CS4102 Algorithms

Spring 2019

Warm up

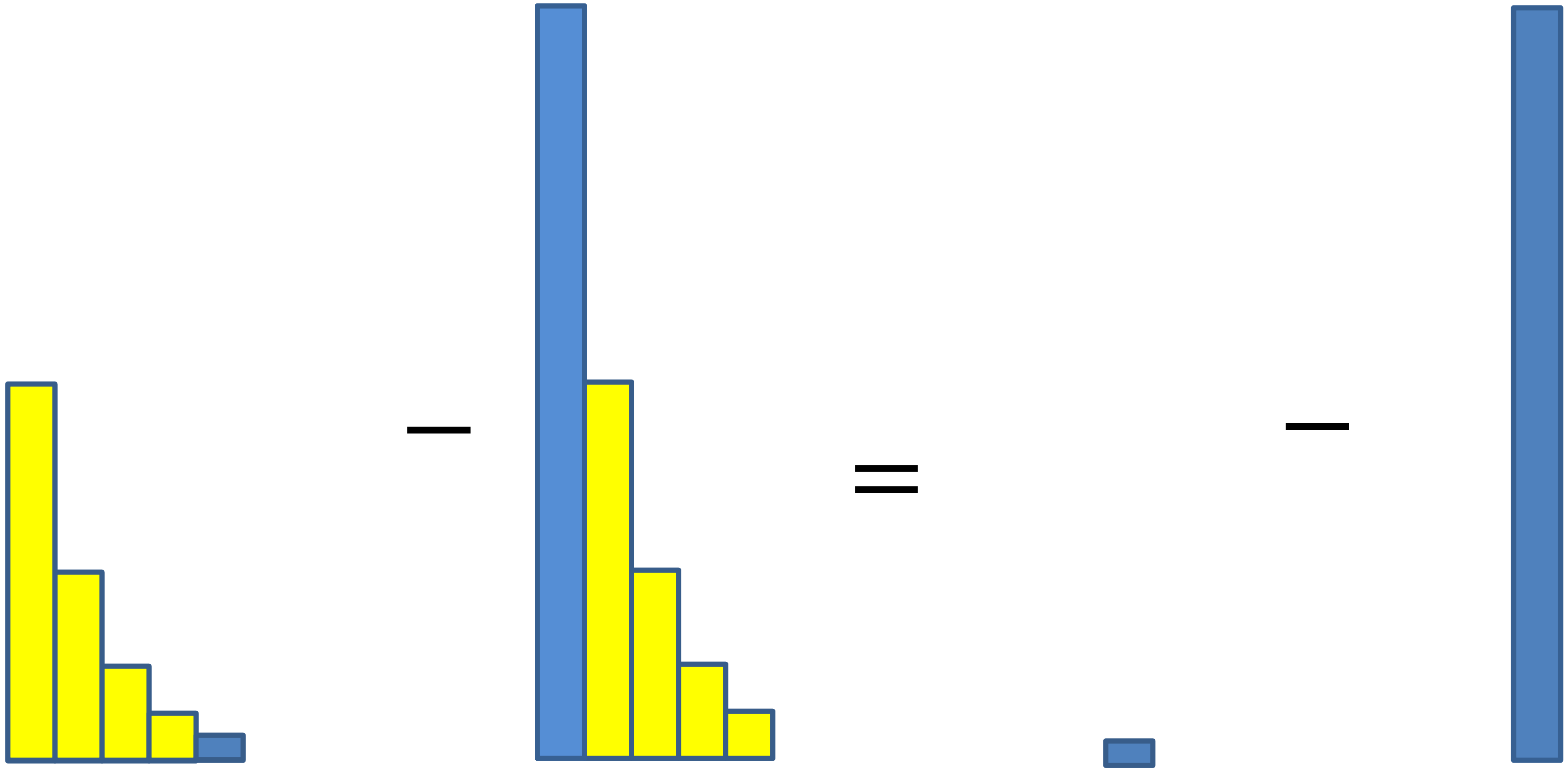
Simplify:

$$(1 + a + a^2 + a^3 + a^4 + \cdots + a^L)(a - 1) = ?$$

$$(\cancel{a} + \cancel{a^2} + \cancel{a^3} + \cancel{a^4} + \cancel{a^5} + \cdots + \cancel{a^L} + a^{L+1}) +$$
$$(-\cancel{a} - \cancel{a^2} - \cancel{a^3} - \cancel{a^4} - \cancel{a^5} - \cdots - \cancel{a^L} - 1) =$$
$$a^{L+1} - 1$$

