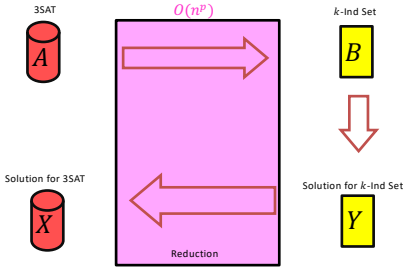
$3SAT \leq_p kIndSet$



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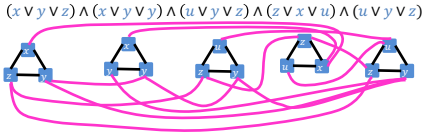
Instance of 3SAT to Instance of kIndSet

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (u \vee y \vee z)$



For each clause, produce a triangle graph with its three variables as nodes

Connect each node to all of its opposites

Let k = number of clauses

There is a k-IndSet in this graph **iff** there is a satisfying assignment

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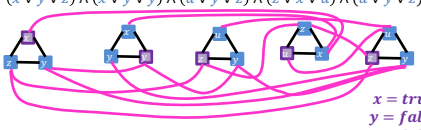
kIndSet ⇒ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (u \vee y \vee z)$



$x = true$
$y = false$
$z = false$
$u = true$

One node per triangle is in the Independent set:
because we can have exactly k total in the set,
and 2 in a triangle would be adjacent

If x is selected in some triangle, x is not selected in any triangle:
Because every x is adjacent to every x

Set the variable which each included node represents to "true"

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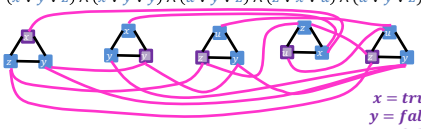
Satisfying Assignment ⇒ kIndSet

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (u \vee y \vee z)$



$x = true$
$y = false$
$z = false$
$u = true$

Use one true variable from the assignment for each triangle

The independent set has k nodes, because there are k clauses

If any variable x is true then x cannot be true

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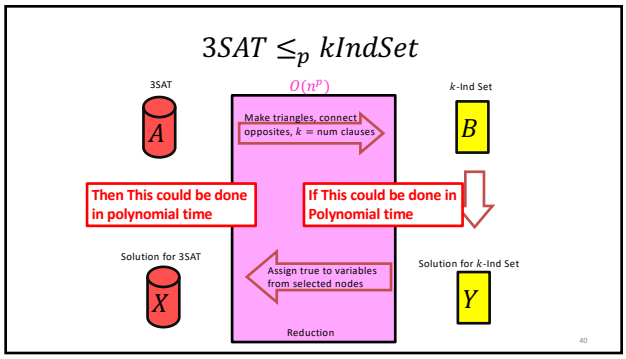

k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier (see earlier slide)
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We showed $kIndSet \leq_p kVertCov$
     - (Last Class)

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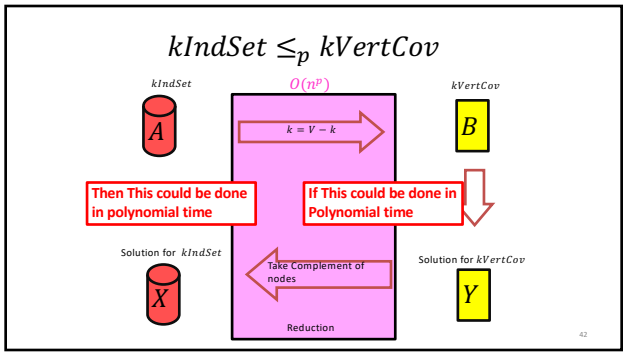
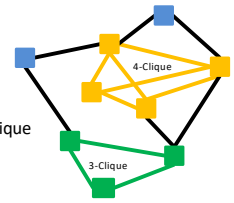


k-Clique Problem

- Clique: A complete subgraph
- k-Clique Problem:
  - Given a graph G and a number k, is there a clique of size k?



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k-Clique is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We will show $3SAT \leq_p kClique$

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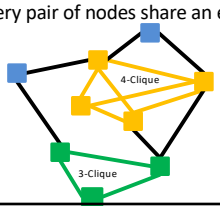
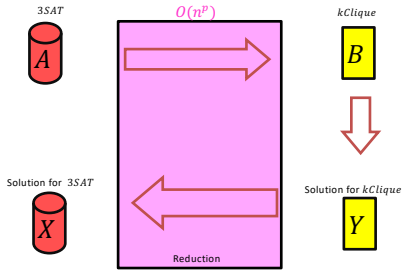
k-Clique is NP

1. Given a Graph and a potential solution
2. Check that the solution has k nodes
3. Check that every pair of nodes share an edge



45

$3SAT \leq_p kClique$



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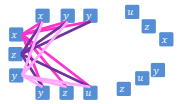
Instance of 3SAT to Instance of kClique

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (x \vee y \vee z)$



(also do this for the other clauses, omitted due to clutter)

For each clause, produce a node for each of its three variables

Connect each node to all non-contradictory nodes in the other clauses (i.e., anything that's not its negation)