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$$

Finite Geometric Series $a < 1$



The series multiplied by a

$(1 + a + a^2 + \cdots + a^L)a$

The series

$(1 + a + a^2 + \cdots + a^L)1$

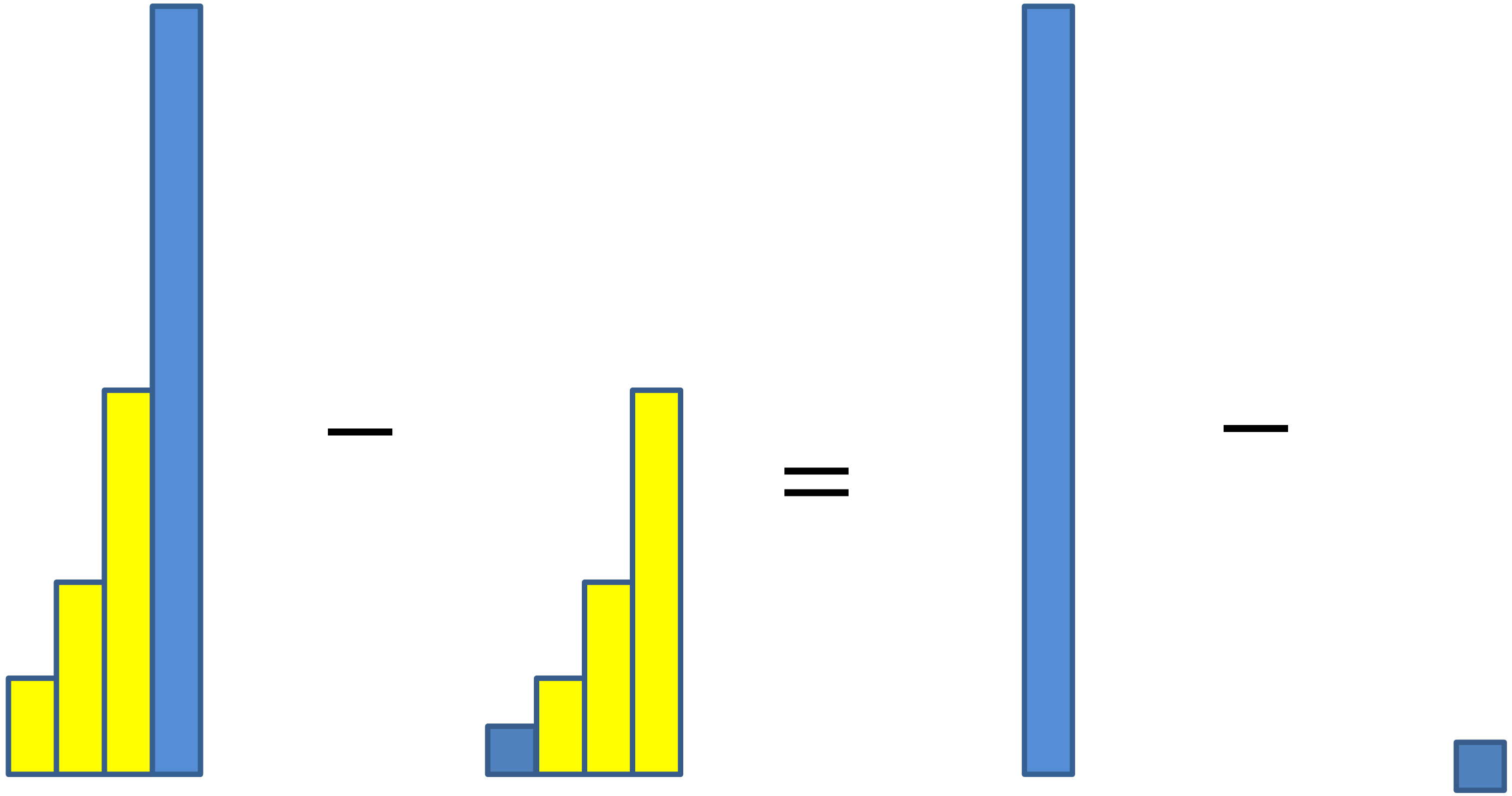
The next term in the series

a^{L+1}

The first term

1

Finite Geometric Series $a > 1$



The series multiplied by a

$(1 + a + a^2 + \cdots + a^L)a$

The series

$(1 + a + a^2 + \cdots + a^L)1$

The next term in the series

a^{L+1}

The first term

1

Today's Keywords

- Divide and Conquer
- Recurrences
- Merge Sort
- Karatsuba
- Tree Method

CLRS Readings

- Chapter 4

Homeworks

- Hw1 due Wed, January 30 at 11pm
  - Start early!
  - Written (use Latex!) – Submit BOTH pdf and zip!
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer

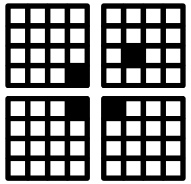
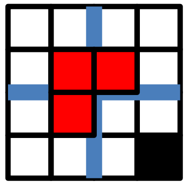
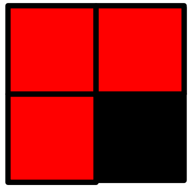
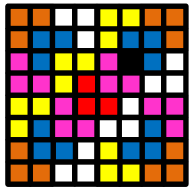
Homework Help Algorithm

- Algorithm: How to ask a question about homework (efficiently)
  1. Check to see if your question is already on piazza
  2. If it’s not on piazza, ask on piazza
  3. Look for other questions you know the answer to, and provide answers to any that you see
  4. TA office hours
  5. Instructor office hours
  6. Email, set up a meeting

Sun 27
Mon 28
Tue 29
Wed 30
Thu 31
Fri 1
Sat 2
all-day
Groundhog Day
9 AM
10 AM
11 AM
Noon
1 PM
2 PM
3 PM
4 PM
5 PM
6 PM
7 PM
8 PM
9 AM Trent's OH Rice 436
2 PM Alex Lehmann OH Rice 436
3:30 PM CS4102 Class Rice 130
5 PM Branden Kim's OH
6 PM Sarah's OH Rice 436
8:30 AM Trent's OH Rice 436
6 PM Sarah's OH Rice 436
10:30 AM Robbie's Office Hours Rice 210
2 PM Alex Lehmann OH Rice 436
3:30 PM CS4102 Class Rice 130
5 PM Nate Saxe OH Rice 436
8:30 AM Trent's OH Rice 436
10 AM Regrade Office Hours
2 PM Alex Lehmann OH Rice 436
2 PM Robbie's 2110 Office Hours
11 AM Alex Lehmann OH Rice 436
2 PM Alex Lehmann OH Rice 436
4 PM Nate Saxe OH Rice 436

Rice 436

Divide and Conquer*

- **Divide**:
  
  - Break the problem into multiple subproblems, each smaller instances of the original
- **Conquer**:
  - If the subproblems are "large":
    
    - Solve each subproblem recursively
  - If the subproblems are "small":
    
    - Solve them directly (base case)
- **Combine**:
  - Merge together solutions to subproblems
  

*CLRS Chapter 4

Analyzing Divide and Conquer

1. Break into smaller subproblems
2. Use recurrence relation to express recursive running time
3. Use asymptotic notation to simplify

- **Divide**: $D(n)$ time,
- **Conquer**: recurse on small problems, size s
- **Combine**: $\mathrm{C}(n)$ time
- **Recurrence:**
  - $T(n) = D(n) + \sum T(s) + C(n)$

Recurrence Solving Techniques

Tree

get a picture of recursion

Guess/Check

guess and use induction to prove

"Cookbook"

MAGIC!

Substitution

substitute in to simplify

Merge Sort

- **Divide:**
  - Break n-element list into two lists of $^n/_2$ elements
- **Conquer:**
  - If $n > 1$:
    - Sort each sublist recursively
  - If $n = 1$:
    - List is already sorted (base case)
- **Combine:**
  - Merge together sorted sublists into one sorted list

Merge

- **Combine:** Merge sorted sublists into one sorted list
- We have:
  - 2 sorted lists (L_1, L_2)
  - 1 output list (L_{out})

While (L_1 and L_2 not empty):

    If $L_1[0] \leq L_2[0]$:

        L_{out}.append(L_1.pop())

    Else:

        L_{out}.append(L_2.pop())

L_{out}.append(L_1)

L_{out}.append(L_2)

$O(n)$

Analyzing Merge Sort

1. Break into smaller subproblems
2. Use recurrence relation to express recursive running time
3. Use asymptotic notation to simplify

- **Divide**: 0 comparisons
- **Conquer**: recurse on 2 small subproblems, size $\frac{n}{2}$
- **Combine**: n comparisons
- **Recurrence:**
  - $T(n) = 2T(\frac{n}{2}) + n$