Let k = number of clauses

There is a k-Clique in this graph **iff** there is a satisfying assignment

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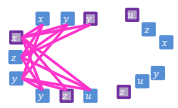
kClique ⇒ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (x \vee y \vee z)$



$x = true$
$y = false$
$z = false$
$u = true$

There are k triplets in the graph, and no two nodes in the same triplet are adjacent

To have a k-Clique, must have one node from each triplet

Cannot select a node for both a variable and its negation

Therefore selection of nodes is a satisfying assignment

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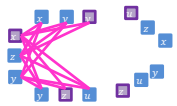
Satisfying Assignment⇒ kClique

$(x \vee y \vee z) \wedge (x \vee y \vee y) \wedge (u \vee y \vee z) \wedge (z \vee x \vee u) \wedge (x \vee y \vee z)$



$x = true$
$y = false$
$z = false$
$u = true$

Select one node for a true variable from each clause

There will be k nodes selected
We can't select both a node and its negation
All nodes will be non-contradictory, so they will be pairwise adjacent

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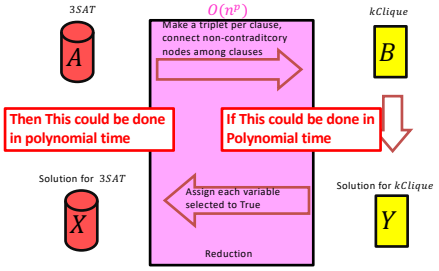
$3SAT \leq_p kClique$



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Academic Integrity

Differential Privacy

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Differential Privacy

- Gives a way to probabilistically answer questions about data without giving away its content
- You can get statistical certainty on the answer
- We're going to use a simple example

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Scheme

- Flip a coin:
  - If Heads, respond "yes"
  - If Tails, truthfully answer an embarrassing question:
- Questions

-have you ever cheated?
-have I looked up a qu on S.O?
-do you like KPop?
- anything illegal?

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How does it work

- Assume everyone participates honestly
- We know 50% of "yes" answers were from the coin landing heads
  - If 100 people participate, eliminate 50 "yes" responses
  - Proportion of "yes" answers given by remaining "yes" answers over 50
- Consider a person who answers "no"
  - We know this person didn't cheat
- Consider a person who answers "yes"
  - Most people who answered "yes" only did so because the coin landed heads
  - It's still more likely that this person did not cheat

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Example: How many people have streaked the lawn?

- Flip a coin:
  - If Heads, respond "yes"
  - If Tails, truthfully answer an embarrassing question:
    - Have you ever streaked the lawn?
      - On the slip of paper, put a 1 in ~~column 1~~ left side, put a 1 in column 2 on right side if you answered yes (else a 0 in column 2)
      - Pass the slip to your left

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Does P=NP?

	$P \neq NP$	$P = NP$	Ind	DC	DK	DK and DC	other
2002	61(61%)	9(9%)	4(4%)	1(1%)	22(22%)	0(0%)	3(3%)
2012	126 (83%)	12 (9%)	5 (3%)	5 (3%)	1(0.6%)	1 (0.6%)	1 (0.6%)

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When Will P=NP be resolved?

	02–09	10–19	20–29	30–39	40–49	50–59	60–69	70–79
2002	5(5%)	12(12%)	13(13%)	10(10%)	5(5%)	12 (12%)	4(4%)	0(0%)
2012	0(0%)	2(.01%)	17(11%)	18(12%)	5(3%)	10 (6.5%)	10 (6.5%)	9(6%)

	80–89	90–99	100–109	110–119	150–159	2200–3000	4000–4100
2002	1(1%)	0(0%)	0(0%)	0(0%)	0(0%)	5(5%)	0(0%)
2012	4(3%)	5(3%)	2(1.2%)	5(3%)	2(1.2%)	3(2%)	3(2%)

	Long Time	Never	Don't Know	Sooner than 2100	Later than 2100
2002	0(0%)	5(5%)	21(21%)	62(62%)	17 (17%)
2012	22(14%)	5(3%)	8(5%)	81(53%)	63 (41%)

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Notable Statements on P vs NP

Scott Aaronson I believe P $\neq$ NP on basically the same grounds that I think I won't be devoured tomorrow by a 500-foot-tall robotic marmoset from Venus, despite my lack of proof in both cases.

Suggested rephrased question:
will humans manage to prove P $\neq$ NP *before they either kill themselves out or are transcended by superintelligent cyborgs? And if the latter, will the cyborgs be able to prove* P $\neq$ NP*?*

Neil Immerman P $\neq$ NP will be resolved somewhere between 2017 and 2034, using some combination of logic, algebra, and combinatorics.

Donald Knuth: (Retired from Stanford) It will be solved by either 2048 or 4096. I am currently somewhat pessimistic. The outcome will be the truly worst case scenario: namely that someone will prove "P=NP because there are only finitely many obstructions to the opposite hypothesis"; hence there will exists a polynomial time solution to SAT but we will never know its complexity!

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