Recurrence Solving Techniques


Tree

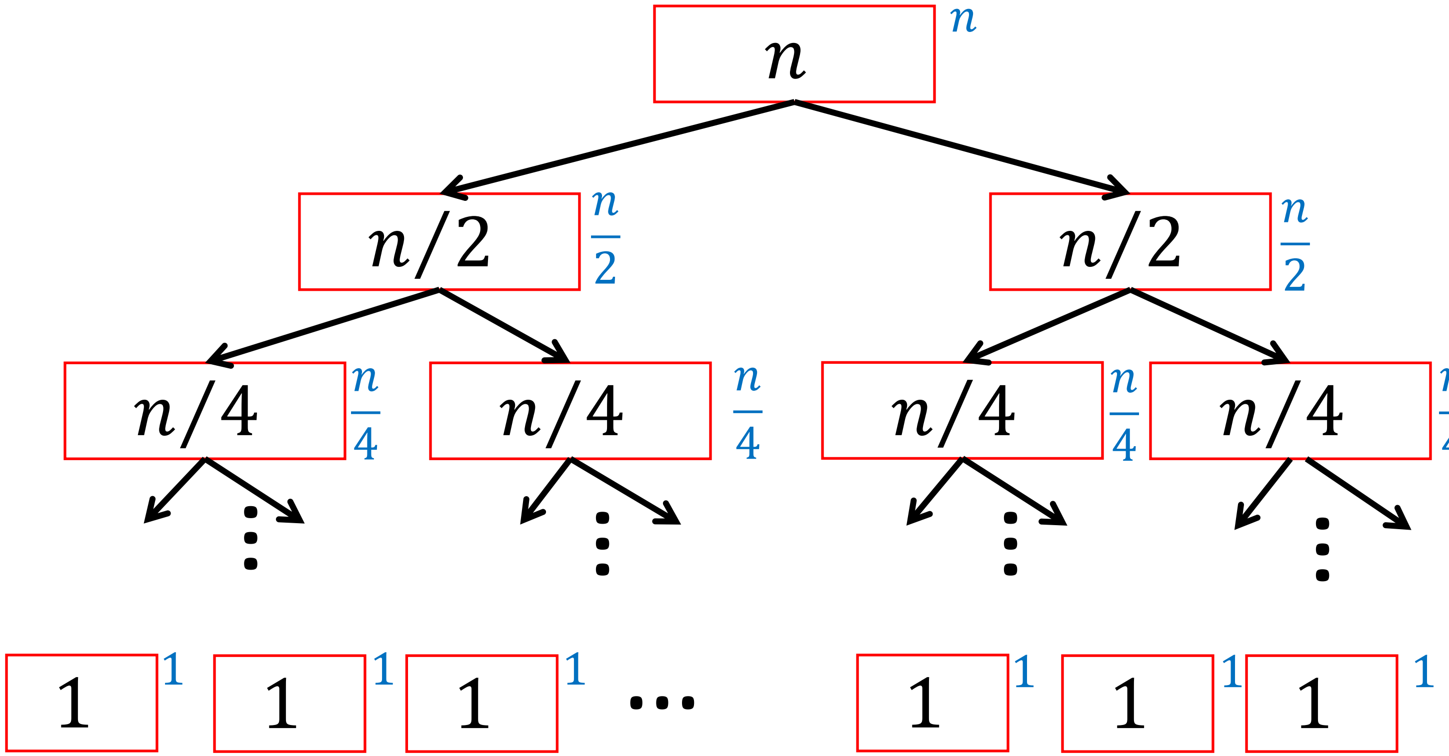
?✓ Guess/Check

"Cookbook"

Substitution

Tree method

$$T(n) = 2T(\frac{n}{2}) + n$$



$\Rightarrow n$ total / level

$\log_2 n$ levels of recursion

$$= n \sum_{i=1}^{\log_2 n} 1 = n \log n$$

$$T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n$$

Multiplication

- Want to multiply large numbers together

$$\begin{array}{r} 4\,1\,0\,2 \\ \times\,1\,8\,1\,9 \\ \hline \end{array}$$

n-digit numbers

- What makes a “good” algorithm?
- How do we measure input size?
- What do we “count” for run time?

“Schoolbook” Method

How many total multiplications?

n-digit numbers

$$
\begin{array}{r}
4\,1\,0\,2 \\
\times\ \boxed{1}\,\boxed{8}\,\boxed{1}\,\boxed{9} \\
\hline
3\,6\,9\,1\,8 \\
4\,1\,0\,2 \\
3\,2\,8\,1\,6 \\
+\ 4\,1\,0\,2 \\
\hline
7\,4\,6\,1\,5\,3\,8
\end{array}
\qquad
\left.\begin{array}{l}
n \text{ mults} \\
n \text{ mults} \\
n \text{ mults} \\
n \text{ mults}
\end{array}\right\} n \text{ levels}
$$

$\Rightarrow \theta(n^2)$

Divide and Conquer method

1. Break into smaller subprojects

$$\underbrace{41}_{a}\underbrace{02}_{b}$$

$4102 = 4100 + 02$

$= 100 \cdot 41 + 02$

$= 10^2 \cdot 41 + 02$

$n = 4, \frac{n}{2} = 2$

$$\boxed{a}\boxed{b} = 10^{\frac{n}{2}}\boxed{a} + \boxed{b}$$

$$\times\ \boxed{c}\boxed{d} = 10^{\frac{n}{2}}\boxed{c} + \boxed{d}$$

$$10^{n}(\boxed{a} \times \boxed{c}) +$$

$$10^{\frac{n}{2}}(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}) +$$

$$(\boxed{b} \times \boxed{d})$$

Divide and Conquer Multiplication

- **Divide**:
  - Break n-digit numbers into four numbers of $^n/_2$ digits each (call them a, b, c, d)
- **Conquer**:
  - If $n > 1$:
    - Recursively compute ac, ad, bc, bd
  - If $n = 1$: (i.e. one digit each)
    - Compute ac, ad, bc, bd directly (base case)
- **Combine**:
  - $10^n(ac) + 10^{\frac{n}{2}}(ad + bc) + bd$

Divide and Conquer method

2. Use recurrence relation to express recursive running time

$$10^n(\boxed{ac}) + 10^{\frac{n}{2}}(\boxed{ad} + \boxed{bc}) + \boxed{bd}$$

Recursively solve

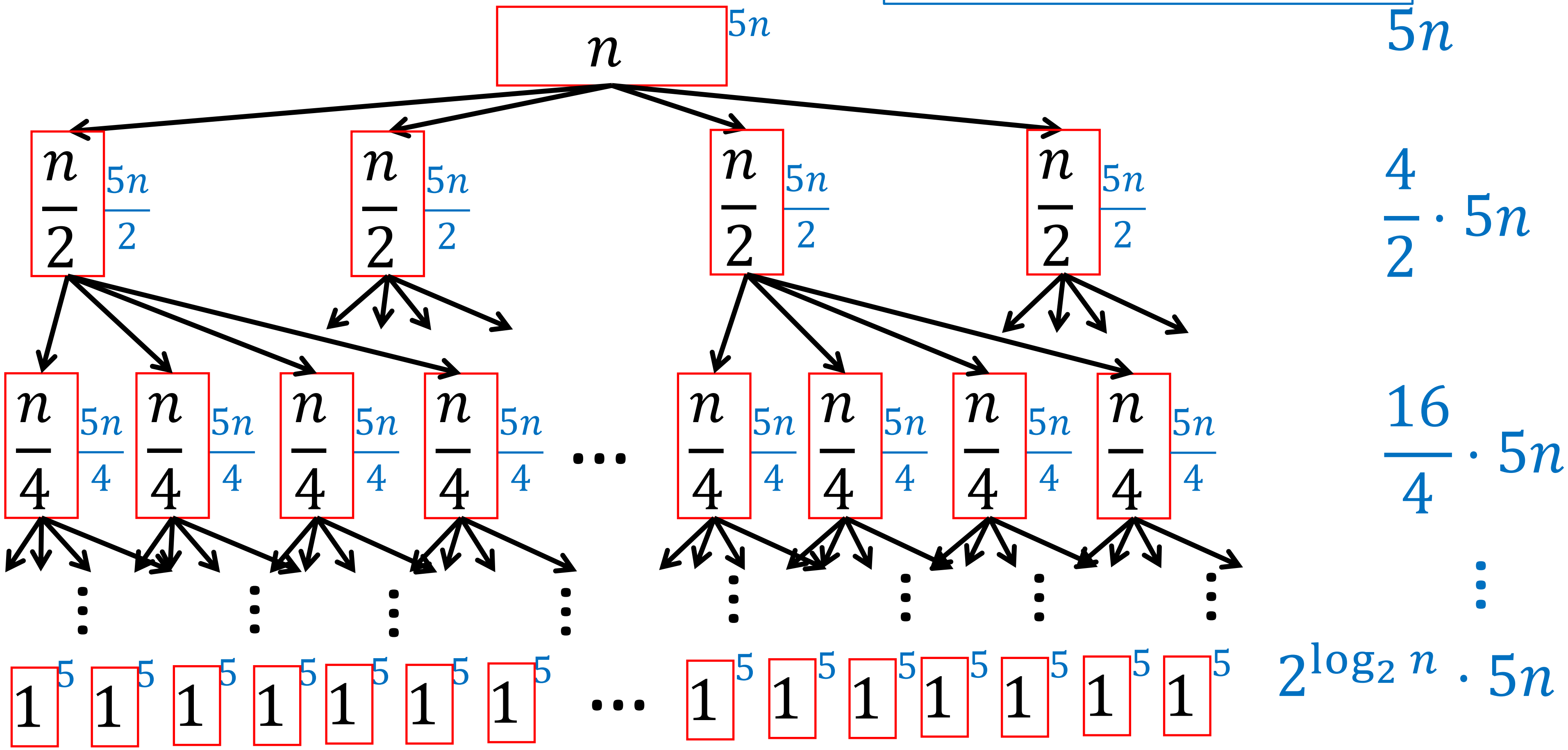
$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Divide and Conquer method

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$



Divide and Conquer method

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$$

$L = \log_2 n$, $a = 2$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$

$$2^{\log_2 n + 1} = 2 \cdot 2^{\log_2 n}$$

$$= 2n$$

$$T(n) = 5n \frac{2^{\log_2 n+1} - 1}{2 - 1}$$

$$T(n) = 5n(2n - 1) \ = \Theta(n^2)$$

Karatsuba

1. Break into smaller subproblems

$$\boxed{a}\boxed{b} = 10^{\frac{n}{2}}\boxed{a} + \boxed{b}$$

$$\times\ \boxed{c}\boxed{d} = 10^{\frac{n}{2}}\boxed{c} + \boxed{d}$$

$$10^{n}(\boxed{a} \times \boxed{c}) +$$

$$10^{\frac{n}{2}}(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}) +$$

$$(\boxed{b} \times \boxed{d})$$

$$\begin{array}{r|c|c|}\hline & a & b \\ \hline \times & c & d \\ \hline\end{array}$$

Karatsuba

$$10^n(\boxed{ac}) + 10^{\frac{n}{2}}(\boxed{ad+bc}) + \boxed{bd}$$

Can't avoid these

This can be simplified

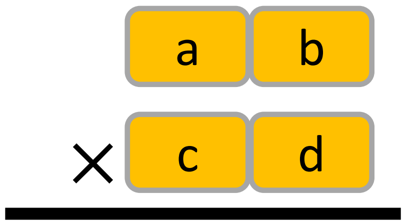
$$(a+b)(c+d) =$$

$$\boxed{ac} + \boxed{ad+bc} + \boxed{bd}$$

$$\boxed{ad+bc} = \boxed{(a+b)(c+d) - \boxed{ac} - \boxed{bd}}$$

Two multiplications

One multiplication



Karatsuba

2. Use recurrence relation to express recursive running time

$$10^n\boxed{(ac)} + 10^{\frac{n}{2}}\left(\boxed{(a+b)(c+d)} - ac - bd\right) + \boxed{bd}$$

Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba

- **Divide**:
  - Break n-digit numbers into four numbers of ${}^{n}/_{2}$ digits each (call them a, b, c, d)
- **Conquer**:
  - If $n > 1$:
    - Recursively compute $ac, bd, (a + b)(c + d)$
  - If $n = 1$:
    - Compute $ac, bd, (a + b)(c + d)$ directly (base case)
- **Combine**:
  - $10^n(ac) + 10^{\frac{n}{2}}\big((a + b)(c + d) - ac - bd\big) + bd$

a b

× c d

Karatsuba Algorithm

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

<u>Pseudo-code</u>

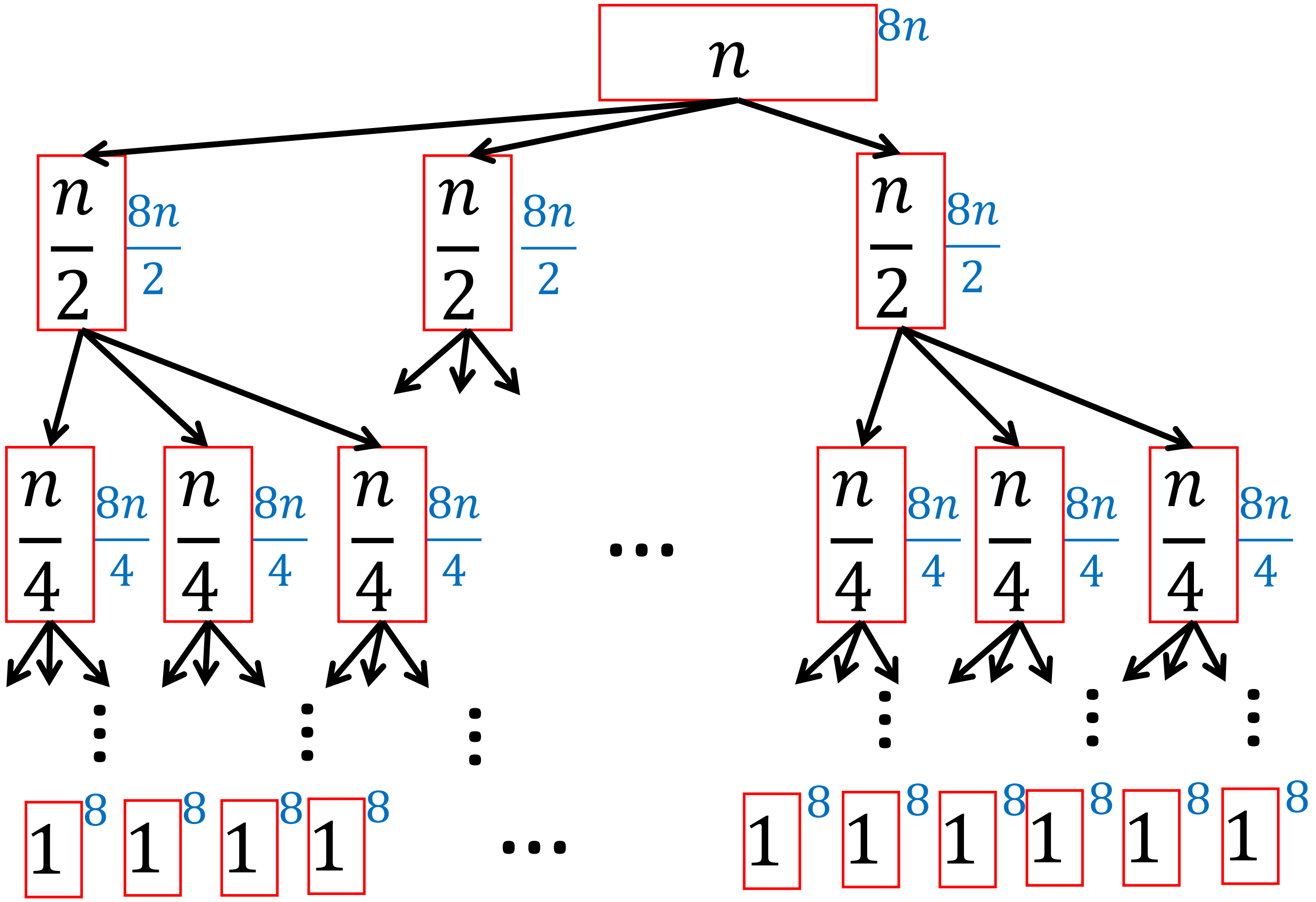
1. x = Karatsuba(a,c)
2. y = Karatsuba(a,d)
3. z = Karatsuba(a+b,c+d)-x-y
4. Return 10^nx +$10^{n/2}$z+y

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}\!/_{2}\right)^{i}$$



$$8 \cdot 1n$$

$$\frac{8}{2} \cdot 3n$$

$$\frac{8}{4} \cdot 9n$$

$$\vdots$$

$$\frac{8}{2^{\log_2 n}} \cdot 3^{\log_2 n} n$$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

Karatsuba

$$T(n) = 8n \frac{\left(\frac{3}{2}\right)^{\log_2 n + 1} - 1}{\cancel{\frac{3}{2}} \quad \frac{1}{2}}$$

$$= 16n\left(\left(\frac{3}{2}\right)^{\log_2 n + 1} - 1\right)$$

$$= 16n\left(\left(2^{\log_2 3 - 1}\right)^{\log_2 n + 1}\right) - 16n$$

$$= 16n\left(2^{\log_2 3 \log_2 n - \log_2 n + \log_2 3 - 1}\right) - 16n$$

$$= 16n\left(\left(2^{\log_2 n}\right)^{\log_2 3} \cdot 2^{-\log_2 n} + 2^{\log_2 3} \cdot 2^{-1}\right) - 16n$$

$$3 = 2^{\log_2 3}$$

$$\frac{2^{\log_2 3}}{2} = 2^{\log_2 3 - 1}$$

Karatsuba

3. Use asymptotic notation to simplify

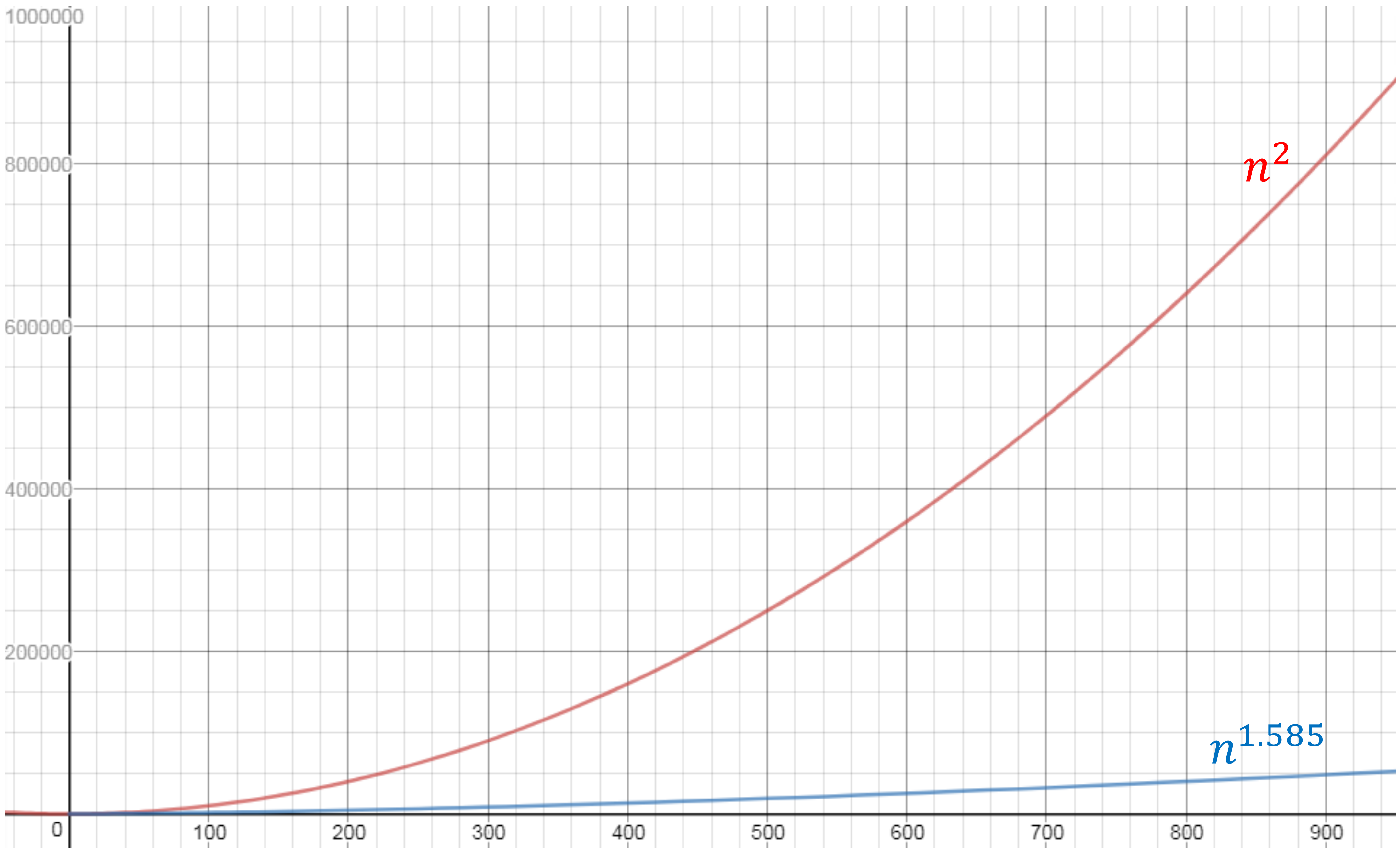
$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

$$T(n) = 24\left(n^{\log_2 3}\right) - 16n \;= \Theta\left(n^{\log_2 3}\right)$$
$$\approx \Theta\left(n^{1.585}\right)$$


1000000
800000
600000
400000
200000
0
100
200
300
400
500
600
700
800
900
n^2
$n^{1.585}$